Quantum chaos, decoherence and quantum computation

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Motivations and outline

Even when a quantum computer is ideally isolated from the environment, its operability is not yet guaranteed: also device imperfections hinder the implementation of quantum protocols

Objectives: determine time scales for reliable quantum computation in the presence of quantum computer hardware imperfections, device strategies to improve the stability of quantum computation

A quantum computer represents a complex system of many coupled qubits, which in general can be viewed as a many-body interacting quantum system: stability of quantum dynamics, transition to chaos, ...

OUTLINE:

- 1) Stability and transition to chaos for the quantum hardware
- 2) Stability of quantum algorithms (simulating complex dynamics) under imperfection effects

Error models

Accuracy of quantum computation measured by fidelity: $f(t) = \langle \psi(t) | \rho_\epsilon(t) | \psi(t) \rangle$

Quantum algorithm: $|\psi_f\rangle = U\,|\psi(0)\rangle, \qquad U = \underbrace{U_{N_g} \cdot \ldots \cdot U_1}_{\text{elementary gates}}$

Errors: $U_j \;\longrightarrow W_\epsilon(j) U_j$

(i) Memoryless unitary errors:

$W_\epsilon(j)$ random and different at each j,

e.g.: random phase fluctuations: $\delta\phi \in [-\epsilon, \epsilon]$ in phase-shift gates

(ii) Static imperfections in the quantum computer itself:

$W_\epsilon(j)$ (random but) constant at each j, e.g. static imperfections Hamiltonian:

$$H_s = \sum_{j=0}^{n-1} \delta_j\, \sigma_j^{(z)} + 2 \sum_{j=0}^{n-2} J_j\, \sigma_j^{(x)}\, \sigma_{j+1}^{(x)} \quad , \quad J_j, \delta_j \in [-\epsilon, \epsilon]$$

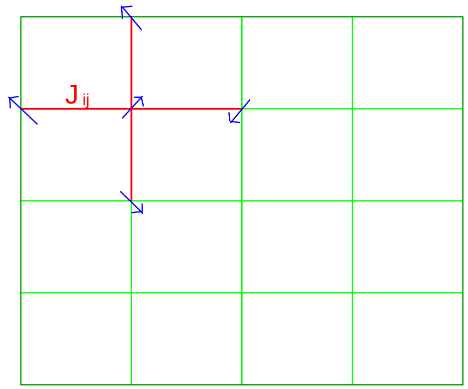
(iii) Non-unitary errors in quantum computation:

$W_\epsilon(j)$ is non-unitary (density matrix and quantum trajectories approach)

A model of quantum computer hardware

Even if the quantum computer is (ideally) decoupled from the environment, internal imperfections can disturb quantum computation

$$H_s = \sum_{i=1}^{n} \Gamma_i \sigma_i^z + \sum_{\langle i<j \rangle} J_{ij} \sigma_i^x \sigma_j^x$$



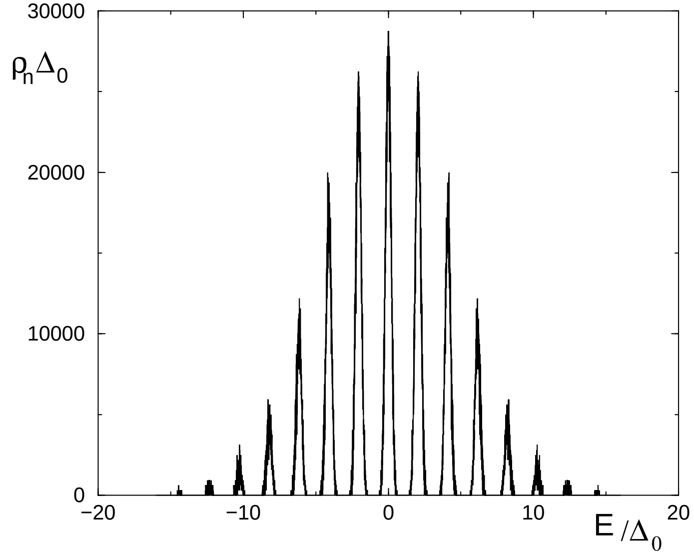
(Georgeot, Shepelyansky, 2000)
$\Gamma_i = \Delta_0 + \delta_i$
$\delta_i \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$ energy fluctuations
$J_{ij} \in [-J, J]$ residual short-range interaction

The non-interacting eigenstates ($J = 0$) are the quantum register states used for computation

H_s quantum computer hardware
$H_s \rightarrow H(\tau) = H_s + H_g(\tau) = H_s + \sum_k \delta(\tau - k\tau_g) h_k$ software (gate operations added)

We consider the limit of small qubit spacing fluctuations and small residual interaction

$$\delta, J \ll \Delta_0$$

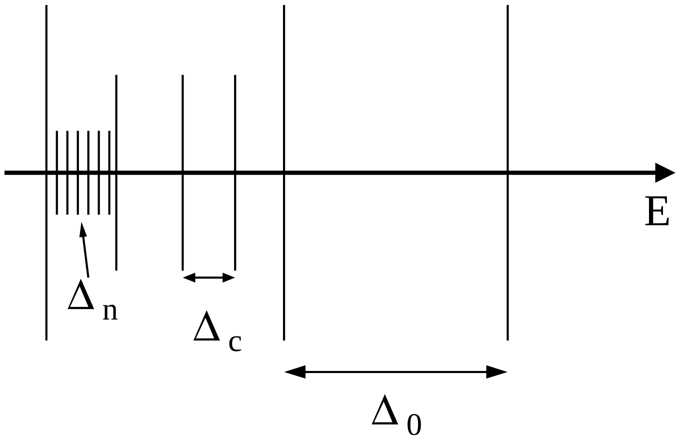


Let us focus on the central band ($n/2$ spins up and $n/2$ spins down), which is the quantum computer core:

