# Complexity of quantum motion, entanglement, and scrambling: A phase-space approach

Giuliano Benenti

Center for Nonlinear and Complex Systems
Univ. Insubria, Como, Italy

In collaboration with:
Jiaozi Wang, Wenge Wang (Hefei)
Giulio Casati (Como)

Ref.: arXiv:1912.07043

# OUTLINE

Understanding, characterising, and measuring the complexity of quantum motion: a fundamental problem for quantum quantum information and quantum technologies

Classical complex systems characterized by exponential instability of motion (chaos, algorithmic complexity, deterministic randomness,...)

Quantum mechanics: the notion of trajectories is forbidden by the Heisenberg uncertainty principle

Phase-space approach: we propose the number **number of harmonics/separability entropy of the Wigner function** as measures of complexity of a quantum state

# Classical chaos: Exponential instability

Classical chaos is characterized by exponential local instability: two nearby trajectories separate exponentially, with rate given by the maximum Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

$d$ length of the tangent vector

# Classical chaos: Trajectories are unpredictable

Chaotic orbits are unpredictable: in order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment and independent of the previous length of the trajectory. The information associated with a segment of trajectory of length $t$ is equal, asymptotically, to

$$\lim_{t\to\infty} \frac{I(t)}{t} = h,$$

$h$ is the KS (Kolmogorov-Sinai) entropy: positive when $\lambda > 0$

# Classical chaos: Statistical description of motion

Exponential instability $\Rightarrow$ Continuous (frequency) Fourier spectrum of motion

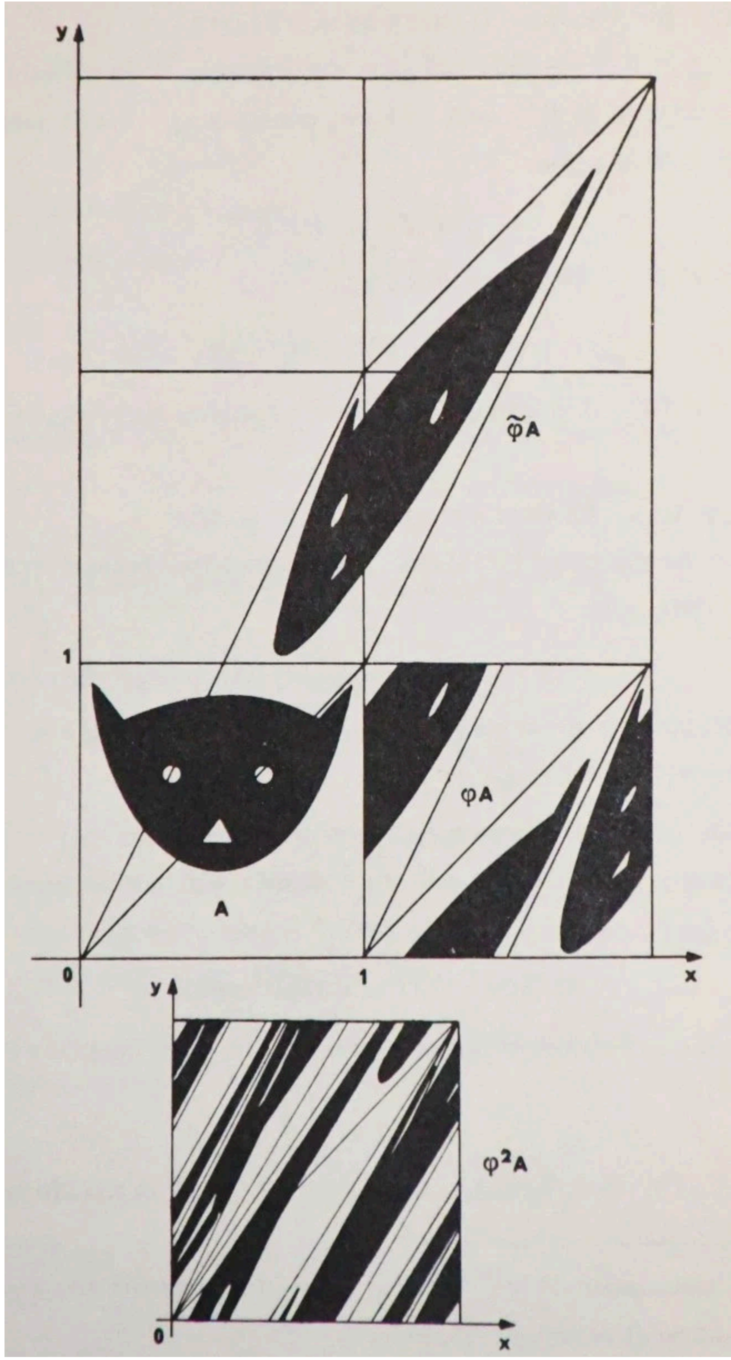
Continuous spectrum (plus undecomposable energy surface) $\Rightarrow$ Decay of correlations (mixing)

Mixing assures the statistical independence of different parts of a trajectory

Mixing (scrambling in quantum information) $\Rightarrow$ Statistical description of chaotic dynamics (diffusion, relaxation, ...)

