OTOC, quantum chaos and the correspondence principle

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OUTLINE

Understanding, characterising, and measuring the complexity of quantum motion: a fundamental problem for quantum quantum information and quantum technologies

Classical complex systems characterized by exponential instability of motion (chaos, algorithmic complexity) vs dynamical stability of quantum motion

How does chaotic classical dynamics emerge from quantum physics? Is OTOC a diagnostic of chaos? Exponential growth of OTOC in classically integrable systems: problems with the correspondence principle?

Classical chaos: Exponential instability

Classical chaos is characterized by exponential local instability: two nearby trajectories separate exponentially, with rate given by the maximum Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

d length of the tangent vector

Classical chaos: Trajectories are unpredictable

Chaotic orbits are unpredictable: in order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment and independent of the previous length of the trajectory. The information associated with a segment of trajectory of length t is equal, asymptotically, to

$$\lim_{t\to\infty} \frac{I(t)}{t} = h,$$

h is the KS (Kolmogorov-Sinai) entropy: positive when $\lambda > 0$

Dynamical stability of quantum motion

The energy and the frequency spectrum of any quantum motion, bounded in phase space, are always discrete⇒ regular motion

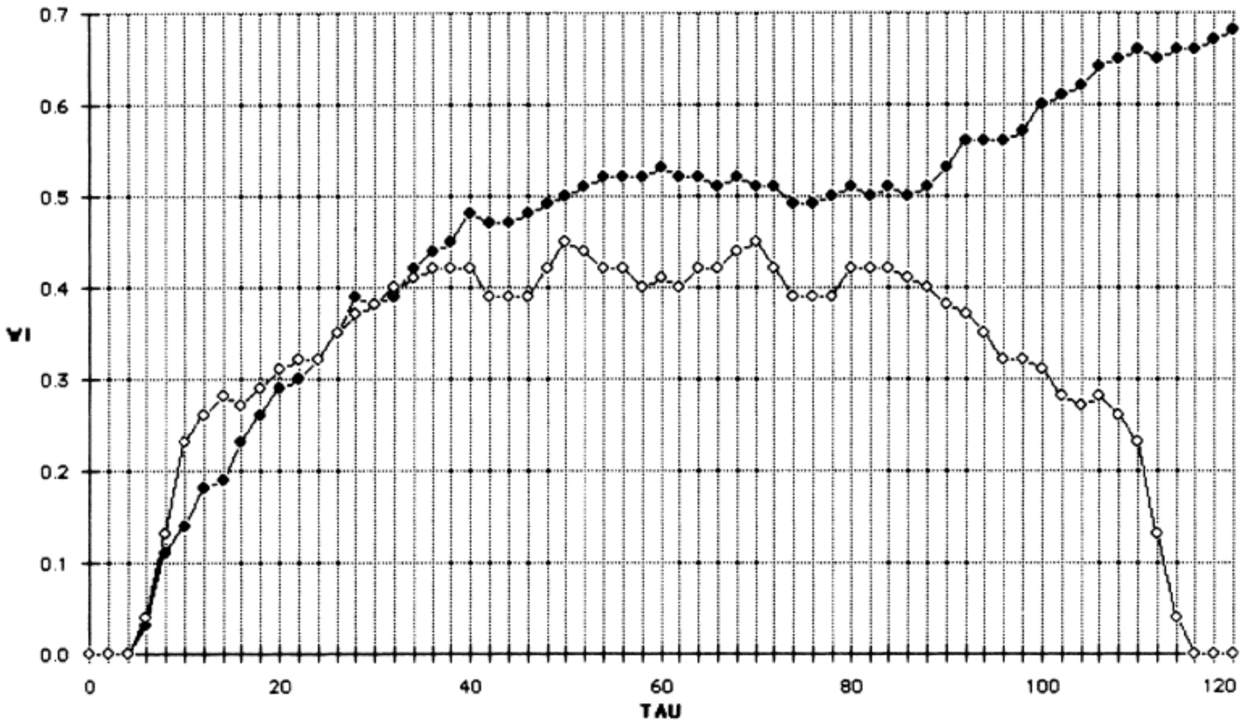


FIG. 3. Classical (solid lozenges) and quantum (open lozenges) ionization probability (excitation above the unperturbed level $n = 150$) as a function of time τ for the case of Fig. 2. Notice the perfect specular symmetry of the quantum curve about the time of reversal $\tau = 60$.

[Casati, Chirikov, Guarneri, Shepelyansky, PRL **56**, 2437 (1986)]

What about the correspondence principle?

"In fact, there was rarely in the history of physics a comprehensive theory which owed so much to one principle as quantum mechanics owed to Bohr's correspondence principle"

[Max Jammer, The conceptual development of quantum mechanics (McGraw-Hill, 1966)]

Ehrenfest time scale

Exponential instability of quantum motion only up to the (logarithmically short) Ehrenfest time scale

$$t_E \sim \frac{1}{\lambda} \left| \ln \hbar_{\mathrm{eff}} \right|$$

[Berman and Zaslavsky, Physica A **91**, 450 (1978)]

t_E diverges at the classical limit, in agreement with the correspondence principle

Out-of-Time-ordered Correlator (OTOC)

(Larkin and Ovchinnikov, 1996; Maldacena and Stanford, 2016,…)

$$\mathcal{C}(t) = \left\langle |[\hat{A}(t), \hat{B}(0)]|^2 \right\rangle$$

Consider:

$$C_{pp}(t) = -\frac{1}{\hbar^2}\langle\psi_0|[\hat{p}_1(t), \hat{p}_1(0)]^2|\psi_0\rangle$$