a) higher density of states $\longrightarrow$ quantum parallelism

b) quantum chaos first appears in this band

Transition to chaos



- $\Delta_n \propto 2^{-n}$ exponentially small level spacing

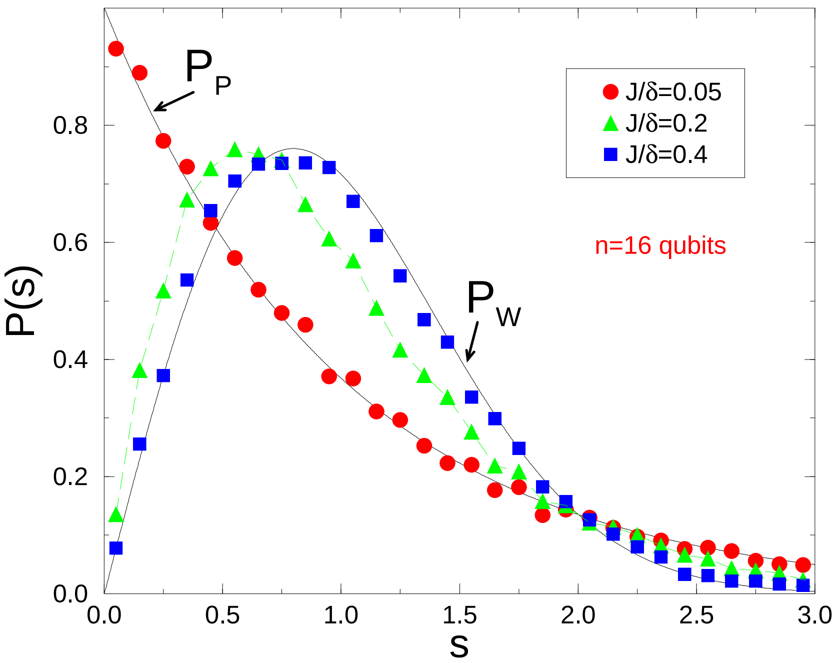
- The interaction has a two-body nature: $\Delta_c \sim \frac{\delta}{n} \gg \Delta_n$ energy spacing between directly coupled states

- Transition to chaos when the typical interaction matrix element is of the order of the spacing between directly coupled states

$$J_c \sim \frac{\delta}{n}$$

Emergence of Fermi-Dirac thermalization in the quantum computer core,
Eur. Phys. J. D **17**, 265 (2001)

Level spacing statistics



$P(s)$ distribution of spacings between adjacent eigenvalues

$P_P(s) = \exp(-s)$
Poisson statistics (integrable systems)

$P_W(s) = \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right)$
Wigner surmise
(Random Matrix Theory, quantum chaos)

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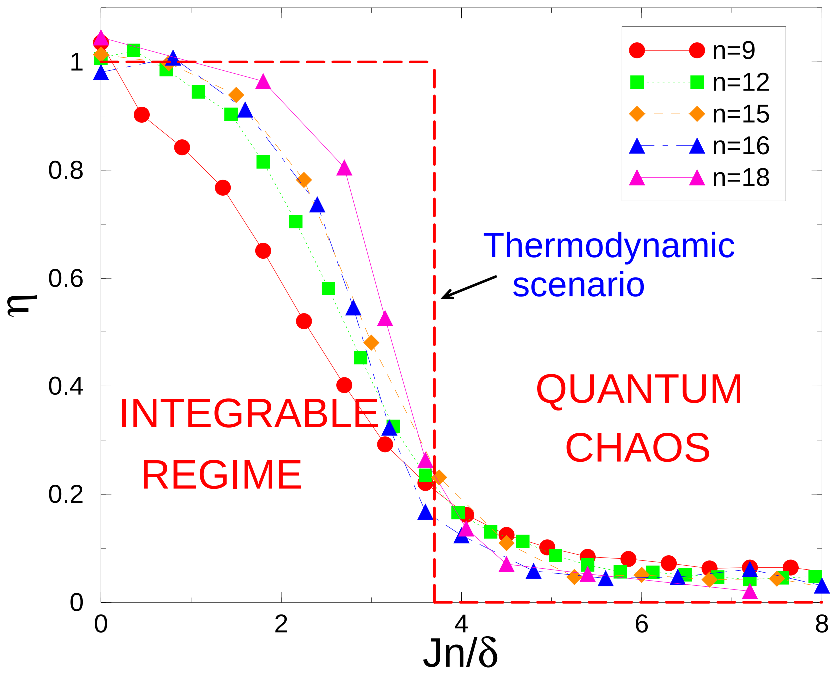
Emergence of Fermi-Dirac thermalization in the quantum computer core,
Eur. Phys. J. D **17**, 265 (2001)

Poisson to Wigner crossover



$$\eta = \frac{\int_0^{s_0=0.47\ldots}(P(s) - P_W(s))ds}{\int_0^{s_0}(P_P(s) - P_W(s))ds}$$

The Poisson to Wigner crossover sharpens when the number of qubits increases

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Consequences for the stability of the quantum computer hardware