Integrable systems $\Rightarrow$ Nearby points separate only linearly

# Loss of memory in the Arnold cat map




$$\varphi : \begin{cases} \bar{x} = x + y \pmod 1, \\ \bar{y} = x + 2y \pmod 1 \end{cases}$$

$$h = \lambda = \ln\left(\frac{3+\sqrt{5}}{2}\right) > 0$$

Stretching and folding of the cat in phase space

Any amount of error rapidly effaces the memory of the initial distribution

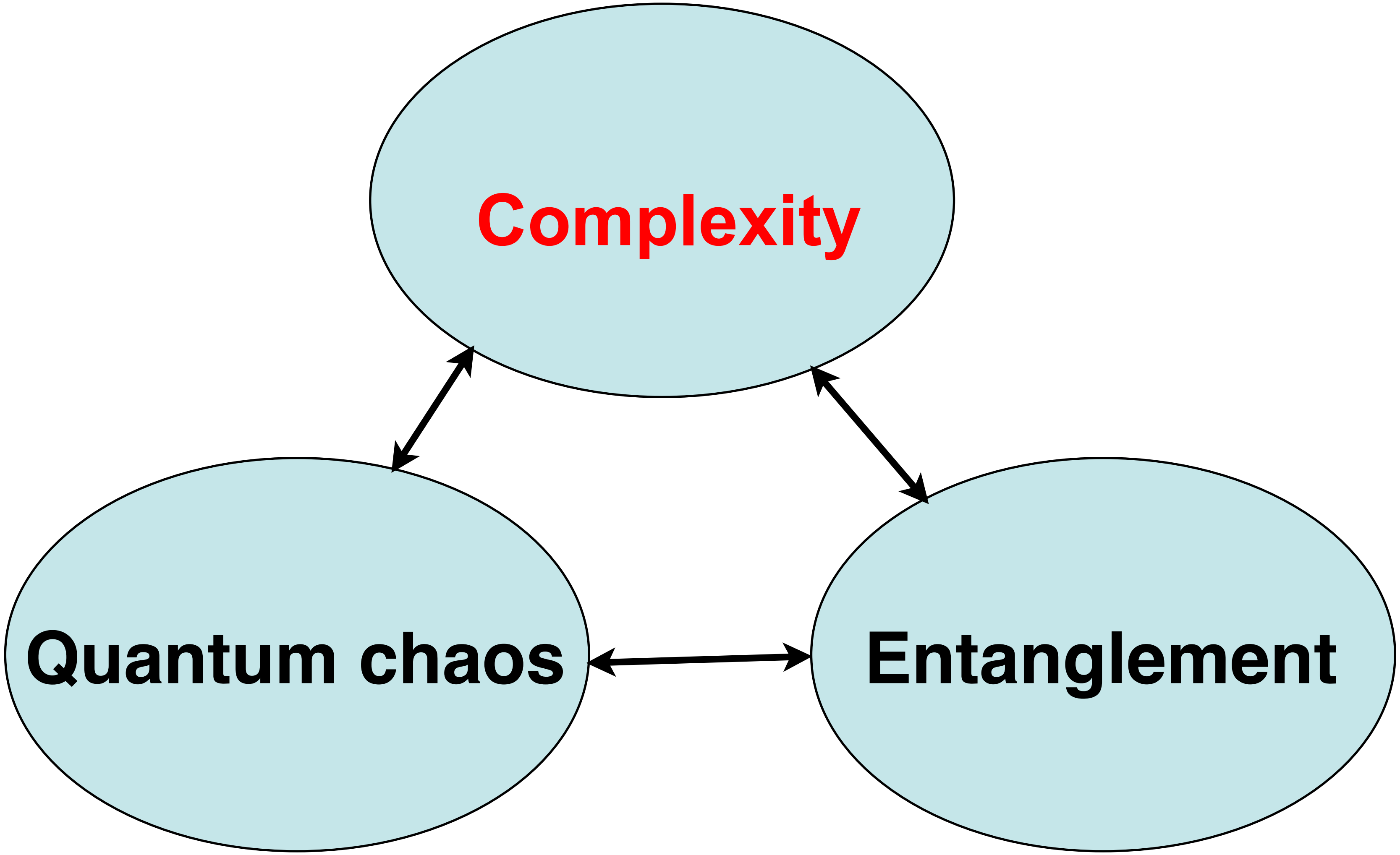
(Arnold and Avez, *Ergodic problems of classical mechanics*)

# Quantum chaos?

The alternative of exponential or power-law divergence of trajectories disappears in quantum mechanics, Heisenberg's uncertainty principle forbidding the notion of trajectories

The energy and the frequency spectrum of any quantum motion, bounded in phase space, are always DISCRETE $\Rightarrow$ regular motion

"Discreteness of the phase space": the uncertainty principle implies a finite size of an elementary phase space cell


Complexity
Quantum chaos
Entanglement

**Requirements for quantum complexity quantifiers:**

(i) to provide a unified description of both one- and many-body dynamics;

(ii) to reproduce at the classical limit the well-known notion of classical complexity based on the local exponential instability of chaotic dynamics;

(iii) to be applicable to both pure and mixed states;

(iv) to be practically useful, that is, convenient for numerical investigations.

# Number of harmonics

In classical mechanics the number of harmonics (i.e., components in the Fourier space) of the classical distribution function in phase space is an estimate of the classical computing resources needed for accurate simulation of Liouville dynamics

The (growth rate of the) number of harmonics is a measure of classical complexity (Gu, Brumer, Pattanayak, Gong,…)

The **number of harmonics of the Wigner function** is a suitable measure of the complexity of a quantum state

# Number of harmonics in classical mechanics

Hamiltonian of a *N*-particle system:

$$H = H_0 + H_I$$

$\boldsymbol{I} = (I_1, ..., I_N)$ and $\boldsymbol{\theta} = (\theta_1, ..., \theta_N)$ action-angle variables of $H_0$

$H_I$ integrable or non-integrable perturbation

$\rho(\boldsymbol{I}, \boldsymbol{\theta}; t)$ classical distribution function

$$\rho(\boldsymbol{I}, \boldsymbol{\theta}; t) = \frac{1}{\pi^N} \sum_{\boldsymbol{m}} \rho_{\boldsymbol{m}}(\boldsymbol{I}; t) \exp(i\boldsymbol{m} \cdot \boldsymbol{\theta}),$$

Evaluate the number of harmonics by the second moment of harmonics distribution:

$$\mathcal{M}_2^{cl}(t) = \frac{\sum_{\boldsymbol{m}} \boldsymbol{m}^2 \int_0^\infty dI |\rho_{\boldsymbol{m}}(\boldsymbol{I}; t)|^2}{\sum_{\boldsymbol{m}} \int_0^\infty dI |\rho_{\boldsymbol{m}}(\boldsymbol{I}; t)|^2}$$