Classical limit:

$$C^{cl}_{pp}(t) = \int d\gamma(0)\rho_0[\gamma(0)]\{p_1(t), p_1(0)\}^2_{PB}$$

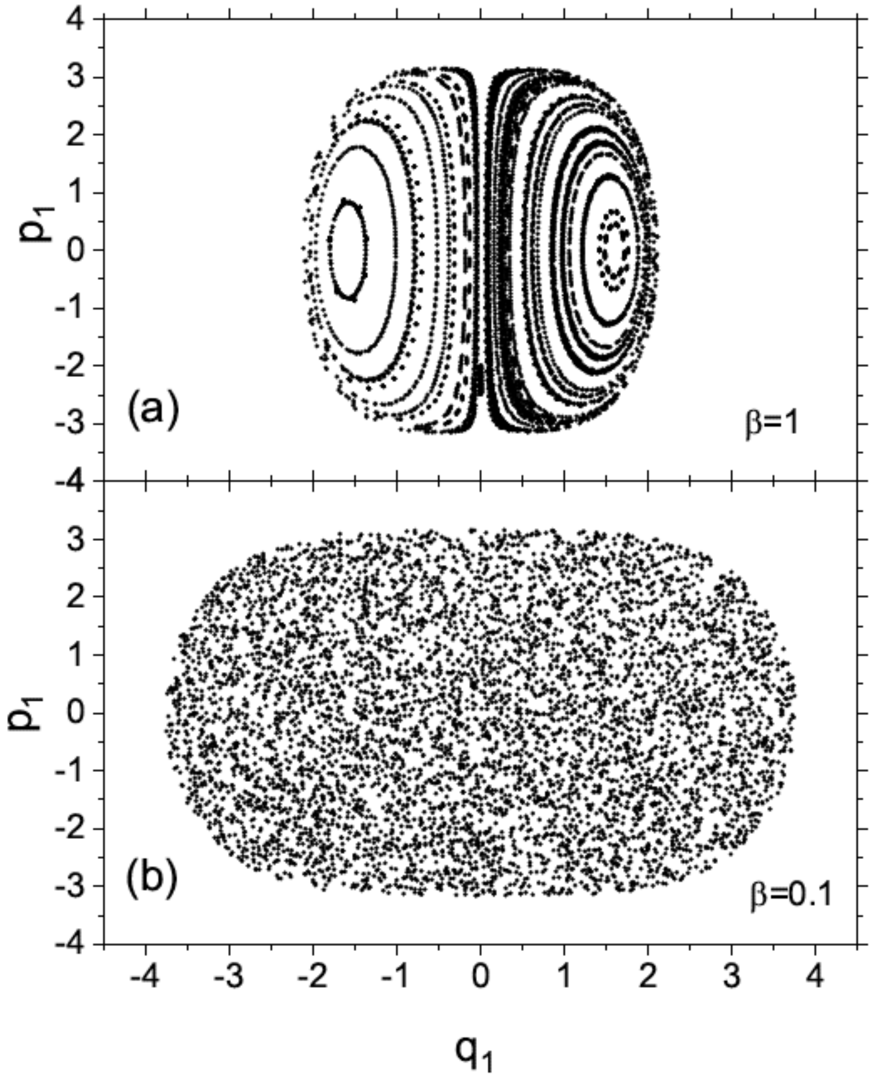
$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \to \{A, B\}_{\text{PB}}$

$$= \int d\gamma(0)\rho_0[\gamma(0)]\left(\frac{\delta p_1(t)}{\delta q_1(0)}\right)^2 \propto \exp[2\lambda_L(r_0)t]$$

Exponential growth in the chaotic case

Numerical illustration: coupled nonlinear oscillators

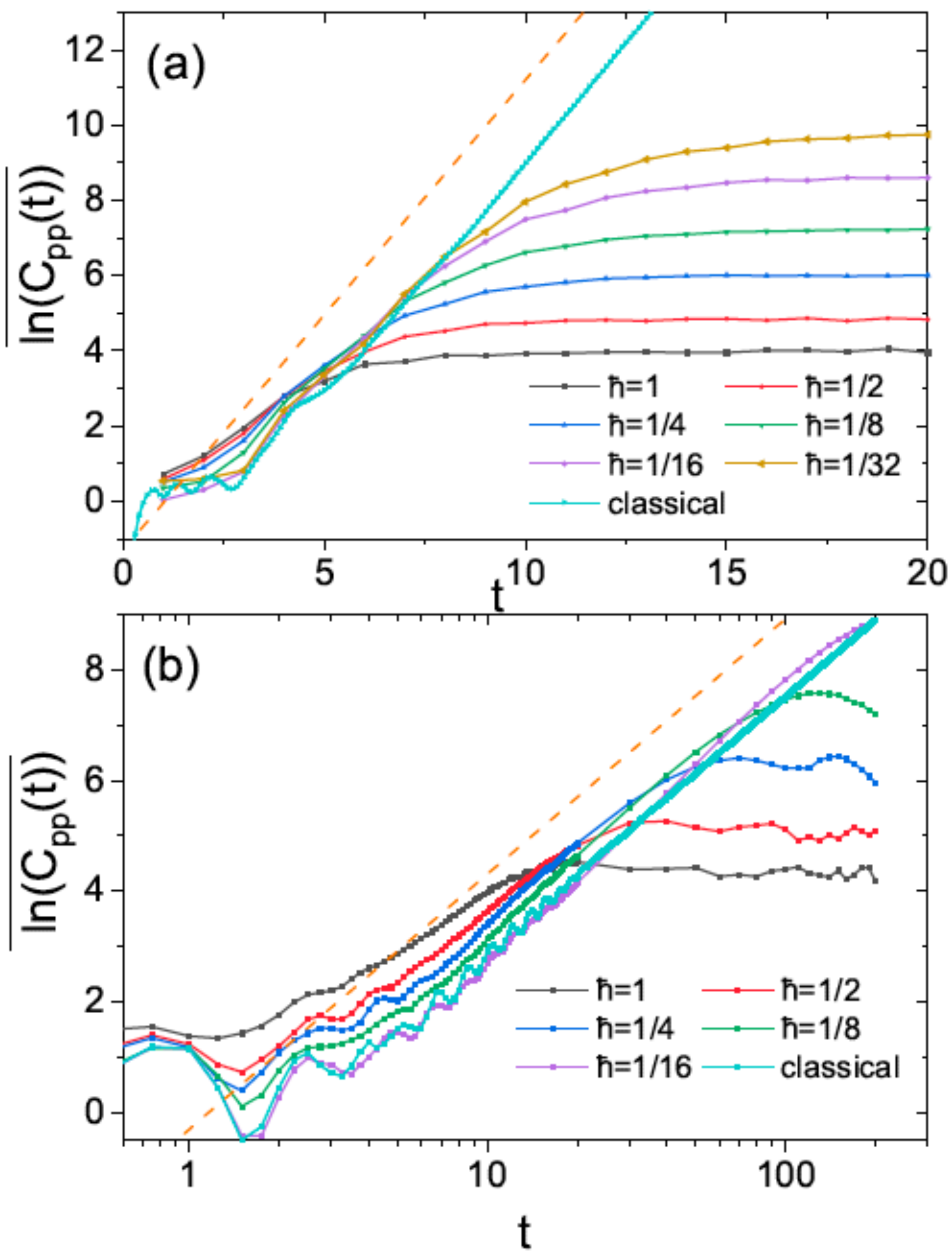
$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{\beta}{4}(\hat{q}_1^4 + \hat{q}_2^4) + \frac{1}{2}\hat{q}_1^2\hat{q}_2^2$$



