Complexity of quantum motion: A phase-space approach

Giuliano Benenti

Center for Nonlinear and Complex Systems
Univ. Insubria, Como, Italy

In collaboration with:
Gabriel Carlo (Buenos Aires)
Tomaž Prosen (Ljubljana)

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OUTLINE

How to measure the complexity of quantum motion?
(lack of a simple description of the evolution of a quantum system)

Classical complex systems characterized by exponential instability of motion (chaos, algorithmic complexity, deterministic randomness,...)

In quantum mechanics the notion of trajectories is forbidden by the Heisenberg uncertainty principle

Phase space approach: we propose the **separability entropy of the Wigner function** as a measure of complexity of a quantum state

Classical chaos: Exponential instability

Classical chaos is characterized by exponential local instability: two nearby trajectories separate exponentially, with rate given by the maximum Lyapunov exponent

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

d length of the tangent vector

Classical chaos: Trajectories are unpredictable

Chaotic orbits are unpredictable: in order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment and independent of the previous length of the trajectory. The information associated with a segment of trajectory of length t is equal, asymptotically, to

$$\lim_{t \to \infty} \frac{I(t)}{t} = h,$$

where h is the KS (Kolmogorov-Sinai) entropy which is positive when $\lambda > 0$

Classical chaos: Statistical description of motion

Exponential instability $\Rightarrow$ Continuous (frequency) Fourier spectrum of motion

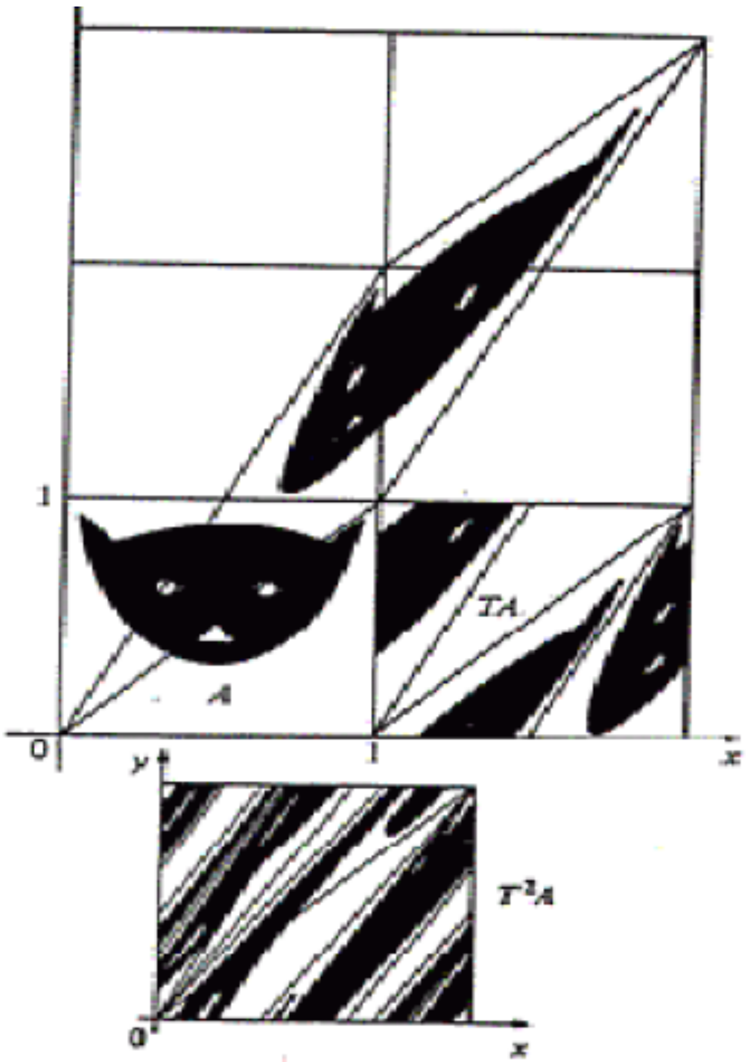
Continuous spectrum $\Rightarrow$ Decay of correlations (mixing)

Mixing assures the statistical independence of different parts of a trajectory

Mixing $\Rightarrow$ Statistical description of chaotic dynamics
(diffusion, relaxation, ...)