*Finite-time results depend on the choice of the basis, not the asymptotic growth rate, determined by the largest Lyapunov exponent, which is basis-independent*

# Number of harmonics in quantum mechanics

$$\hat{H} = \hat{H}_0 + \hat{H}_1,$$
$$\hat{H}_0 = \hat{H}_0(\hat{n}_1, \cdots, \hat{n}_N)$$
$$\hat{H}_1 = \hat{H}_1(\hat{a}_1^\dagger, \cdots, \hat{a}_N^\dagger, \hat{a}_1, \cdots, \hat{a}_N; t)$$
$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0,\ [\hat{a}_i^\dagger, \hat{a}_j] = \delta_{ij},\quad \hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$

Wigner function:

$$W(\boldsymbol{\alpha}, \boldsymbol{\alpha}^*; t) = \frac{1}{\pi^{2N}\hbar^N}\int d^2\boldsymbol{\eta} \exp\left(\frac{\boldsymbol{\eta}^* \cdot \boldsymbol{\alpha}}{\sqrt{\hbar}} - \frac{\boldsymbol{\eta} \cdot \boldsymbol{\alpha}^*}{\sqrt{\hbar}}\right) \mathrm{Tr}[\hat{\rho}(t)\hat{D}(\boldsymbol{\eta})],$$

$$\hat{D}(\boldsymbol{\eta}) = \exp\left[\sum_{i=1}^{N}\left(\eta_i \hat{a_i}^\dagger - \eta_i^* \hat{a}_i\right)\right]$$ displacement operator

$$W(\boldsymbol{\alpha},\boldsymbol{\alpha}^*;t)=\frac{1}{\pi^N}\sum_{\boldsymbol{m}} W_{\boldsymbol{m}}(\boldsymbol{I};t)\exp(i\boldsymbol{m}\cdot\boldsymbol{\theta}) \ ,$$

$$\alpha_k = \sqrt{I_k}\exp(i\theta_k)$$

Evaluate the number of harmonics by the second moment of (Wigner) harmonics distribution:

$$\mathcal{M}_2(t) = \sum_{\boldsymbol{m}} |\boldsymbol{m}|^2 \mathcal{W}_{\boldsymbol{m}}(t) \ ,$$

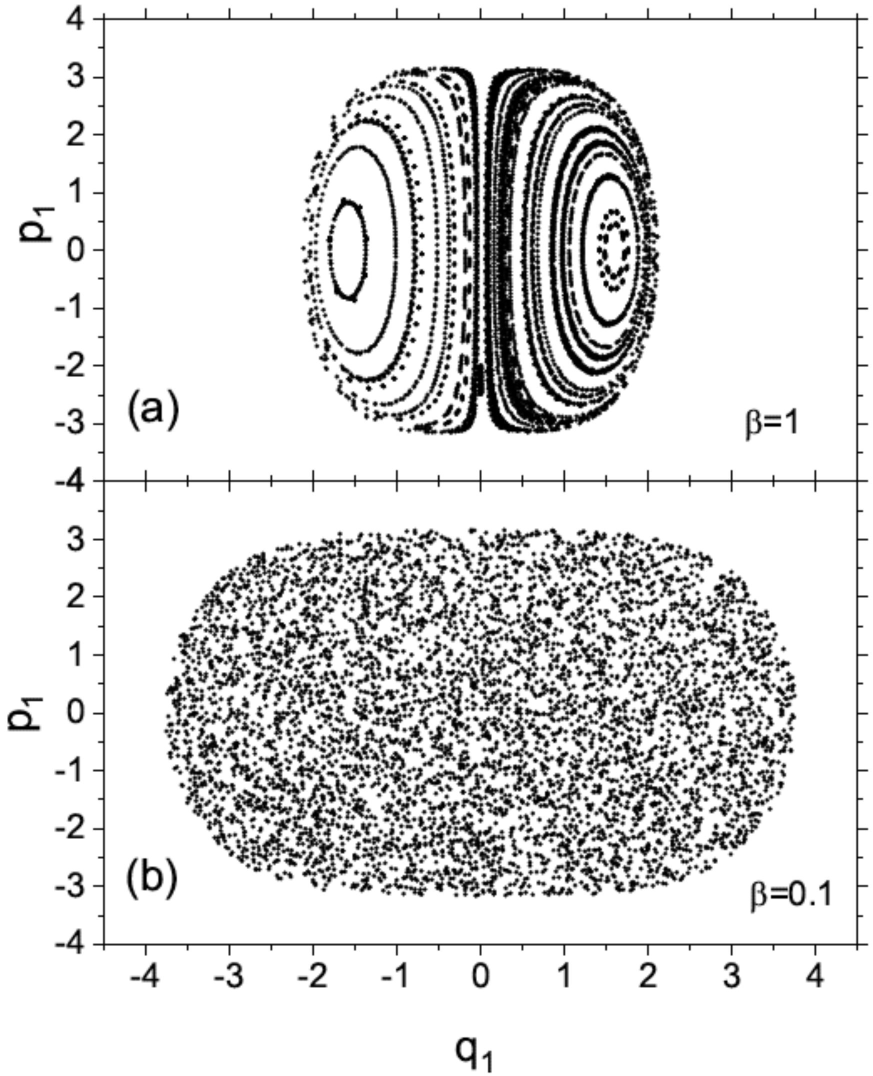
$$\mathcal{W}_{\boldsymbol{m}}(t) = \frac{\int d\boldsymbol{I}|W_{\boldsymbol{m}}(\boldsymbol{I};t)|^2}{\sum_{\boldsymbol{m}}\int d\boldsymbol{I}|W_{\boldsymbol{m}}(\boldsymbol{I};t)|^2}$$

In terms of the density operator one can show that

$$\mathcal{M}_2(t) = \frac{\sum_k \mathrm{Tr}(|[\hat{\rho}(t),\hat{I}_k]|^2)}{\hbar^2\mathrm{Tr}(\hat{\rho}^2(t))}, \quad \hat{I}_k = \hbar\hat{n}_k$$