Integrable regime

Chaotic regime

OTOC growth



Chaotic regime

(exponential growth, with rate twice the largest Lyapunov exponent)

Integrable regime

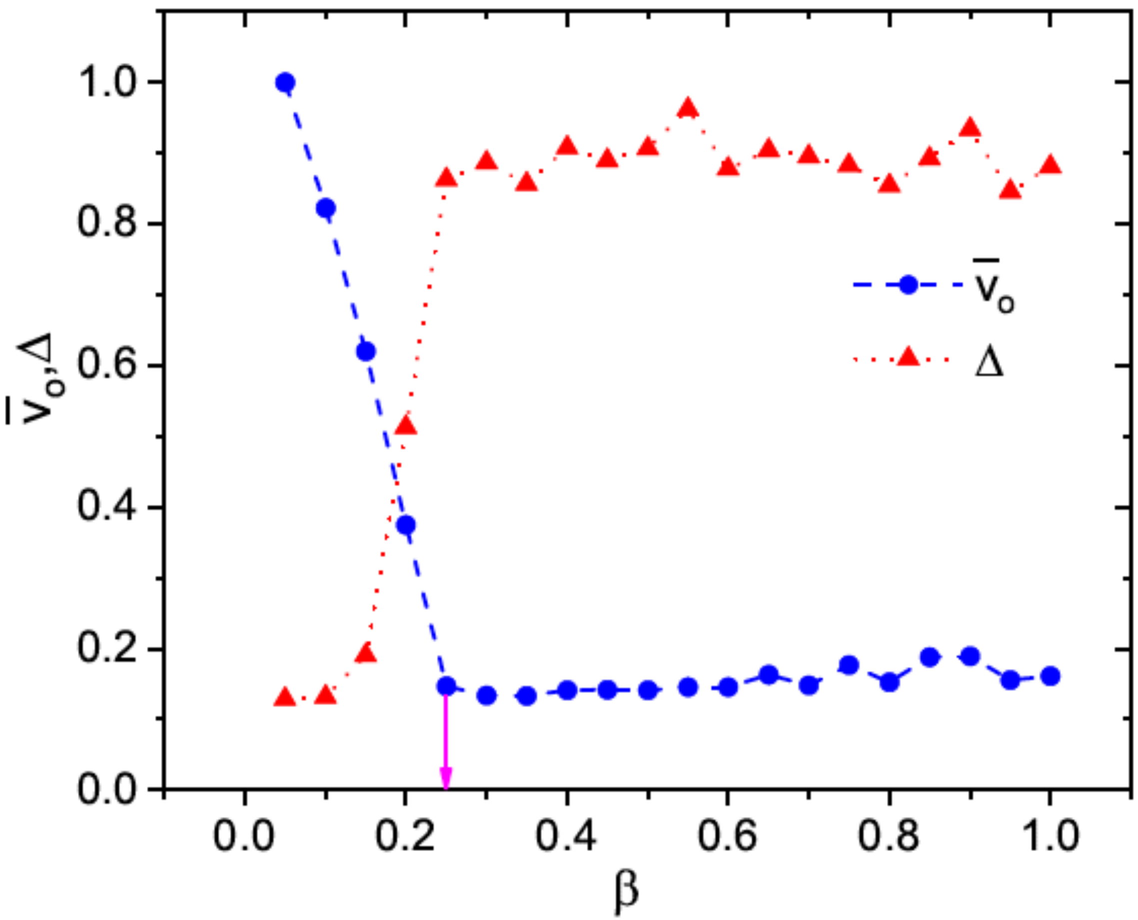
(quadratic growth)

(Coupled oscillators model)

Detect transition to chaos in the time domain

$$H = \frac{1}{2}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{\beta}{4}(\hat{q}_1^4 + \hat{q}_2^4) + \frac{1}{2}\hat{q}_1^2\hat{q}_2^2$$

Coupled nonlinear oscillators model



Δ from level spacing statistics

$$\Delta = \frac{\int_0^\infty |P(s) - P_w(s)| ds}{\int_0^\infty |P_p(s) - P_w(s)| ds}$$

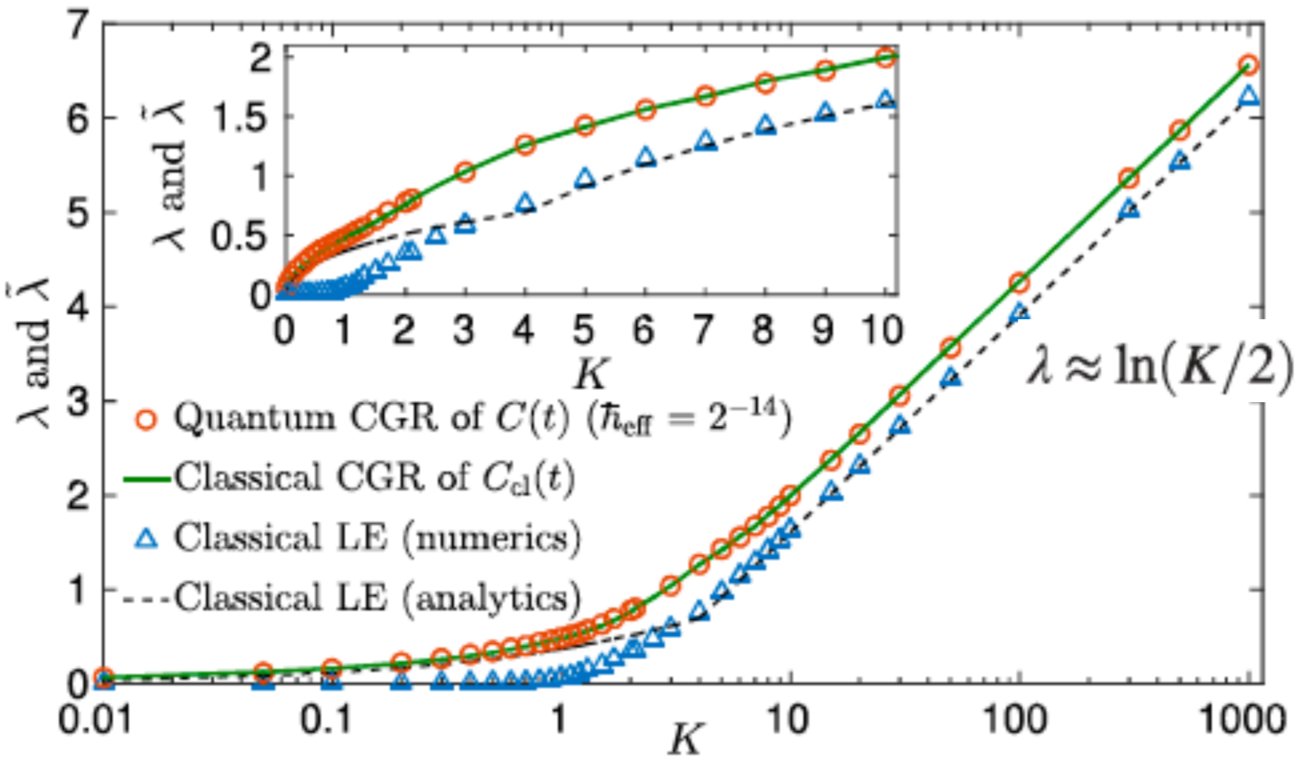
$\overline{v}_o$ average velocity in the growth of OTOC