In the quantum chaos regime, an eigenstate is composed of exponentially many quantum register states, mixed inside the band width:

$$\Gamma \sim J^2 n/\delta \ (\delta/n < J < \delta); \qquad \Gamma \sim J\sqrt{n} \ (J > \delta)$$

The residual interaction destroys a quantum register state after the chaotic time scale

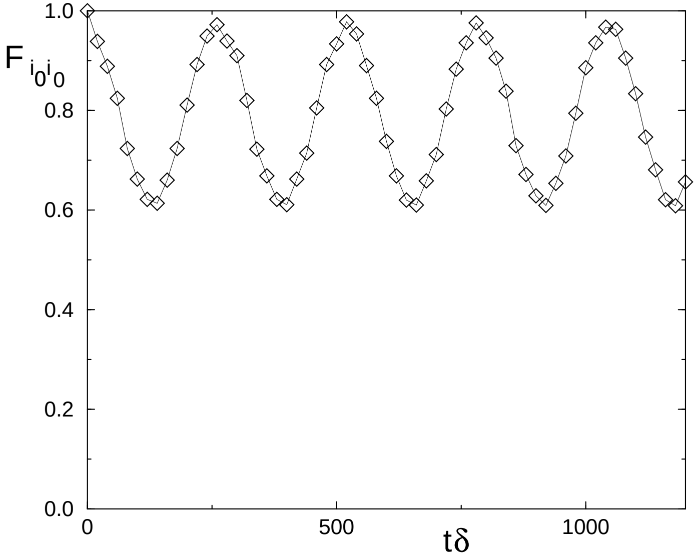
$$\tau_\chi \sim \frac{1}{\Gamma}$$

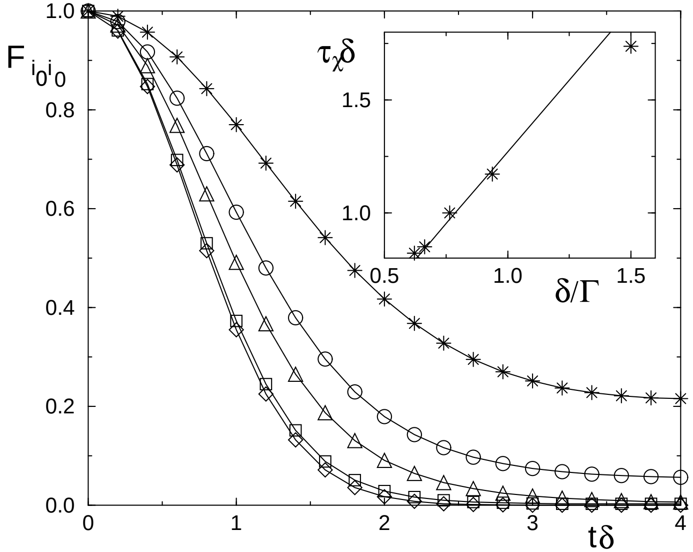
A first requirement for the operability of the quantum computer is that one can operate many gates inside this time scale:

$$\tau_g \ll \tau_\chi$$

Fidelity $\quad F(t) = |\langle\psi(t)|\psi_0\rangle|^2$

A measure of the stability of the quantum memory, that is, of a state loaded on a quantum computer with imperfections





(Georgeot and Shepelyansky, PRE **62**, 6366 (2000))

Below the chaos border: Rabi oscillations;

Above the chaos border: exponential (for $\delta/n < J < \delta$) or Gaussian ($J > \delta$) decay

A testing ground for quantum computation: the quantum sawtooth map model

Kicked Hamiltonian $H(I,\theta,\tau) = \frac{I^2}{2} - \frac{k}{2}(\theta-\pi)^2 \sum_{m=-\infty}^{+\infty} \delta(\tau - mT)$

$$\text{CLASSICAL SAWTOOTH MAP} \begin{cases} \overline{I} = I + k(\theta - \pi), \\ \overline{\theta} = \theta + T\overline{I} \quad (0 \leq \theta < 2\pi) \end{cases}$$

$$\text{After the rescaling } I \to p = IT : \begin{cases} \overline{p} = p + K(\theta - \pi), \\ \overline{\theta} = \theta + \overline{p} \end{cases}$$

Classical dynamics depens only on $K = kT$

* $-4 < K < 0$ stable motion
* $K < -4$ and $K > 0$ chaotic dynamics

KAM theorem does not apply, for any $K \neq 0$ the motion is not bounded by invariant KAM tori:

* $K > 0$ diffusive motion, $< (\Delta p)^2 > \approx D(K)t$:
(i) $K > 1$ random phase approximation, $D(K) \propto K^2$
(ii) $0 < K < 1$ cantori diffusion, $D(K) \propto K^{5/2}$

* $-4 < K < 0$ anomalous diffusion, $< (\Delta p)^2 > \approx t^\alpha$ ($\alpha = 0.57$ when $K = -0.1$)

* $K = -1, -2, -3$ integrable regime

QUANTUM SAWTOOTH MAP:

$$\overline{\psi} = \hat{U}\psi = e^{ik(\hat{\theta}-\pi)^2/2} e^{-iT\hat{I}^2/2}\psi, \quad (\hat{I} = -i\partial/\partial\theta)$$

Classical limit: $k \to \infty$, $T \to 0$, $K = kT =$ const ($[\theta, p] = T[\theta, I] = iT = i\hbar_{\text{eff}}$)

Quantum localization effects: dynamical localization and cantori localization

One-period unitary evolution operator $\hat{U} = \hat{U}_k \hat{U}_T$:

* $\hat{U}_k = \exp(ik(\hat{\theta} - \pi)^2/2)$ (<u>kick</u>)

diagonal in the θ- representation

* $\hat{U}_T = \exp(-iT\hat{I}^2/2)$ (<u>free rotation</u>)

diagonal in the I- representation

On a classical computer, the time evolution is simulated via forward/backward Fast Fourier Transforms, in $O(N \log N)$ operations (N number of levels)

Efficient quantum computing of complex dynamics,
Phys. Rev. Lett. **87**, 227901 (2001)

Quantum algorithm for the sawtooth map

(i) Free rotation $|\psi\rangle = \sum_{I=0}^{N-1} a_I |I\rangle \longrightarrow |\psi\rangle = \hat{U}_T |\psi\rangle = \sum_I a_I \exp(-iTI^2/2)|I\rangle$
$I = \sum_{j=0}^{n-1} \alpha_j 2^j$ (binary code, $\alpha_j = 0, 1$, $n = \log_2 N$ number of qubits)
$\exp(-iTI^2/2) = \prod_{j_1,j_2} \exp(-iT\alpha_{j_1}\alpha_{j_2} 2^{j_1+j_2-1})$

This step can be performed in n^2 controlled-phase shift gates: $\begin{cases} |00\rangle \longrightarrow |00\rangle \\ |01\rangle \longrightarrow |01\rangle \\ |10\rangle \longrightarrow |10\rangle \\ |11\rangle \longrightarrow \exp(-iT2^{j_1+j_2-1})|11\rangle \end{cases}$

(ii) Quantum Fourier Transform
QFT can be performed in n Hadamard gates and $n(n-1)/2$ controlled phase-shift gates

(iii) Kick This step is similar to (i) since now $|\psi\rangle$ is given in the θ representation, where $\hat{U}_k = \exp(ik(\hat{\theta} - \pi)^2/2)$ is diagonal and for the sawtooth map analogous to $\hat{U}_T$.

(iv) Inverse QFT $\longrightarrow$ back to the momentum basis

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Advantages of this quantum algorithm

- The simulation of time evolution is exponentially faster than classical computation: it requires $n_g = O(n^2 = (\log_2 N)^2)$ quantum gates per map iteration instead of $O(N \log_2 N)$ elementary operations

- Optimum use of qubits: no extra work space qubits

- Complex dynamics can be simulated with less than 10 qubits (less than 40 qubits would be sufficient to make simulations inaccesible to present-day supercomputers)

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Phys. Rev. Lett. **87**, 227901 (2001)

We can reach exponentially fast with the number of qubits two distinct limits:

- Classical limit:

$$T = \frac{2\pi L}{N} = \frac{2\pi L}{2^n}, \quad K = \text{const}$$

The number of levels inside the interval $-\pi L \leq p < \pi L$ grows exponentially
The effective Planck constant $\hbar_{\text{eff}} = T \sim 1/N = 1/2^n \to 0$ when $N \to \infty$

- Thermodynamic limit:

$$k, K \ \ \text{constant}, \quad L = \frac{TN}{2\pi} = \frac{T2^n}{2\pi}$$

The thermodynamic limit corresponds to the system size (number of cells) $L \to \infty$
The effective Planck constant is fixed

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Dynamical localization simulated on a few-qubit quantum computer,
Phys. Rev. A **67**, 052312 (2003)

Quantum computing of dynamical localization

We have shown that quantum computers can simulate efficiently the quantum localization of classical chaos

Quantum dynamical localization: suppression of diffusion due to quantum interference effects

Dynamical localization is one of the most interesting phenomena that characterize the quantum behavior of classically chaotic systems: quantum interference effects suppress classical diffusion leading to exponentially localized wave functions

This phenomenon was first observed in the quantum kicked rotor model and has profound analogies with Anderson localization of electronic transport in disordered materials

Experiments: Dynamical localization has been observed experimentally in the microwave ionization of Rydberg atoms and with cold atoms

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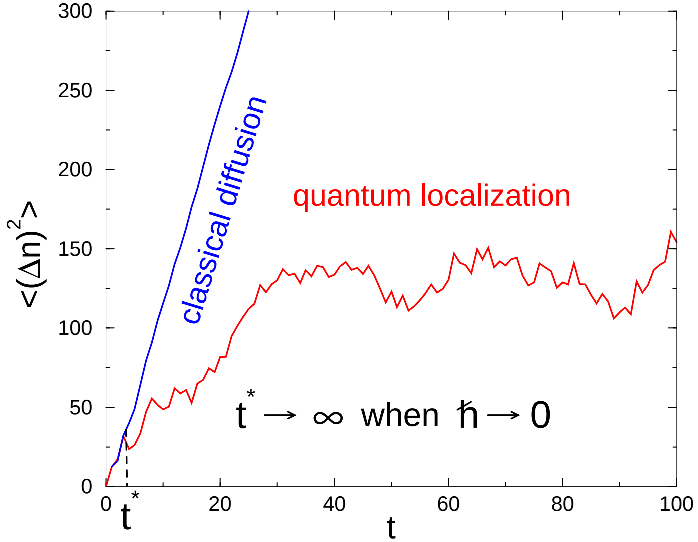
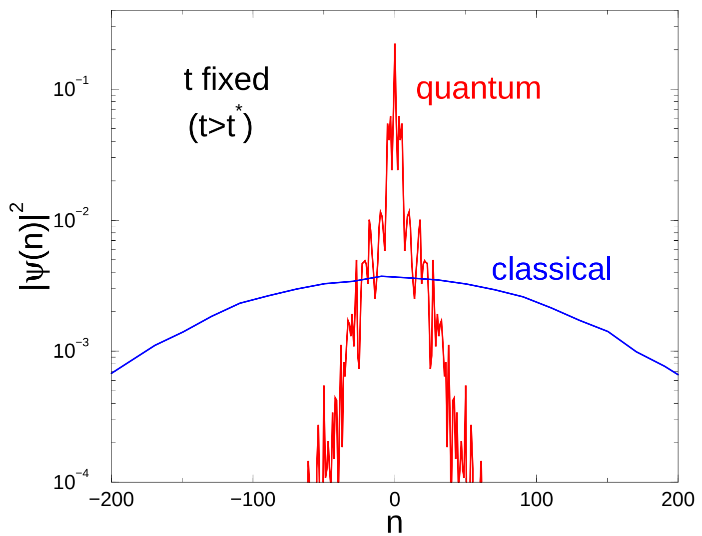
Dynamical localization simulated on a few-qubit quantum computer,
Phys. Rev. A **67**, 052312 (2003)