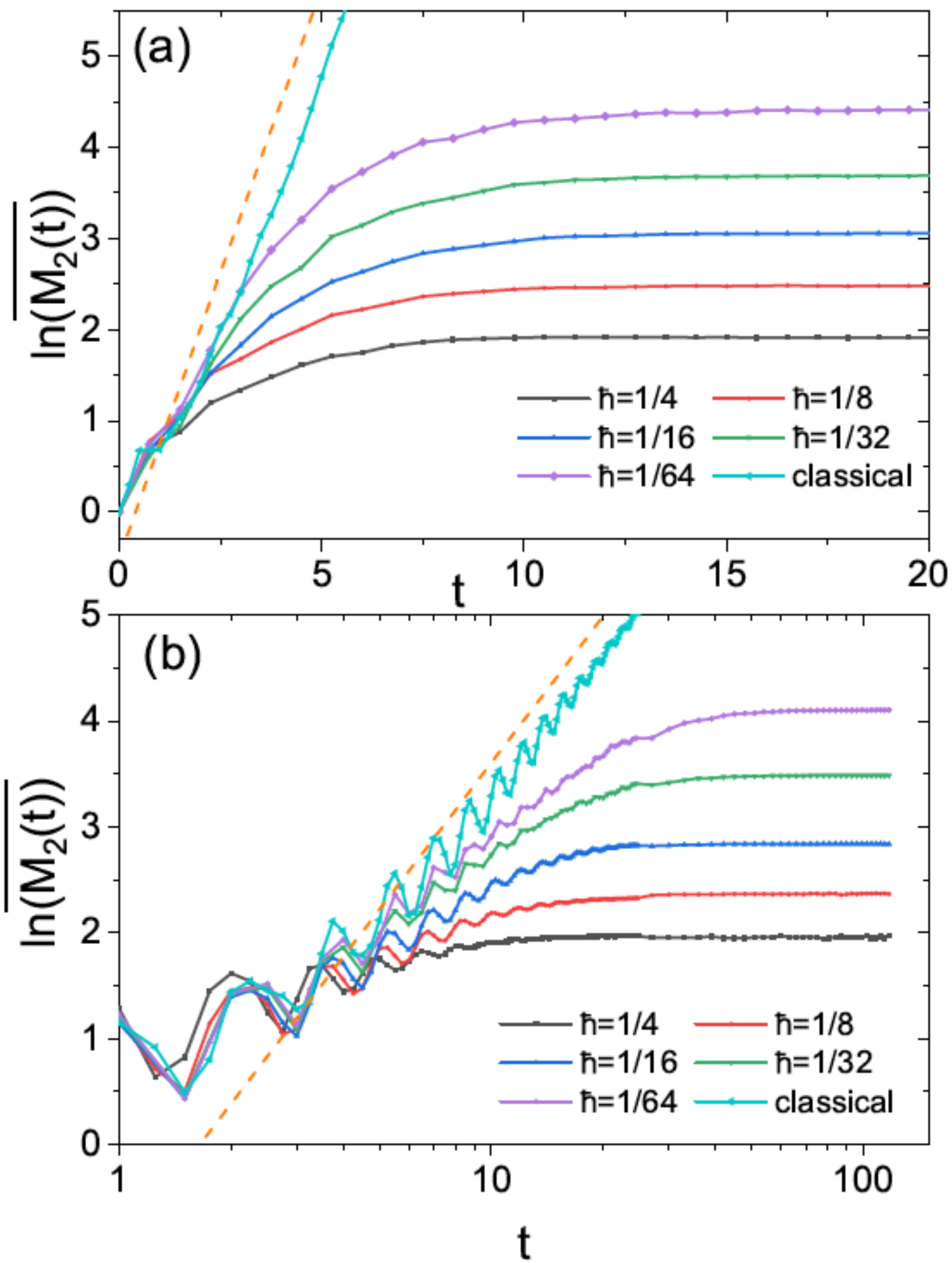
# Numerical illustration: coupled oscillators

$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{\beta}{4}(\hat{q}_1^4 + \hat{q}_2^4) + \frac{1}{2}\hat{q}_1^2\hat{q}_2^2$$




Integrable regime

Chaotic regime




Chaotic regime

(exponential growth, with rate twice the largest Lyapunov exponent)

Integrable regime

(quadratic growth)

# Wigner separability entropy

## Schmidt (singular value) decomposition of the Wigner function

Arbitrary phase space decomposition, $\Omega = \Omega_1 \oplus \Omega_2$, into two set of coordinates, $z \equiv (x, y)$; normalization constraint $\int \mathrm{d}z W(z) = 1$ :

$$W(\boldsymbol{x}, \boldsymbol{y}) = \sum \mu_n v_n(\boldsymbol{x}) w_n(\boldsymbol{y})$$

with $n \in \mathbb{N}$, $\{v_n\}$ and $\{w_n\}$ orthonormal bases for $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively, and the Schmidt coefficients (singular values) $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ satisfying $\sum_n \mu_n^2 = \int \mathrm{d}z W^2(z)$.

*Definition (Wigner separability entropy):*

$$h[W] = -\sum_n \tilde{\mu}_n^2 \ln \tilde{\mu}_n^2, \qquad \tilde{\mu}_n \equiv \frac{\mu_n}{\sqrt{\int \mathrm{d}z W^2(z)}}$$

# Connection with operator space entanglement entropy (for bipartite systems)

$\mathrm{Tr}(\hat{\rho}^2) \leqslant 1 \Rightarrow$ the density operator is a Hilbert-Schmidt operator

$$\|\hat{\rho}\|_{\mathrm{HS}} = \sqrt{\mathrm{Tr}(\hat{\rho}^2)} \qquad \langle \hat{A}, \hat{B} \rangle_{\mathrm{HS}} = \mathrm{Tr}(\hat{A}^\dagger \hat{B})$$

Therefore, given $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the density operator has a Schmidt decomposition:

$$\hat{\rho} = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n,$$

where $\{\hat{\sigma}_n\}$ and $\{\hat{\tau}_n\}$ are orthonormal $[\mathrm{Tr}(\hat{\sigma}_m^\dagger \hat{\sigma}_n) = \delta_{mn}$, $\mathrm{Tr}(\hat{\tau}_m^\dagger \hat{\tau}_n) = \delta_{mn}]$ bases for $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$, respectively, and the Schmidt coefficients $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ satisfying $\sum_n \mu_n^2 = \mathrm{Tr}(\hat{\rho}^2) = \|\hat{\rho}\|_{\mathrm{HS}}^2$.