[J.Wang, GB, G.Casati, W. Wang, PRR **2**, 043178 (2020)]

OTOC growth vs. Lyapunov exponent

Integrability to chaos crossover in the kicked rotor model

$$H(I,\theta,\tau) = \frac{1}{2}I^2 + k\cos\theta \sum_{m=-\infty}^{+\infty} \delta(\tau - mT)$$



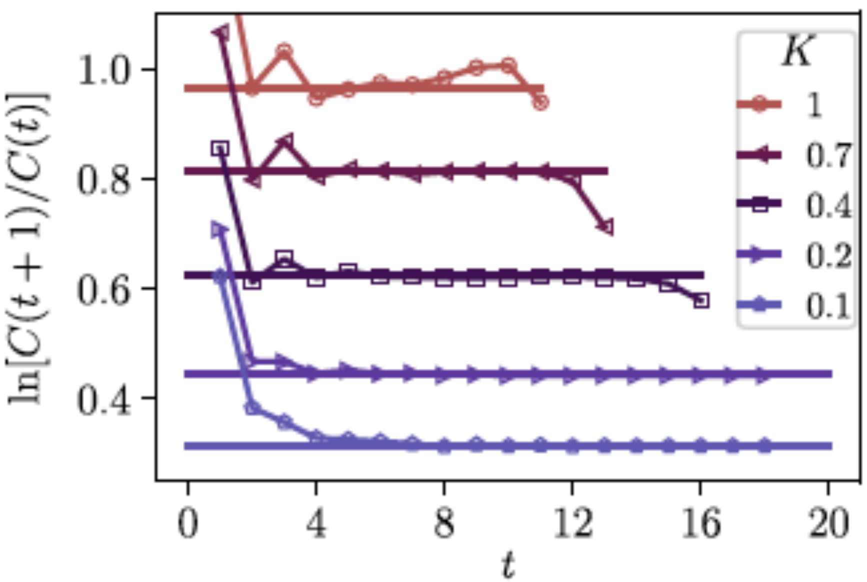
$\lambda \approx \ln(K/2)$

$K = kT$

[Rozembaum, Ganeshan, Galitski, PRL **118**, 086801 (2017)]

Exponential growth of OTOC without chaos

Exponential growth due to unstable fixed points, also for integrable systems



fixed point at (x,p)=(0,0) for the kicked rotator model

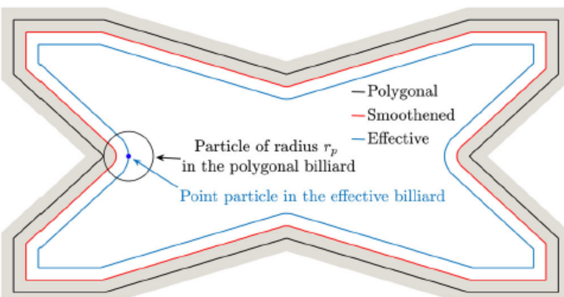
$$\omega(K) = \log\left(1 + \frac{K}{2} + \sqrt{K^2 + \frac{K}{4}}\right)$$

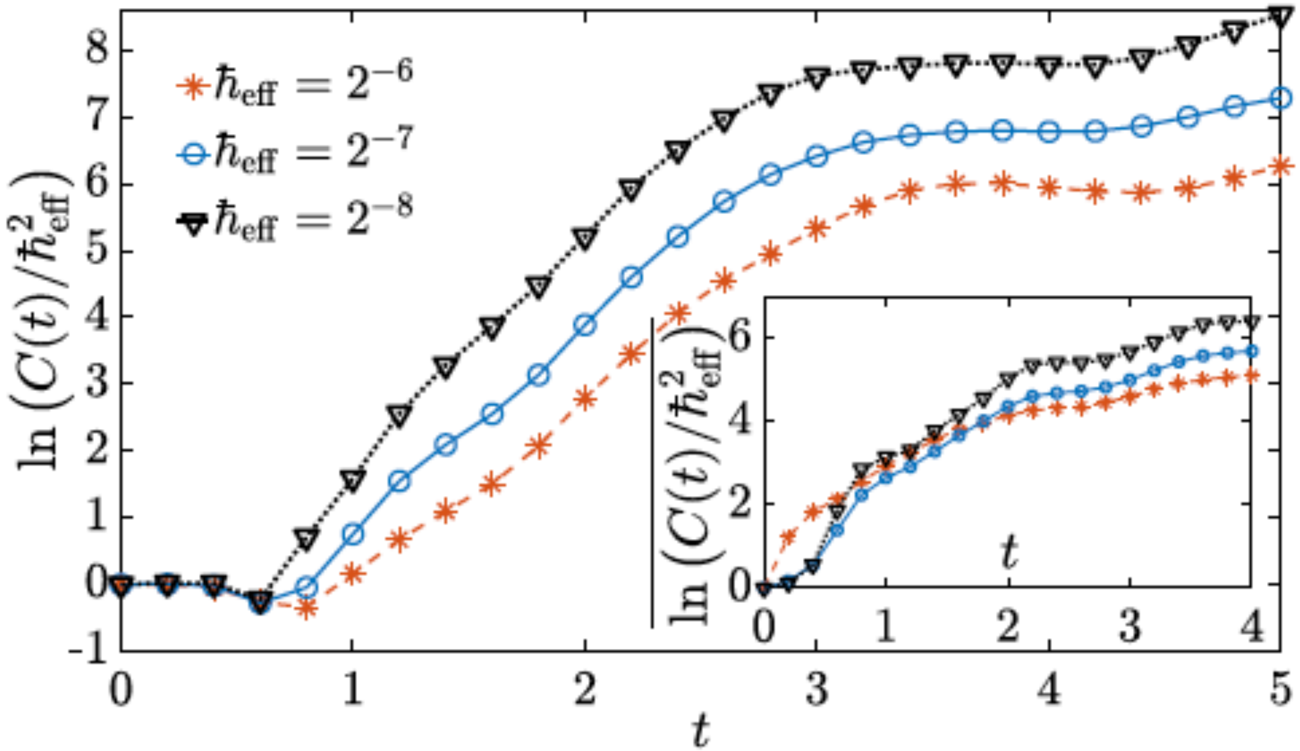
[Xu, Scaffidi, Cao, PRL **124**, 140602 (2020);
see also Pilatovsky-Cameo *et al.*, PRE **101**, 010202 (2020)]