Dynamical localization observed in periodically driven Hamiltonian systems (a kicked rotator, an hydrogen atom under a microwave field, cold ions in an optical lattice, ...)

$$H = H_0(n) + \epsilon V(\theta, t), \quad V(\theta, t+T) = V(\theta, t)$$





- Exponential quantum localization
- No disorder in the Hamiltonian
- Also in conservative systems

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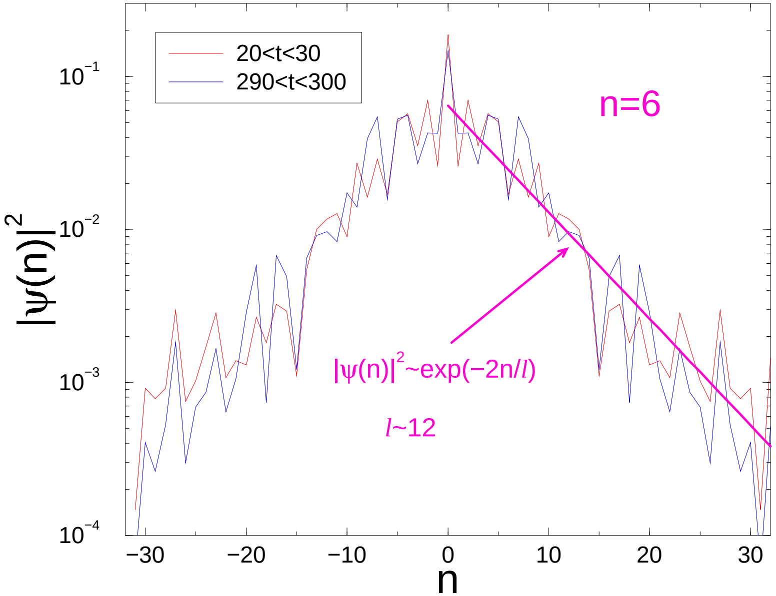
Dynamical localization simulated on a few-qubit quantum computer,
Phys. Rev. A **67**, 052312 (2003)

Computing dynamical localization in the sawtooth map model

Using the quantum algorithm for the sawtooth map, it is possible to study localization effects on a quantum computer with exponentially large system sizes $N = 2^{n_q}$ (n_q number of qubits) and quadratic speedup with respect to classical computation



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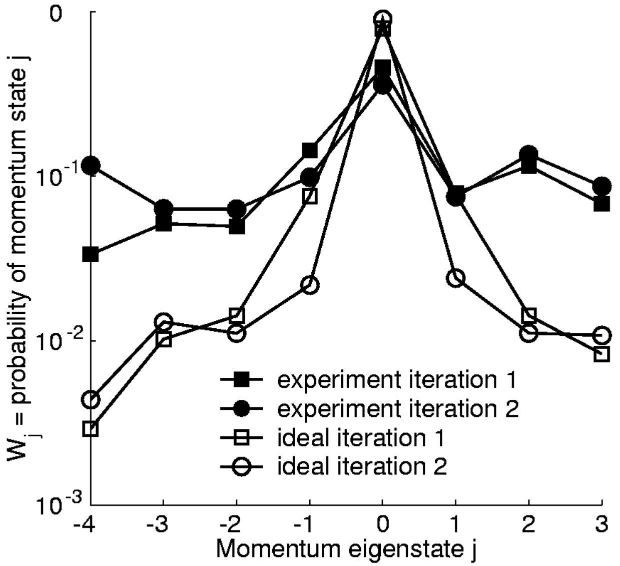
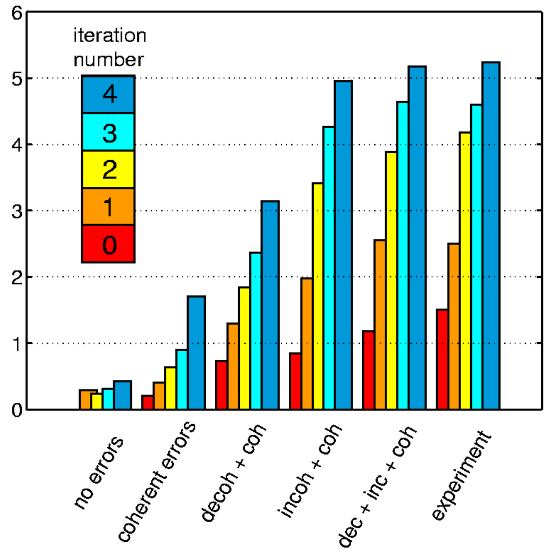
First (NMR-based) experimental implementation