Integrable systems $\Rightarrow$ Nearby points separate only linearly

Loss of memory in the Arnold cat map



$$T : \begin{cases} \bar{x} = x + y \pmod 1, \\ \bar{y} = x + 2y \pmod 1 \end{cases}$$

$$h = \lambda = \ln\left(\frac{3+\sqrt{5}}{2}\right) > 0$$

Stretching and folding of the cat in phase space

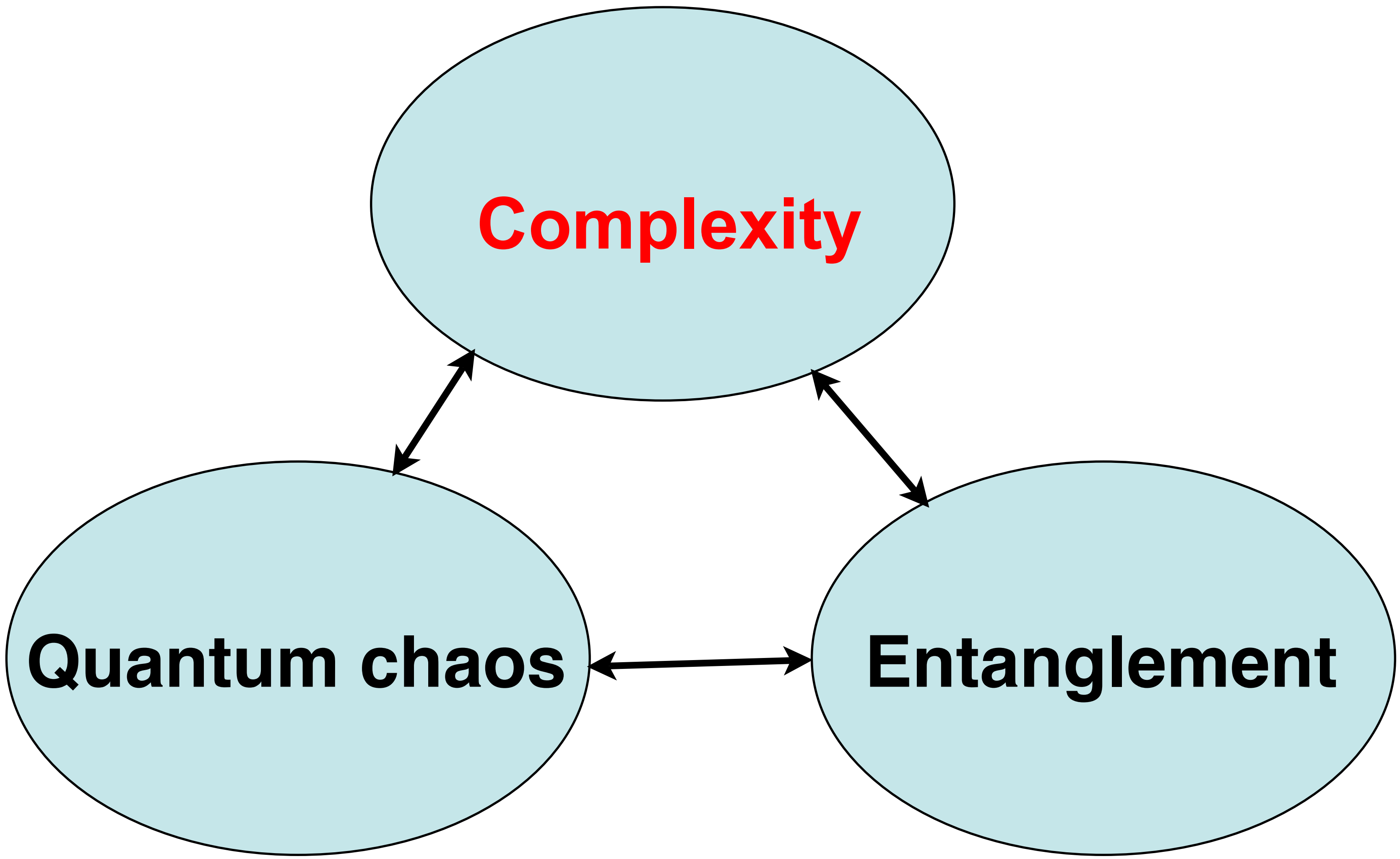
Any amount of error rapidly effaces the memory of the initial distribution