Is it still possible to use OTOC as a diagnostic of chaos?

OTOC and the correspondence principle

Exponential growth of OTOC in quantum polygonal billiards





[Rozembaum, Bunimovich, Galitski, PRL **125**, 014101 (2020)]

The growth rate increases with reducing the effective Planck constant. What about the correspondence principle?

A simple model to investigate previous questions

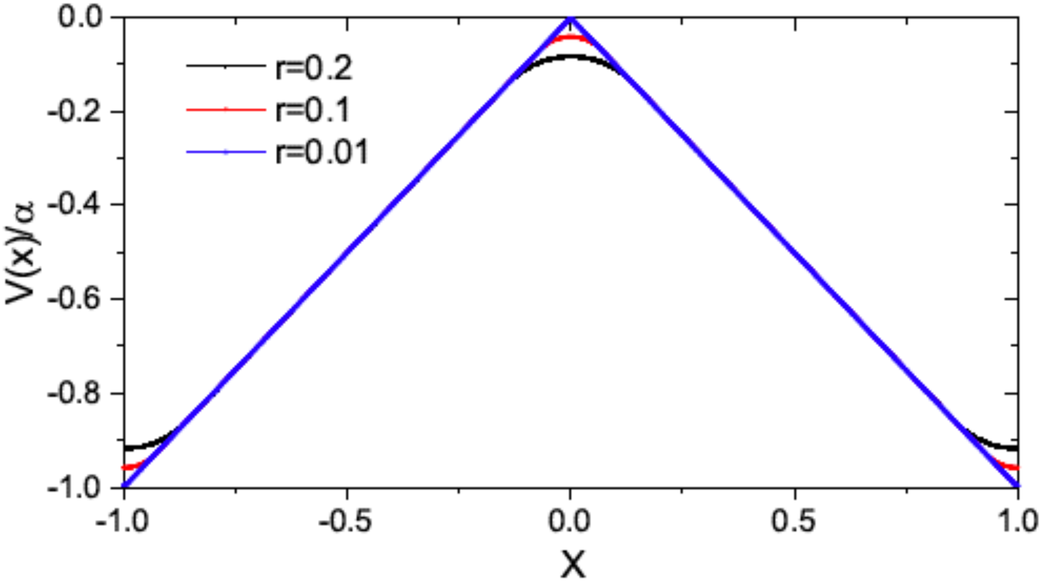
Triangle map (mixing but not chaotic)

$$\begin{cases} p_{n+1} = p_n - V'(x_n) & (\text{mod } 2) \\ x_{n+1} = x_n + p_{n+1} & (\text{mod } 2) \end{cases}$$

$$(x, p) \in [-1, 1) \times [-1, 1)$$

$$V(x) = -\alpha|x| - \beta$$

Round-off triangle map (chaotic)



Lyapunov exponent

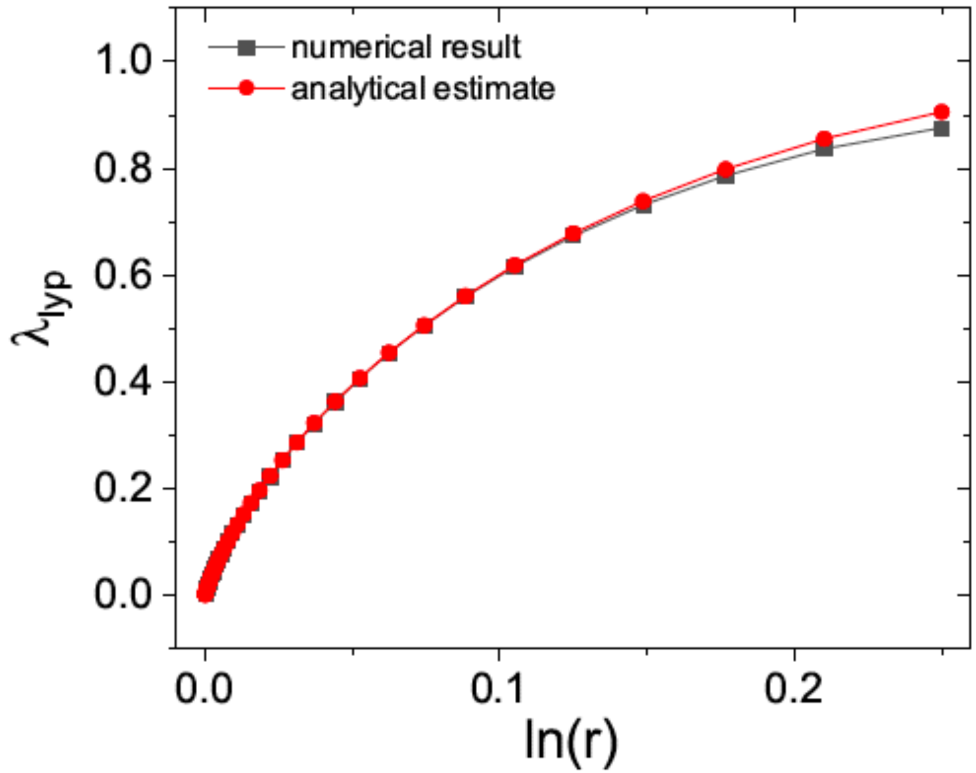
Dominated by passages of trajectories in region