(M.K. Henry, J. Emerson, D.G. Cory, quant-ph/0512204)

Localization is a purely quantum effect, quite fragile in the presence of noise: a significant degree of quantum control has been achieved

Efficient quantum computing of complex dynamics,
Phys. Rev. Lett. **87**, 227901 (2001)

Study of static imperfections

Numerical study of the effect of static imperfections for a many-body/qubit quantum computer simulating the quantum algorithm for the sawtooth map

We assume that:
1) The quantum computer is decoupled from the environment
2) Short-range, instantaneous and perfect one- and two-qubit gates separated by a time interval τ_g
3) The hardware Hamiltonian contains static imperfections, giving unwanted phase rotations and qubit couplings

$$H_s \to H(\tau) = H_s + H_g(\tau) = H_s + \sum_k \delta(\tau - k\tau_g) h_k$$

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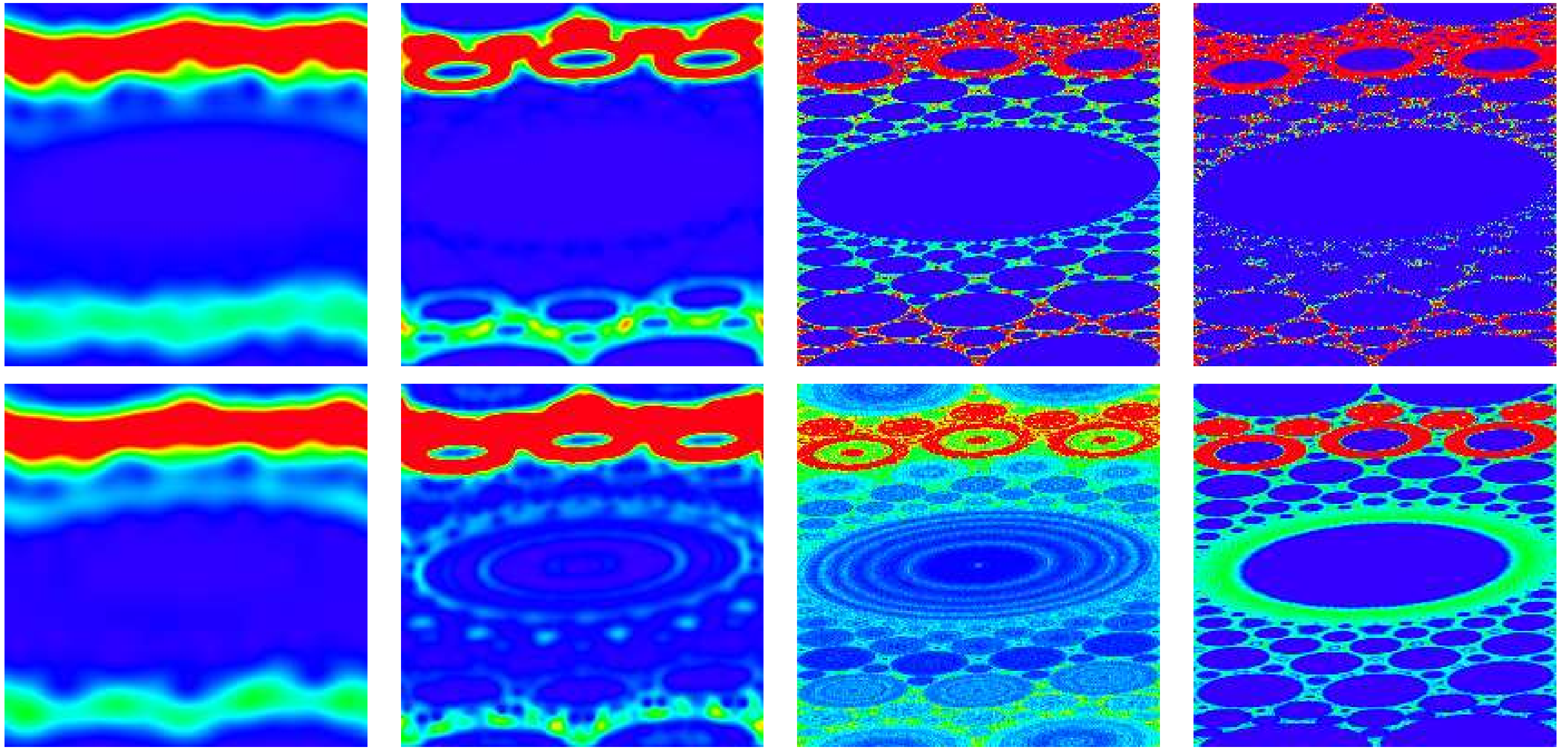
Phys. Rev. Lett. **87**, 227901 (2001)

Quantum versus classical errors



- Quantum errors are "nonlocal" in phase space: they can induce direct transfer of probability on a large distance in phase space: injection of quantum probabilities inside integrable islands
- Classical round-off errors: slow diffusive spreading inside integrable islands