# Operator space entanglement entropy

$$h[\hat{\rho}] = -\sum_n \tilde{\mu}_n^2 \ln \tilde{\mu}_n^2, \quad \tilde{\mu}_n \equiv \frac{\mu_n}{\|\hat{\rho}\|_{\mathrm{HS}}}$$

The Weyl correspondence establishes an isomorphism between Hilbert-Schmidt operators and $L^2(\Omega)$ functions on classical phase space. One can prove that:

$$h[\hat{\rho}] = h[W]$$

[Benenti, Carlo, Prosen, PRE **85**, 051129 (2012)]

Put on a broader context the complexity of the classical simulation of quantum dynamics

# Pure states

Schmidt decomposition of $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$|\psi\rangle = \sum_j \lambda_j |\phi_j\rangle \otimes |\xi_j\rangle$$

Schmidt decomposition of $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\hat{\rho} = \sum_{j,k} \lambda_j \lambda_k |\phi_j\rangle\langle\phi_k| \otimes |\xi_j\rangle\langle\xi_k| = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n$$

$$h[W] = -\sum_n \mu_n^2 \ln \mu_n^2 = -\sum_{j,k} \lambda_j^2 \lambda_k^2 \ln\left(\lambda_j^2 \lambda_k^2\right)$$

$$= -2 \sum_j \lambda_j^2 \ln \lambda_j^2 = -2S(\hat{\rho}_1) = -2S(\hat{\rho}_2),$$

$$\hat{\rho}_1 = \mathrm{Tr}_2(\hat{\rho}) \text{ and } \hat{\rho}_2 = \mathrm{Tr}_1(\hat{\rho})$$

For pure states the Wigner separability entropy is twice the entanglement entropy (i.e., reduced von Neumann entropy)

$$h[W] = 2\,E(|\psi\rangle) = I(1:2)$$

The quantum mutual information measures correlations (both of classical and quantum nature) between subsystems 1 and 2

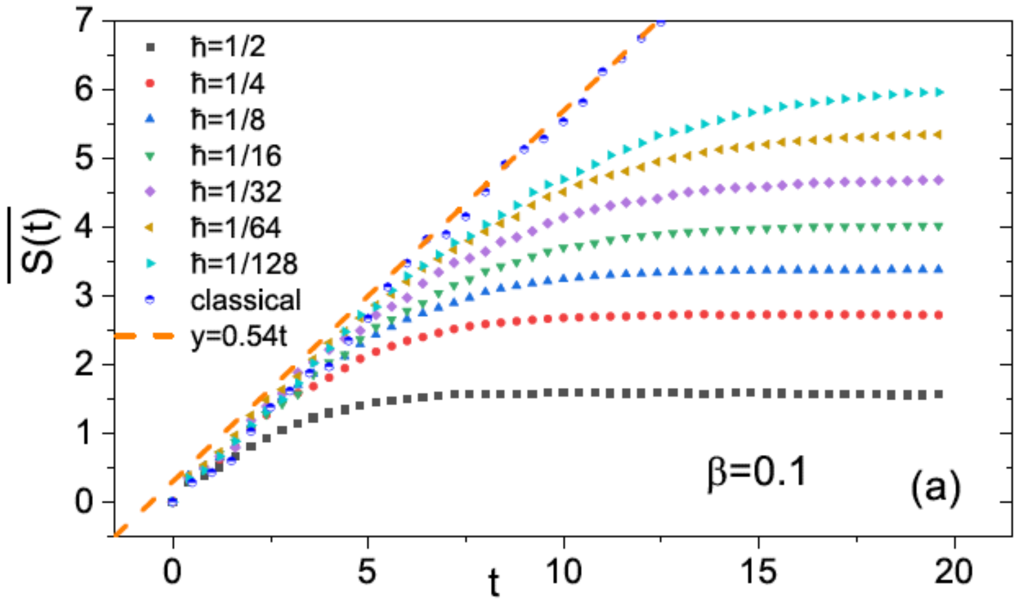
$$I(1:2) = S(\hat{\rho}_1) + S(\hat{\rho}_2) - S(\hat{\rho})$$

# Numerical illustration

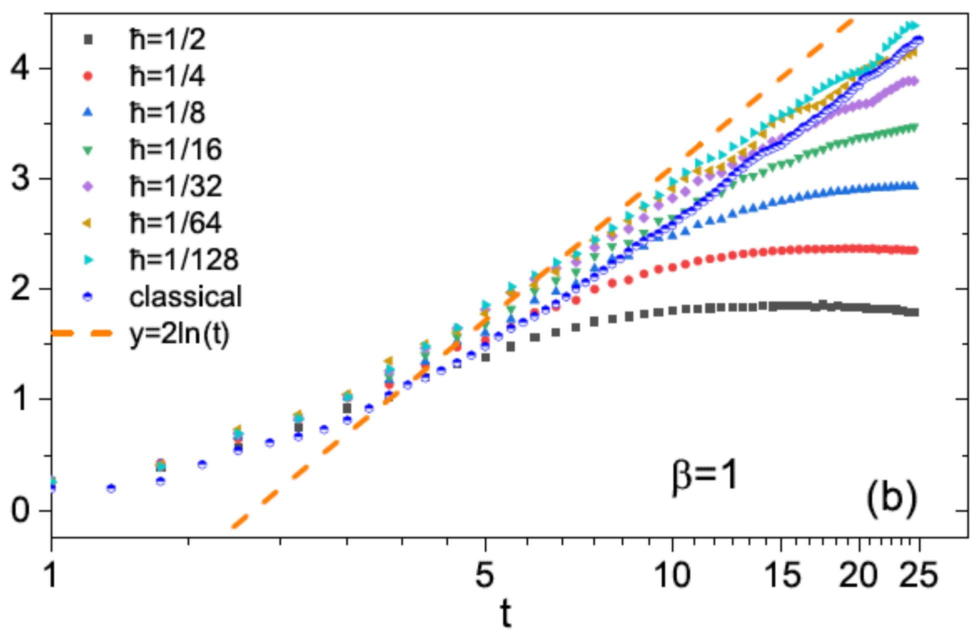
$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{\beta}{4}(\hat{q}_1^4 + \hat{q}_2^4) + \frac{1}{2}\hat{q}_1^2\hat{q}_2^2$$