Quantum chaos?

The alternative of exponential or power-law divergence of trajectories disappears in quantum mechanics, Heisenberg's uncertainty principle forbidding the notion of trajectories

The essential conditions for classical chaos are violated in quantum mechanics. Indeed the energy and the frequency spectrum of any quantum motion, bounded in phase space, are always DISCRETE $\Rightarrow$ regular motion

The ultimate origin of this fundamental quantum property is the discreteness of the phase space: the uncertainty principle implies a finite size of an elementary phase space cell


Complexity
Quantum chaos
Entanglement

Requirements for quantum complexity quantifiers:

(i) to provide a unified description of both one- and many-body dynamics;

(ii) to reproduce at the classical limit the well-known notion of classical complexity based on the local exponential instability of chaotic dynamics;

(iii) to be applicable to both pure and mixed states;

(iv) to be practically useful, that is, convenient for numerical investigations.

Wigner harmonics entropy

In classical mechanics the number of harmonics of the classical distribution function in phase space provide un upper bound to classical computing resources needed for accurate simulation of Liouville dynamics

$$\tilde{\rho}^t(\boldsymbol{k}) \equiv (2\pi)^{-2d} \int \mathrm{d}^{2d} z e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{z}} \rho^t(\boldsymbol{z}), \boldsymbol{k} \in \mathbb{Z}^{2d}$$

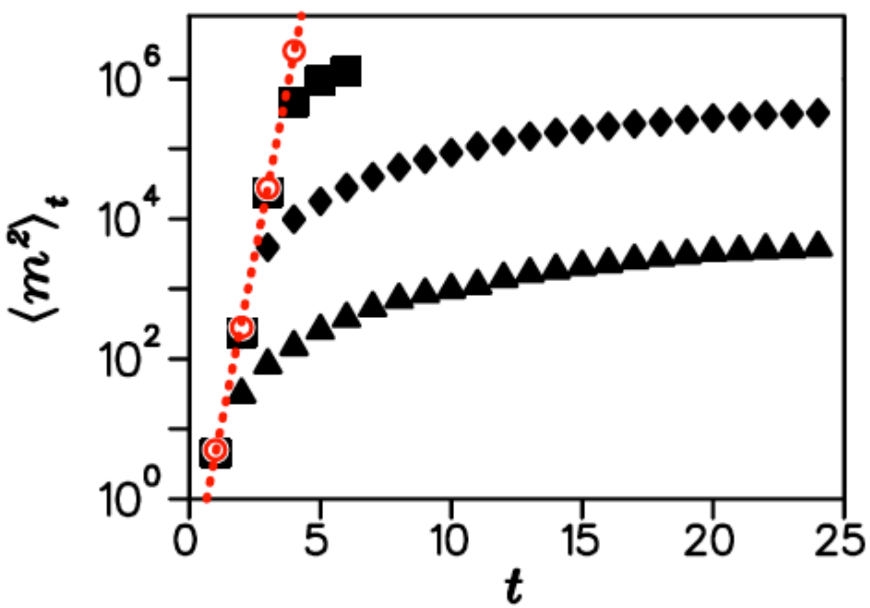
$$g[\rho^t] = -\sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} |\tilde{\rho}^t(\boldsymbol{k})|^2 \ln |\tilde{\rho}^t(\boldsymbol{k})|^2$$

The (growth rate of the) number of harmonics is a measure of classical complexity