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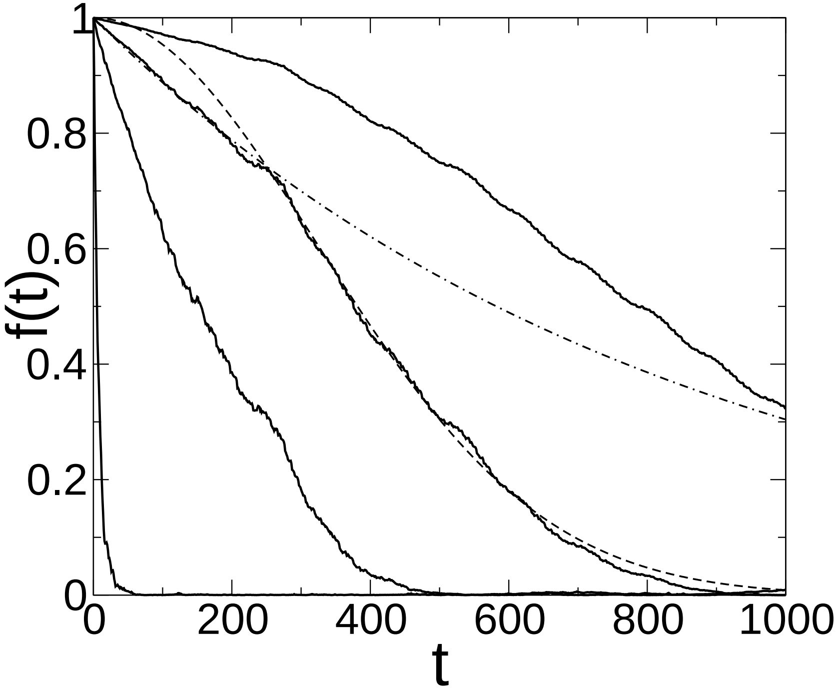
Phys. Rev. Lett. **87**, 227901 (2001)

Static imperfections vs. noisy gates



Static imperfections: from exponential fidelity decay (Fermi golden rule, $f(t) \approx \exp(-A\epsilon^2 t)$) to Gaussian fidelity decay ($f(t) \approx \exp(-B\epsilon^2 t^2)$)

The exponential to Gaussian crossover takes place at the Heisenberg time $t_H \sim N$, given by the inverse mean level spacing: Before t_H the system does not resolve the discreteness of the spectrum. Therefore, the density of states can be treated as a continuous and the Fermi golden rule can be applied.

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Phys. Rev. Lett. **87**, 227901 (2001)

"Noisy gates": modeled by

$$H = \sum_i (\Delta_0 + \delta_i(t))\sigma_i^z + \sum_{<i,j>} J_{ij}(t)\sigma_i^x\sigma_j^x$$

with δ_i and J_{ij} fluctuating without memory from gate to gate

In the case of noisy gates the fidelity decay is exponential. This corresponds to the Fermi golden rule regime, where at each gate operation a probability of order ϵ^2 is transferred from the ideal state to other states. Since there are no correlations between consecutive noisy gates, the population of the ideal noiseless state decays exponentially. We can write

$$f(t) \approx \exp(-C\epsilon^2 N_g)$$

where $N_g = n_g t$ is the total nomber of gates required to evolve t steps of the sawtooth map

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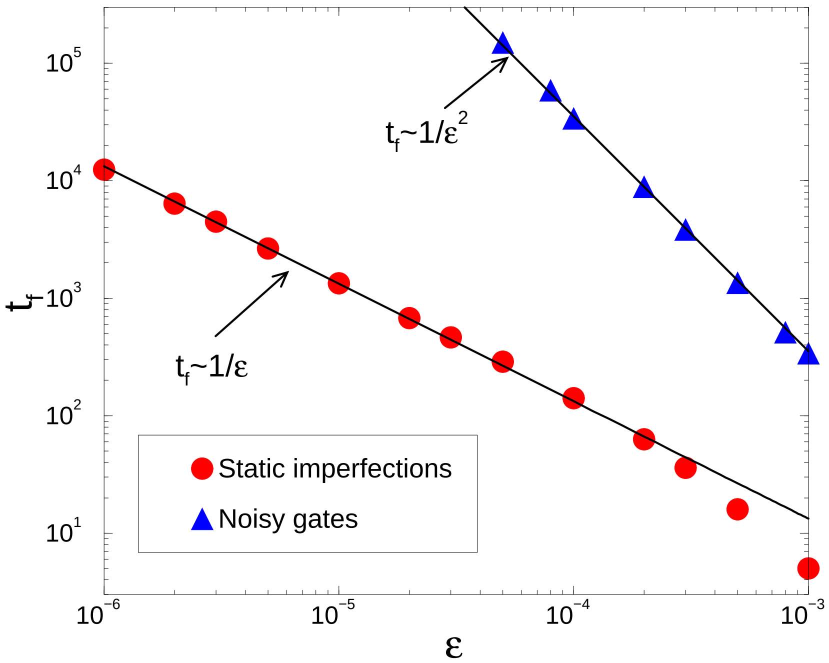
Phys. Rev. Lett. **87**, 227901 (2001)

Fidelity time scale



t_f obtained from the condition $f(t_f) = 0.9$

The static imperfections give shorter time scales t_f and therefore can be considered more dangerous for quantum computation

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Random matrix theory approach

Assuming that the Floquet operator U can be modeled by a random matrix one obtains [Frahm et al., EPJD **29**, 139 (2004)]

$$-\ln f(t) \approx \frac{t}{t_c} + \frac{t^2}{t_c t_H}$$

$t_c \approx 1/(\epsilon^2 n n_g^2)$ characterizes the effective strength of the perturbation
$t_H = N = 2^n$ Heisenberg time

This relation is valid as long as ϵ and t are sufficiently small so that $1 - f \ll 1$