Coupled oscillators model

Chaotic regime




Integrable regime




$$\overline{S(t)} \propto (\lambda_1 + \lambda_2)t$$

# Connection with out-of-time-ordered correlates (OTOC)

$$\mathcal{C}(t) = \left\langle |[\hat{A}(t), \hat{B}(0)]|^2 \right\rangle$$

The number of harmonics can be seen as an OTOC with:

$$\hat{A}(t) = \hat{\rho}(t) \qquad \hat{B}(0) = \hat{I}_k \qquad \hat{I}_k = \hbar \hat{n}_k$$

Standard OTOC: $C_{pp}(t) = -\frac{1}{\hbar^2}\langle\psi_0|[\hat{p}_1(t), \hat{p}_1(0)]^2|\psi_0\rangle$

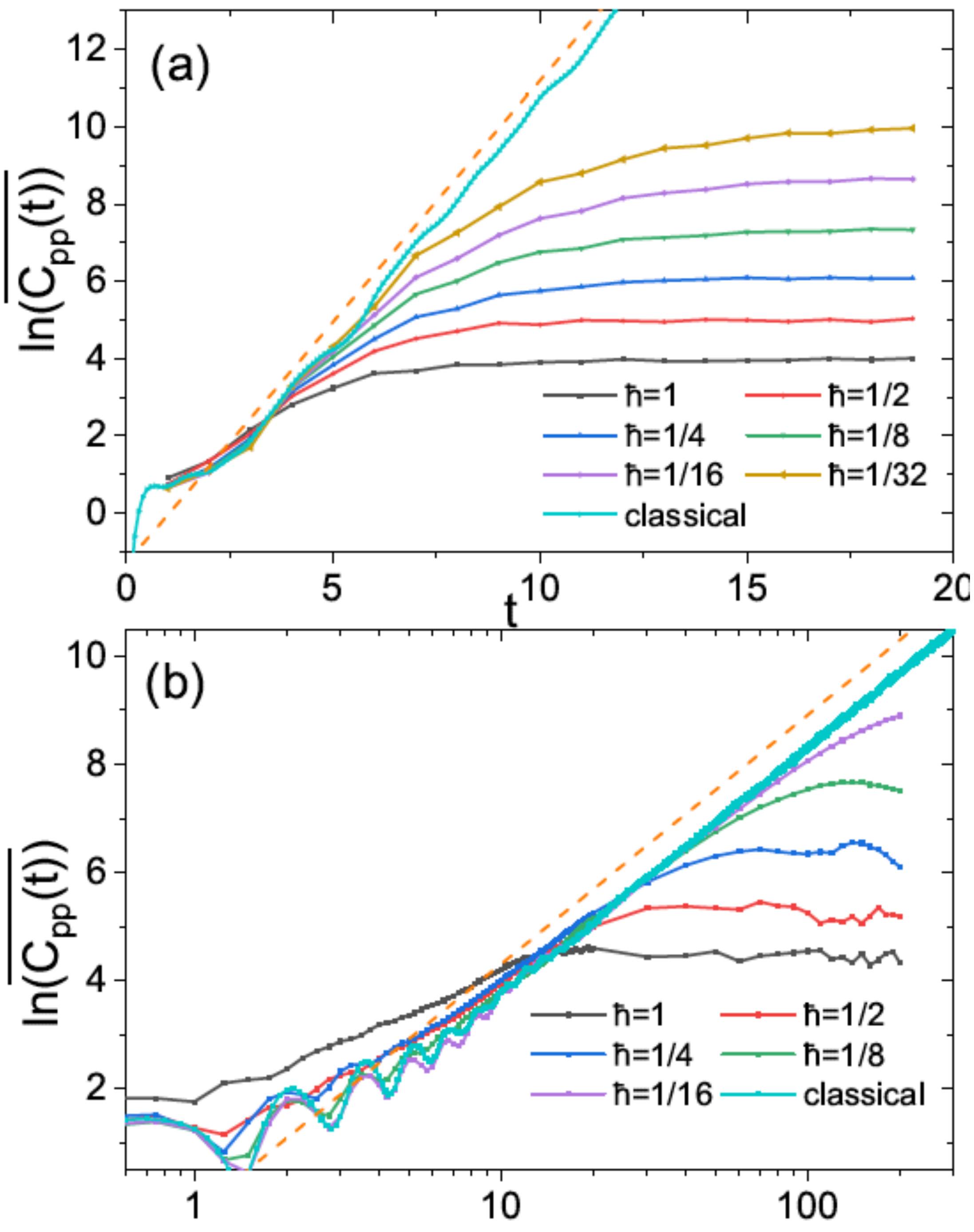
Classical limit:

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \to \{A, B\}_{\mathrm{PB}},$$

$$C^{cl}_{pp}(t) = \int d\gamma(0)\rho_0(\gamma(0))\{p_1(t), p_1(0)\}^2_{PB}$$
$$= \int d\gamma(0)\rho_0(\gamma(0))\left(\frac{\delta p_1(t)}{\delta q_1(0)}\right)^2,$$

Exponential growth in the chaotic case (Larkin and Ovchinnikov, 1996; Maldacena and Stanford, 2016,…)