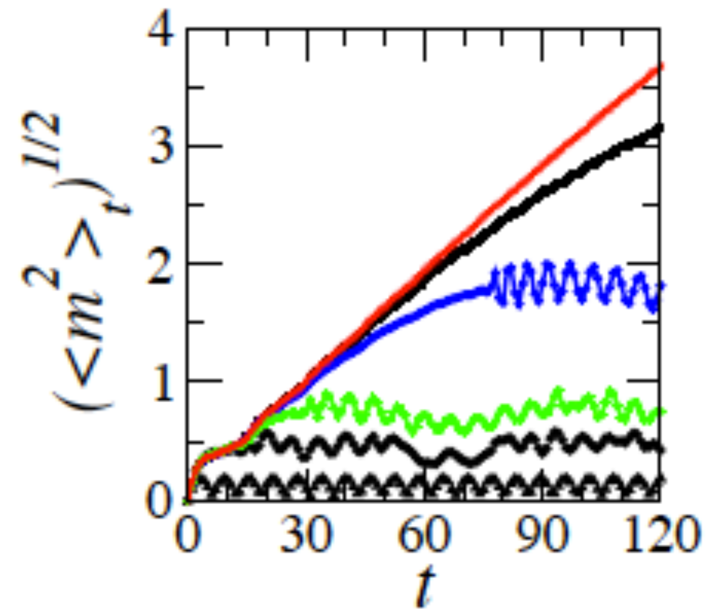
Since the phase space approach can be equally used for both classical and quantum mechanics, the **number of harmonics of the Wigner function** could be a suitable measure of the complexity of a quantum state

Numerical illustration

Chaotic regime



Integrable regime

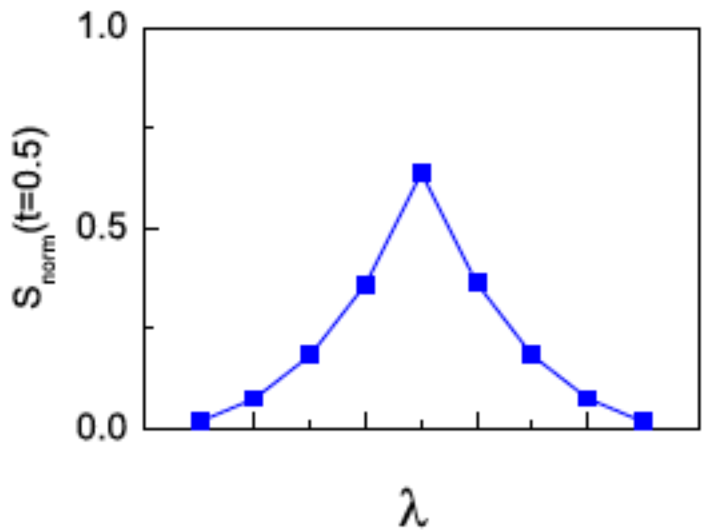


Kicked quartic oscillator model

$$\hat{H} = \hbar\,\omega_0 \hat{n} + \hbar^2\,\hat{n}^2 - \sqrt{\hbar}\,g(t)(\hat{a}+\hat{a}^\dagger),$$

$$g(t) = g_0 \sum \delta(t-s), \quad \hat{n} = \hat{a}^\dagger \hat{a},\ [\hat{a},\hat{a}^\dagger] = 1$$

Many-body quantum systems



The number of harmonics can detect quantum phase transitions

Examples: Ising chain in a transverse field

Drawbacks:

1) Basis-dependent quantity

2) For both integrable and chaotic quantum spin chains the number of harmonics grows exponentially with time

Complexity of time-dependent DMRG

Matrix product operator representation of states/observables in terms of 4L (L systems size) matrices of size D:

$$O_{\text{MPO}} = \sum_{s_j} \operatorname{tr}\left(A_0^{s_0} A_1^{s_1} \cdots A_{L-1}^{s_{L-1}}\right) \sigma_0^{s_0} \sigma_1^{s_1} \cdots \sigma_{L-1}^{s_{L-1}}$$

For local operators (or sum of local operators) the size D required for accurate simulation of dynamics up to time t grows exponentially with t for non-integrable spin chains and only polynomially for linear integrable chains [Prosen and Znidaric, PRE 75, 015202 (R) (2007)]

Here quantum (algorithmic) complexity is measured by the computation complexity of the best (known) classical simulation of quantum dynamics.
Can we put this result in a broader context?