$$E = E_0 \cup E_{|1|} \qquad |x| < \frac{\sqrt{2}}{2} r \text{ or } |x| > 1 - \frac{\sqrt{2}}{2} r$$

mean time between consecutive passages: $\overline{\tau} \simeq \frac{\sqrt{2}}{2r}$

given the distribution of return times one can estimate

$$\lambda_{\text{lyp}} = 2r^2 \sum_{\tau=1}^{\infty} (1 - \sqrt{2}r)^{\tau-1} \ln\left(\frac{\sqrt{2}\alpha}{r}\tau\right)$$

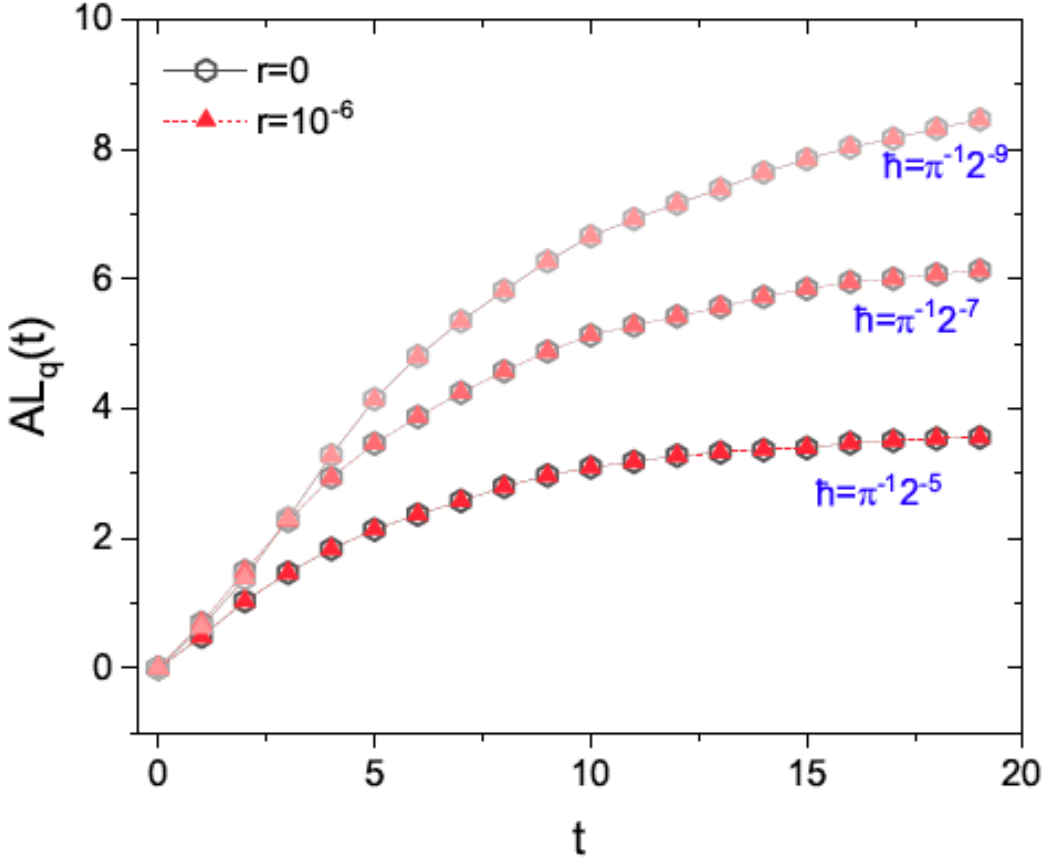


[J.Wang, GB, G.Casati, W. Wang, arXiv:2010.10360]

Problems with the correspondence principle?

$$AL_q(t) = \frac{1}{N} \sum_{k=1}^{N} \ln \left(\langle \psi_k | [\hat{x}(t), \hat{p}(0)]^2 | \psi_k \rangle \right) ,$$

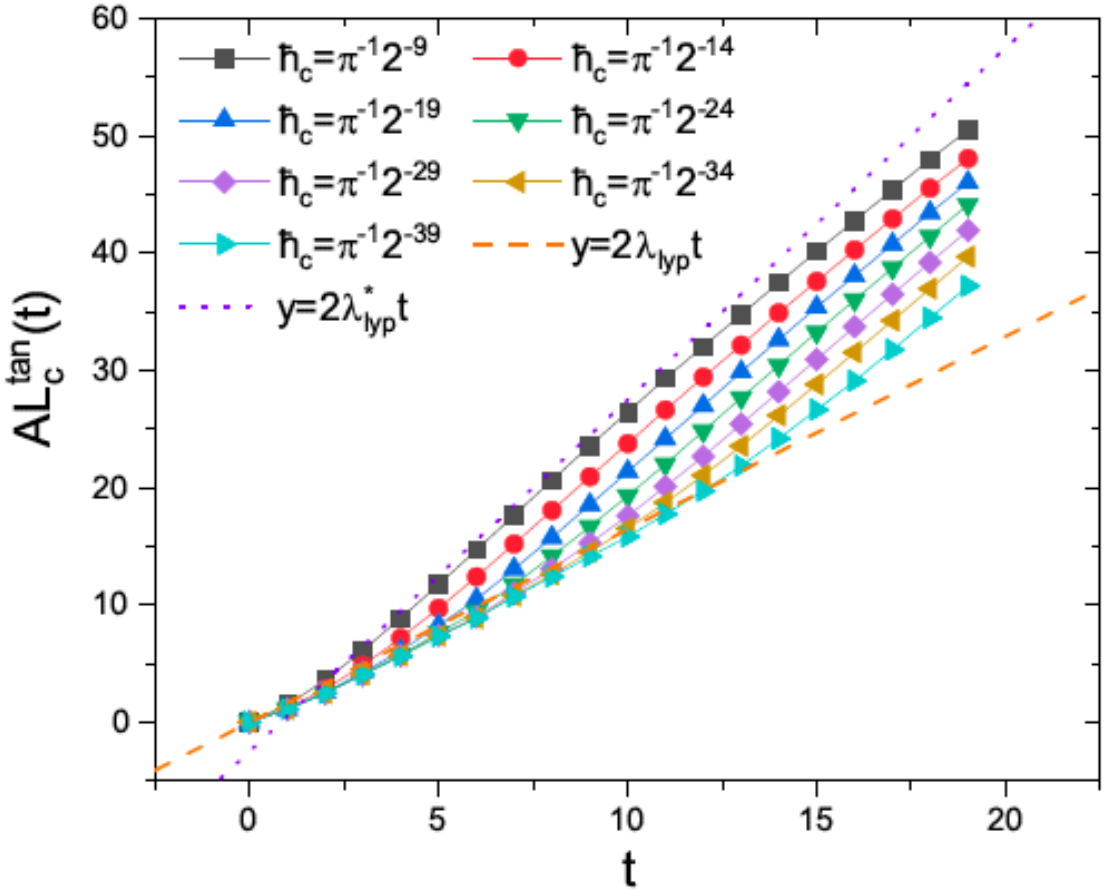
(average over a large number N of initial coherent states)