Perturbative regime for $\epsilon < \epsilon_c \approx 1/(\sqrt{n2^n} n_g)$, that is, for $t_c > t_H$, the fidelity decay is dominated by the quadratic term in the above expression: The decay essentially takes place after the Heisenberg time and is Gaussian, $f(t) \approx \exp(-t^2/t_c t_H)$

Fidelity time scale $t_f \sim \sqrt{t_c t_H} \approx \frac{\sqrt{2^n}}{\epsilon\sqrt{n} n_g}$

Number of gates inside this time $(N_g)_f = t_f n_g \sim \frac{\sqrt{2^n}}{\epsilon\sqrt{n}}$

Quantum chaos regime for $\epsilon > \epsilon_c \approx 1/(\sqrt{n2^n} n_g)$, that is, for $t_c < t_H$, the fidelity decay is dominated by the linear term in the above expression: The decay is exponential, $f(t) \approx \exp(-t/t_c)$, and occurs before the Heisenberg time

Fidelity time scale $t_f \sim t_c \approx \frac{1}{\epsilon^2 n n_g^2}$

Number of gates inside this time $(N_g)_f = t_f n_g \sim \frac{1}{\epsilon^2 n n_g}$

The threshold ϵ_c is the chaos border above which static imperfections mix the Floquet eigenstates

How to reduce the impact of static imperfections?

Since random imperfections changing from gate to gate always lead to an exponential decay of the fidelity, it is tempting to try to randomize the static imperfections to slow down the fidelity decay from Gaussian to exponential

This idea can be formalized if one observes that the fidelity can be expressed in terms of a correlation function of the perturbation [Prosen and Žnidarič, J. Phis. A **35**, 1455 (2002)]

$U = U(T)U(T-1)\cdots U(1)$ sequence of ideal quantum gates

$U_\epsilon = e^{-i\epsilon V(T)}U(T)e^{-i\epsilon V(T-1)}U(T-1)\cdots e^{-i\epsilon V(1)}U(1)$ perturbed sequence

$V(t)$ Hermitian operator

ϵ perturbation strength

Fidelity of this quantum algorithm:

$$f(T) = \left| \frac{1}{N} \mathrm{Tr}[U_\epsilon(T,0)U(T,0)] \right|^2$$

the trace average the result over a complete set of initial states (for instance the quantum register states)
$U(t,t') \equiv U(t)U(t-1)\cdots U(t'+1)$ evolution operator from t' to $t > t'$
$U_\epsilon(t,t')$ is defined in the same way for the perturbed evolution

After defining the Heisenberg evolution of the perturbation as $V(t,t') = U^\dagger(t,t')V(t)U(t,t')$, we obtain

$$f(T) = \left| \frac{1}{N} \mathrm{Tr}\left(e^{i\epsilon V(1,0)} e^{i\epsilon V(2,0)} \cdots e^{i\epsilon V(T,0)} \right) \right|^2$$

As we are interested in the case in which the fidelity is close to unity, we can expand it up to the second order in ϵ:

$$f(t) \approx 1 - \epsilon^2 \sum_{t,t'=1}^{T} C(t,t'), \quad C(t,t') = \frac{1}{N}\mathrm{Tr}[V(t',0)V(t,0)]$$

$C(t,t')$ is a two-point time correlation of the perturbation

A quantum algorithm is therefore more stable when the correlation time of the perturbation is smaller. This can be done by devising a "less regular" sequence of gates that realize the transformation U required by the algorithm [Prosen and Znidarič, J. Phis. A **34**, L681 (2001)]

For instance, using the Pauli operators one can change the computational basis repeatedly and randomly during a quantum computation [Kern et al., EPJD **32**, 153 (2005)]

Related work

Dynamics of entanglement in quantum computers with imperfections

[1] S. Montangero, G. Benenti, and R. Fazio, *Dynamics of entanglement in quantum computers with imperfections*, Phys. Rev. Lett. **91**, 187901 (2003).

[2] D. Rossini, G. Benenti and G. Casati, *Entanglement Echoes in Quantum Computation* Phys. Rev. A **69**, 052317 (2004).

[3] C. Mejía-Monasterio, G. Benenti, G. G. Carlo and G. Casati, *Entanglement across a Transition to Quantum Chaos*, Phys. Rev. A **71**, 062324 (2005).

Non-unitary quantum noise effects for quantum computation and communication

[1] G. G. Carlo, G. Benenti, and G. Casati, *Teleportation in a noisy environment: a quantum trajectories approach*, Phys. Rev. Lett. **91**, 257903 (2003).

[2] G. G. Carlo, G. Benenti, G. Casati and C. Mejía-Monasterio, *Simulating noisy quantum protocols with quantum trajectories*, Phys. Rev. A **69**, 062317 (2004).

[3] G. Benenti, S. Felloni and G. Strini, *Effects of single-qubit quantum noise on entanglement purification*, Eur. Phys. J. D **38**, 389 (2006).

G. Benenti, G. Casati and G. Strini, *Principles of quantum computation and information*, vol. 1 (2004) and vol.2 (in press) (World Scientific, Singapore)

Final remarks

Decoherence and imperfections appear to be the ultimate obstacle to the practical realization of a large-scale quantum computer

Numerical simulations including realistic noise parameters, shape of the pulses, and partial memory effects promise to become a valuable tool for quantum hardware design and to determine optimal regimes for the operability of quantum computers

We can expect that a deeper understanding of quantum chaos in many-body quantum computers will suggest strategies suitable for a better working of quantum processors