# Numerical illustration




Chaotic regime

(exponential growth, with rate twice the largest Lyapunov exponent)

Integrable regime

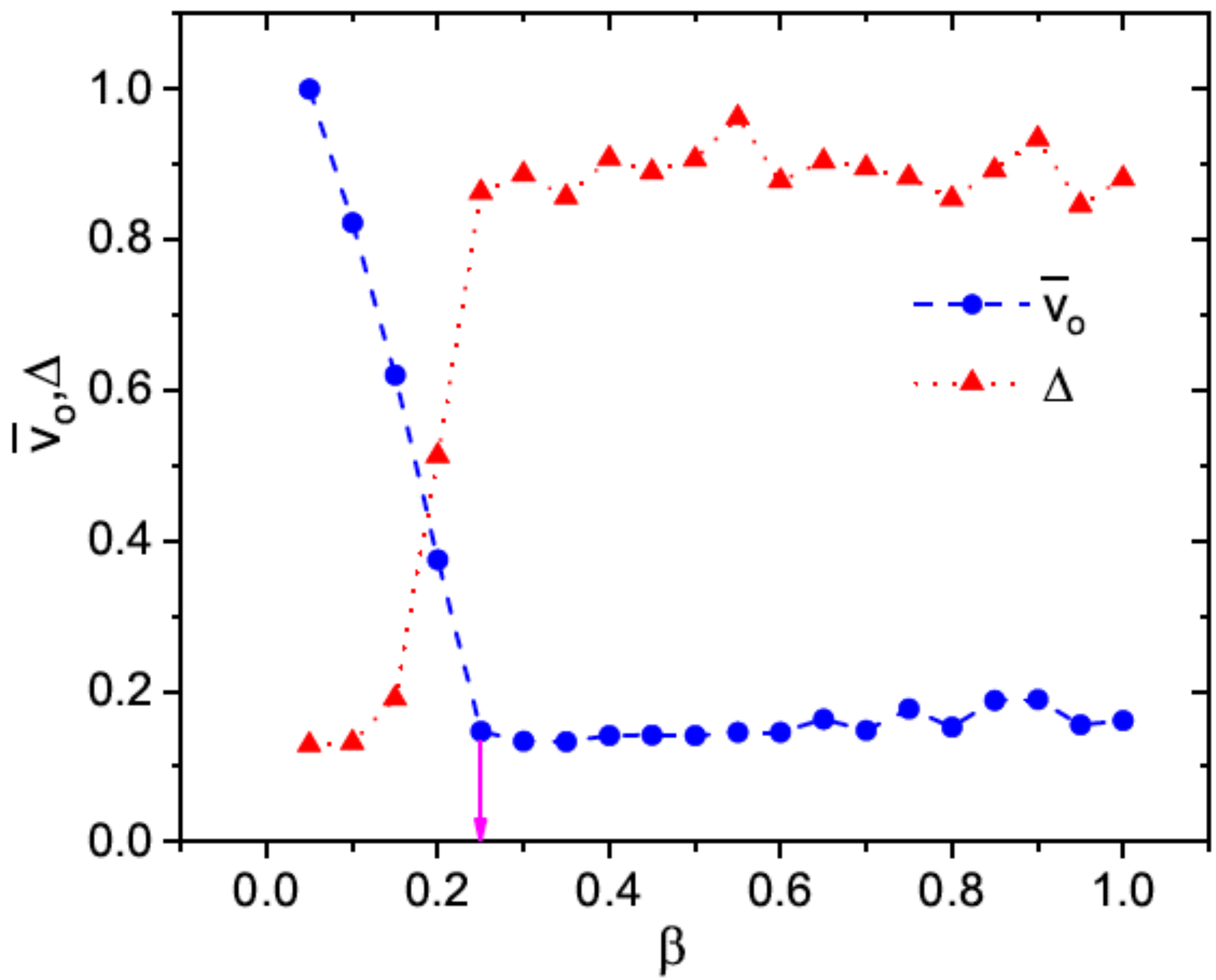
(quadratic growth)

(Coupled oscillators model)

# Detect transition to chaos in the time domain

$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{\beta}{4}(\hat{q}_1^4 + \hat{q}_2^4) + \frac{1}{2}\hat{q}_1^2\hat{q}_2^2$$

Coupled oscillators model




**Δ** from level spacing statistics

$$\Delta = \frac{\int_0^\infty |P(s) - P_w(s)| ds}{\int_0^\infty |P_p(s) - P_w(s)| ds}$$

$\overline{v}_o$ average velocity in the growth of OTOC

# Summary

Complexity of quantum motion can be conveniently characterized in phase space

At the classical limit the classical notion of complexity based on exponential instability is recovered

The relation with entanglement is clear for bipartite pure states, while for the generic case it remains an open problem

Quantum computation and information is a rapidly developing interdisciplinary field. It is not easy to understand its fundamental concepts and central results without facing numerous technical details. This book provides the reader with a useful guide. In particular, the initial chapters offer a simple and self-contained introduction; no previous knowledge of quantum mechanics or classical computation is required.

Various important aspects of quantum computation and information are covered in depth, starting from the foundations (the basic concepts of computational complexity, energy, entropy, and information, quantum superposition and entanglement, elementary quantum gates, the main quantum algorithms, quantum teleportation, and quantum cryptography) up to advanced topics (like entanglement measures, quantum discord, quantum noise, quantum channels, quantum error correction, quantum simulators, and tensor networks).

It can be used as a broad range textbook for a course in quantum information and computation, both for upper-level undergraduate students and for graduate students. It contains a large number of solved exercises, which are an essential complement to the text, as they will help the student to become familiar with the subject. The book may also be useful as general education for readers who want to know the fundamental principles of quantum information and computation.

"Thorough introductions to classical computation and irreversibility, and a primer of quantum theory, lead into the heart of this impressive and substantial book. All the topics – quantum algorithms, quantum error correction, adiabatic quantum computing and decoherence are just a few – are explained carefully and in detail. Particularly attractive are the connections between the conceptual structures and mathematical formalisms, and the different experimental protocols for bringing them to practice. A more wide-ranging, comprehensive, and definitive text is hard to imagine."