Wigner separability entropy

Schmidt (singular value) decomposition of the Wigner function

Arbitrary phase space decomposition, $\Omega = \Omega_1 \oplus \Omega_2$, into two set of coordinates, $z \equiv (x, y)$; normalization constraint $\int \mathrm{d}z\, W(z) = 1$:

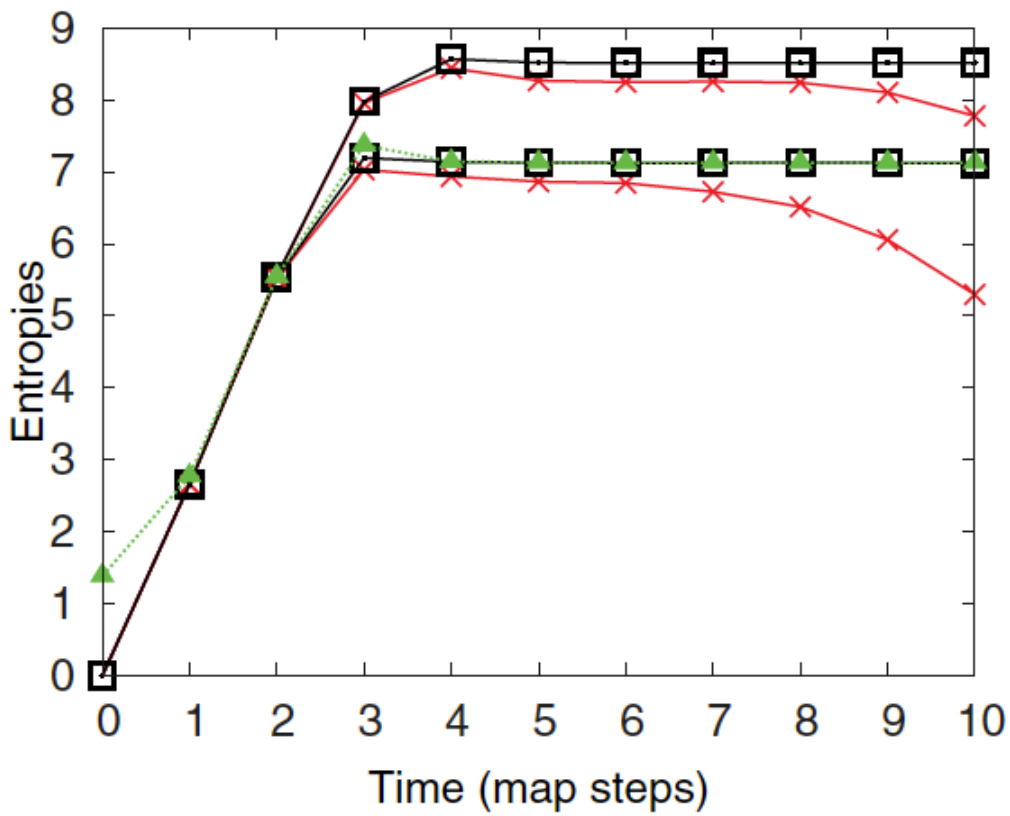
$$W(\boldsymbol{x}, \boldsymbol{y}) = \sum \mu_n v_n(\boldsymbol{x}) w_n(\boldsymbol{y})$$

with $n \in \mathbb{N}$, $\{v_n\}$ and $\{w_n\}$ orthonormal bases for $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively, and the Schmidt coefficients (singular values) $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ satisfying $\sum_n \mu_n^2 = \int \mathrm{d}z\, W^2(z)$.