Classical OTOC

$$AL_c^{\tan}(t) = \frac{1}{N}\sum_{k=1}^{N} \ln\left[\int d\gamma \rho_{\gamma_0^k}(\gamma)\left(\frac{\partial x(t)}{\partial x(0)}\right)^2\right], \quad \gamma = (x,p)$$

(average over a large number N of initial gaussian distributions)



$$\lambda^*_{\rm lyp} \approx \ln\left(\frac{\sqrt{2}\alpha}{r}\right) + \frac{1}{2}\ln(\sqrt{2}r)$$

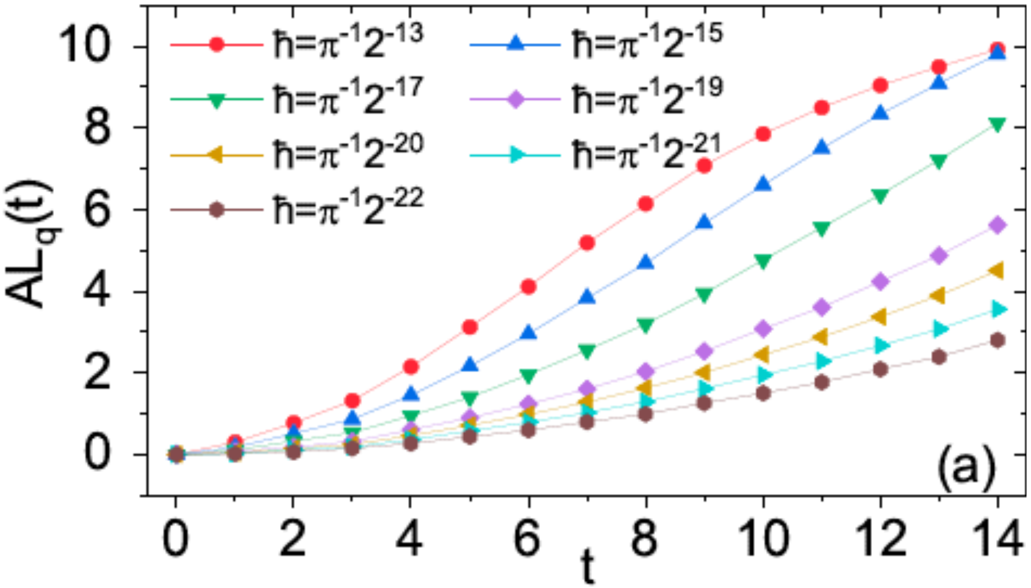
dominated by the largest local Lyapunov exponent for the region E

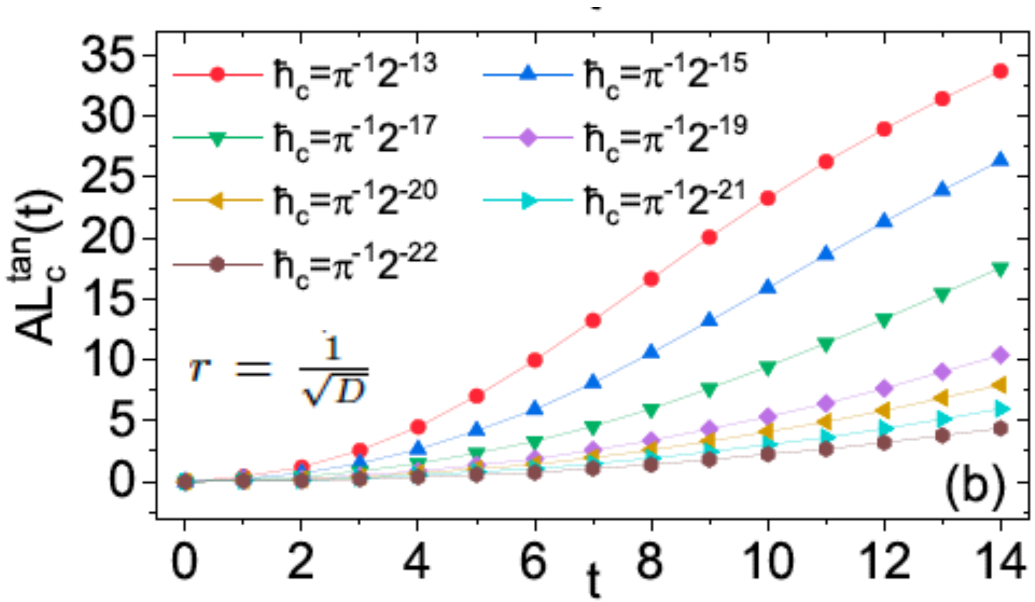
$$AL_c^{\tan}(t) \propto 2\lambda_{\rm lyp} t \quad t \ll t^*$$