— Sir Michael Berry, *University of Bristol, UK*

"This second edition of the textbook is a timely and very comprehensive update in a rapidly developing field, both in theory as well as in the experimental implementation of quantum information processing. The book provides a solid introduction into the field, a deeper insight in the formal description of quantum information as well as a well laid-out overview on several platforms for quantum simulation and quantum computation. All in all, a well-written and commendable textbook, which will prove very valuable both for the novices and the scholars in the fields of quantum computation and information."

— Rainer Blatt, *Universität Innsbruck and IQOQI Innsbruck, Austria*

"The book by Benenti, Casati, Rossini and Strini is an excellent introduction to the fascinating field of quantum information, of great benefit for scientists entering the field and a very useful reference for people already working in it. The second edition of the book is considerably extended with new chapters, as the one on many-body systems, and necessary updates, most notably on the physical implementations."

— Rosario Fazio, *The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

**World Scientific**
www.worldscientific.com
10909 hc

ISBN 978-981-3237-22-3
9 789813 237223

Benenti
Casati
Rossini
Strini

Principles of Quantum Computation and Information
A Comprehensive Textbook

Giuliano Benenti Giulio Casati
Davide Rossini Giuliano Strini

| | | | | |
|---|---|---|---|---|
| S | A | T | O | R |
| A | R | E | P | O |
| T | E | N | E | T |
| O | P | E | R | A |
| R | O | T | A | S |

# Principles of Quantum Computation and Information

## A Comprehensive Textbook

**World Scientific**

# https://tqsp.lakecomoschool.org/

UNIVERSITÀ DEGLI STUDI DI MILANO

Fondazione CARIPLO

## LAKE COMO SCHOOL OF ADVANCED STUDIES

International School on

### THERMODYNAMICS OF QUANTUM SYSTEMS AND PROCESSES

Como (Italy), August 31 – September 4, 2020

**Lectures**

**Alexia Auffèves (Institut Néel, CNRS Grenoble, France)**
*Quantum measurement and thermodynamics*

**Michele Campisi (Università di Firenze, Italy)**
*Fluctuation relations in quantum systems*

**Ronnie Kosloff (Hebrew University of Jerusalem, Israel)**
*Quantum thermal machines*

**Massimo Palma (Università di Palermo, Italy)**
*Non-Markovianity and thermodynamics of open quantum systems*

**Jukka Pekola (Aalto University, Finland)**
*Non equilibrium thermodynamics of quantum circuits*

**Ferdinand Schmidt-Kaler (University of Mainz, Germany)**
*Experimental stochastic and quantum engines: from single particle to many body*

**Robert Whitney (Université Grenoble-Alpes, France)**
*Quantum thermoelectricity*

**Important dates**
Deadline for application: May 15, 2020
Acceptance notification: May 25, 2020

**Organizers**
Giuliano Benenti, Università dell'Insubria (IT)
Dario Gerace, Università di Pavia (IT)
Mauro Paternostro, Queen's University of Belfast (UK)
Jukka Pekola, Aalto University (FI)

**School website**
https://tqsp.lakecomoschool.org/

FONDAZIONE ALESSANDRO VOLTA
CULTURA EDUCAZIONE CIVITAS



Villa del Grumello
Via per Cernobbio 11
22100 Como, Italy
Tel. +39. 031. 579813
Fax +39. 031. 573395
E-mail: info@lakecomoschool.org
www.lakecomoschool.org

https://qtts.ifisc.uib-csic.es

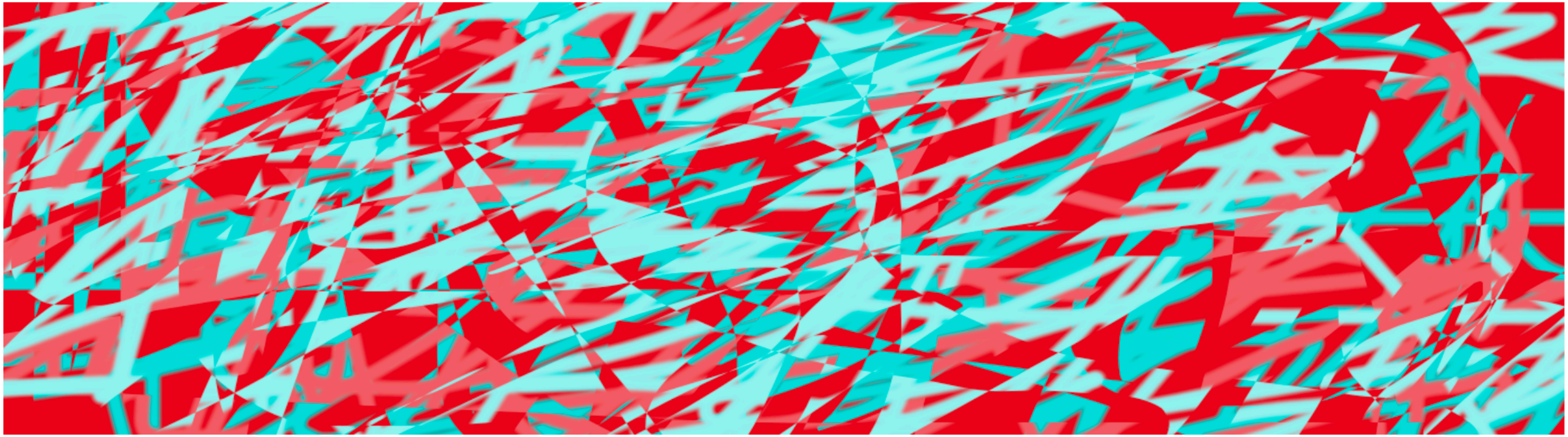
Search

# QTTS

Quantum Transport and Thermodynamics Society

About us | Announcements | Membership | Publications | Links | Contact

Google Custom Search



A network of scientists dedicated to understanding the thermodynamics of quantum systems and quantum transport.