Definition (Wigner separability entropy):

$$h[W] = -\sum_n \tilde{\mu}_n^2 \ln \tilde{\mu}_n^2, \qquad \tilde{\mu}_n \equiv \frac{\mu_n}{\sqrt{\int \mathrm{d}z\, W^2(z)}}$$

Classical-quantum correspondence (unipartite systems)



Perturbed cat map

Classical saturation (and subsequent drop) due to phase-space coarse graining for numerical simulations

Quantum saturation fixed by the finite size of Planck constant

Connection with operator space entanglement entropy (for bipartite systems)

$\mathrm{Tr}(\hat{\rho}^2) \leqslant 1 \Rightarrow$ the density operator is a Hilbert-Schmidt operator

$$\|\hat{\rho}\|_{\mathrm{HS}} = \sqrt{\mathrm{Tr}(\hat{\rho}^2)} \qquad \langle \hat{A}, \hat{B} \rangle_{\mathrm{HS}} = \mathrm{Tr}(\hat{A}^\dagger \hat{B})$$

Therefore, given $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the density operator has a Schmidt decomposition:

$$\hat{\rho} = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n,$$

where $\{\hat{\sigma}_n\}$ and $\{\hat{\tau}_n\}$ are orthonormal $[\mathrm{Tr}(\hat{\sigma}_m^\dagger \hat{\sigma}_n) = \delta_{mn}$, $\mathrm{Tr}(\hat{\tau}_m^\dagger \hat{\tau}_n) = \delta_{mn}]$ bases for $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$, respectively, and the Schmidt coefficients $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0$ satisfying $\sum_n \mu_n^2 = \mathrm{Tr}(\hat{\rho}^2) = \|\hat{\rho}\|_{\mathrm{HS}}^2$.

Operator space entanglement entropy

$$h[\hat{\rho}] = -\sum_n \tilde{\mu}_n^2 \ln \tilde{\mu}_n^2, \quad \tilde{\mu}_n \equiv \frac{\mu_n}{\|\hat{\rho}\|_{\mathrm{HS}}}$$

The Weyl correspondence establishes an isomorphism between Hilbert-Schmidt operators and $L^2(\Omega)$ functions on classical phase space

Since the density operator $\hat{\rho}$ is the integral kernel of a unique linear one-to-one transformation mapping the operators $\hat{\sigma}_n \leftrightarrow \hat{\tau}_n$, $\forall n$, i.e. $\mathrm{Tr}_1((\hat{\sigma}_n^\dagger)_1 \hat{\rho}) = \mu_n \hat{\tau}_n$, and vice versa for the inverse transformation, $\mathrm{Tr}_2((\hat{\tau}_n^\dagger)_2 \hat{\rho}) = \mu_n \hat{\sigma}_n$, and since the Weyl transform of the density operator is the Wigner function, it follows that $\hat{\rho}$ and W have the same Schmidt coefficients, and therefore

$$h[\hat{\rho}] = h[W]$$

Pure states

Schmidt decomposition of $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$$|\psi\rangle = \sum_j \lambda_j |\phi_j\rangle \otimes |\xi_j\rangle$$