$$\propto 2\lambda^*_{\rm lyp} t \quad t \gg t^*$$

$$t^* \sim \frac{1}{\lambda}\ln\frac{r}{\sqrt{\hbar_c}}$$

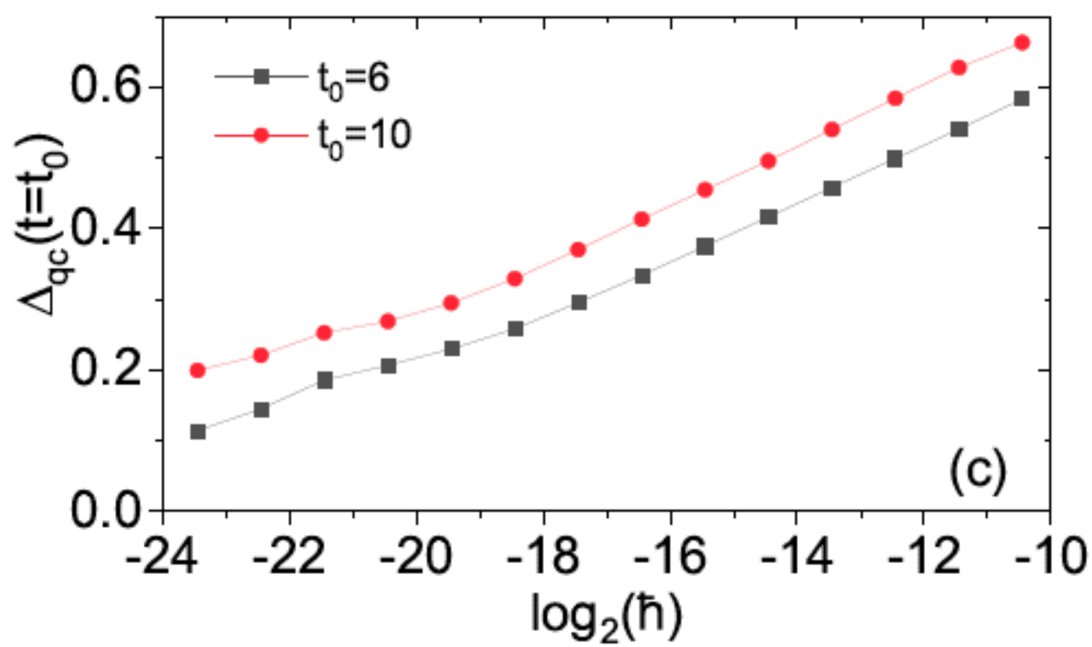
Correspondence principle restored





(the quantum potential "sees" a rounded potential, with r decreasing with the effective Planck constant)

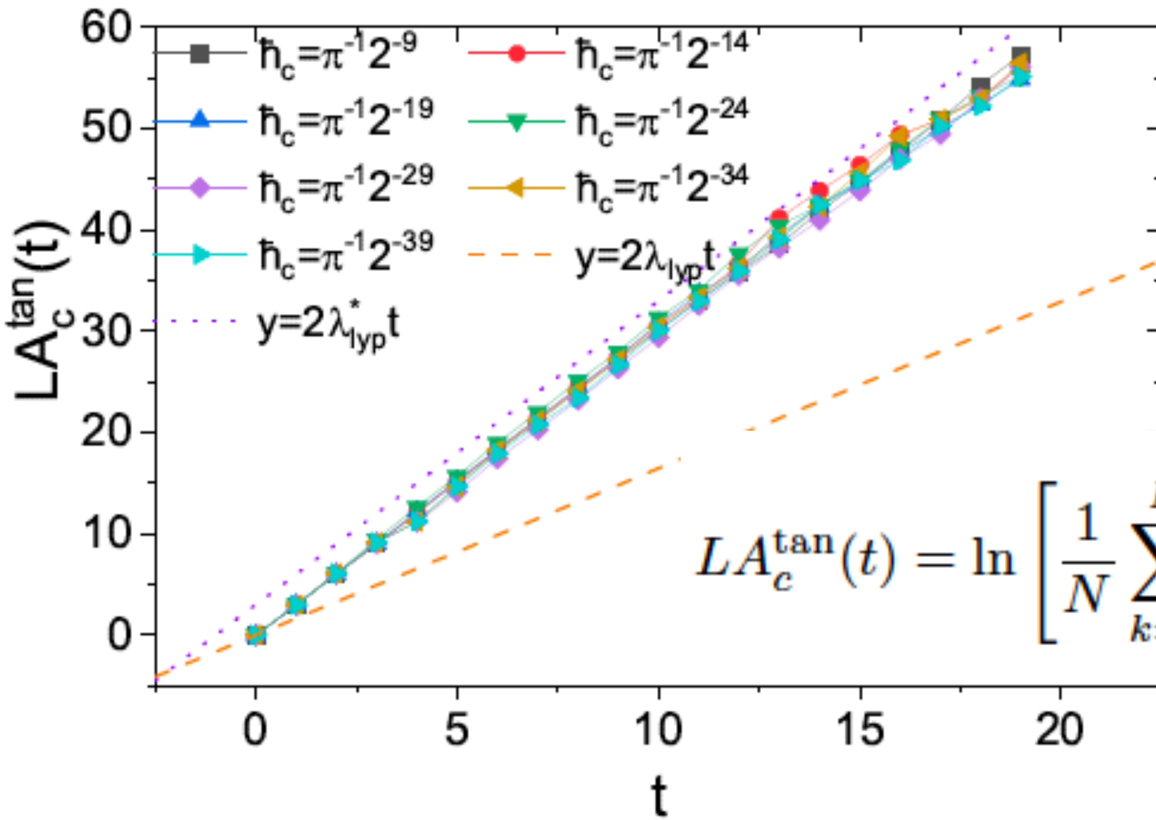
$$\Delta_{qc}(t) = \frac{|AL_q(t) - AL_c^{\tan}(t)|}{AL_q(t) + AL_c^{\tan}(t)}$$



OTOC and chaotic dynamics

A proper averaging is needed to use the OTOC as a diagnostic of chaotic dynamics

Averaging directly OTOC rather than their logs, the growth rate is determined by local fluctuations (unstable fixed points) and not by the Lyapunov exponent



$$LA_c^{\tan}(t) = \ln\left[\frac{1}{N}\sum_{k=1}^{N}\int d\gamma \rho_{\gamma_0^k}(\gamma)\left(\frac{\partial x(t)}{\partial x(0)}\right)^2\right]$$

Summary

OTOC does not lead to any violation of the correspondence principle

OTOC is a diagnostic of quantum chaos (after proper averaging)

Possible extensions of our results to many-body systems?