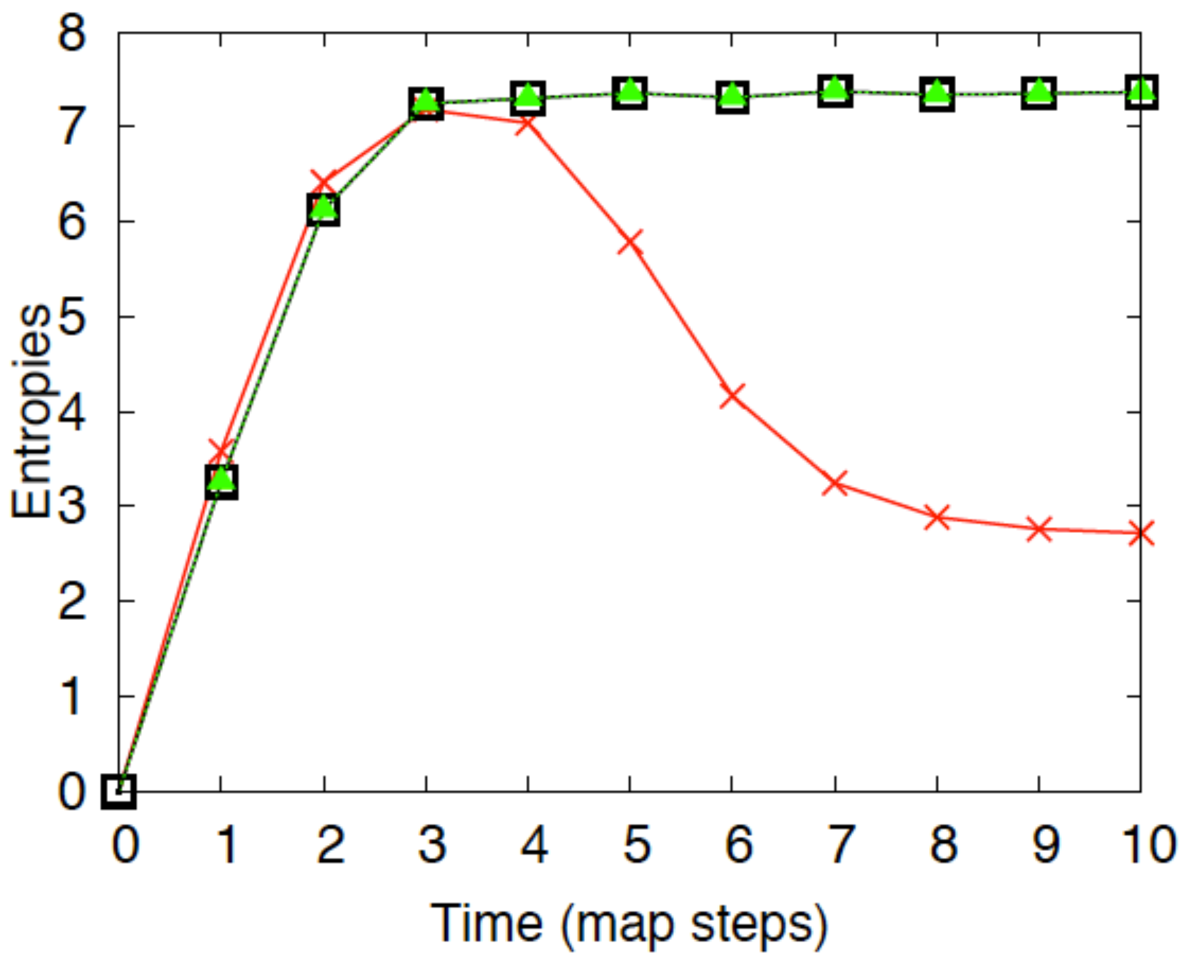
Schmidt decomposition of $\hat{\rho} = |\psi\rangle\langle\psi|$

$$\hat{\rho} = \sum_{j,k} \lambda_j \lambda_k |\phi_j\rangle\langle\phi_k| \otimes |\xi_j\rangle\langle\xi_k| = \sum_n \mu_n \hat{\sigma}_n \otimes \hat{\tau}_n$$

For pure states the Wigner separability entropy is twice the entanglement entropy and equal to the quantum mutual information

$$h[W] = 2\,E(|\psi\rangle) = I(1:2)$$

Classical-quantum correspondence (bipartite systems)



Two coupled cat map

The initial growth of entanglement can be reproduced in the semiclassical regime by purely classical computations

Summary

We have proposed a new measure of complexity of quantum states, the Wigner separability entropy

This quantity also quantifies the minimal amount of computational resources required to simulate the quantum dynamical evolution of a system by means of time-dependent density-matrix renormalization group (operator space entanglement entropy)

Relation with entanglement entropy and classical separability entropy (of Liouville densities)

The Wigner separability entropy is well defined also for mixed states and could therefore be used to quantify the complexity of decoherent quantum dynamics