Quantum capacity of dephasing channels with memory

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Motivations and Outline

- Non-Markovian effects in open quantum systems
- Can memory effects enhance the capacity of quantum channels?

Dephasing channels with memory: quantum capacity maximized by separable input states

1) Markov chain model: explicit computation of the quantum capacity
2) Bosonic bath of oscillators
Non-Markovian effects

- Low-frequency noise noise in solid-state devices (for instance, $1/f$ noise)
- Fluctuating birefringe in optical fibers
- Quantum information transmission across spin chains
- Sending atoms through a resonant cavity
Capacity of a classical channel

![Diagram of a classical channel with Source, Encoding, Input, Classical channel, Output, Decoding, and Receiver stages.]

**MUTUAL INFORMATION**  
\[ I(X : Y) = H(X) + H(Y) - H(X,Y) \]

**Shannon information of the random variable X**  
\[ H(X) = -\sum_x p_x \log_2 p_x \]

**CAPACITY:** maximum rate at which classical information can be reliably transmitted down the channel  
\[ C = \max_{p_x} H(X : Y) \]
Quantum channels

QUANTUM SOURCE: quantum states chosen from the ensemble \( \{\rho_0, \ldots, \rho_k\} \) with a priori probabilities \( \{p_1, \ldots, p_k\} \) are sent through the channel

QUANTUM CHANNEL described by a linear, completely positive, trace preserving (CPT) map \( \mathcal{E} \): 

\[
\rho' = \mathcal{E}(\rho), \quad \rho = \sum_{x=1}^{k} p_x \rho_x
\]

Use quantum states to reliably transmit classical information (classical capacity) or quantum information (quantum capacity)
Entanglement fidelity

How to measure the reliability in the transmission of quantum information?

It is not sufficient to verify that the input state $\rho$ is transmitted with high fidelity.

Ex: send a member of a Bell pair through a completely dephasing channel

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad \rho = \text{Tr}_2(|\psi\rangle_{12}\langle\psi|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E} \left( \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \right) = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

$$\mathcal{E}(\rho) = \rho, \quad \text{but} \quad (\mathcal{E} \otimes \mathcal{I})(|\psi\rangle_{12}\langle\psi|) = \frac{1}{2}(|01\rangle\langle01| + |10\rangle\langle10|)$$

Entanglement is lost
The ENTANGLEMENT FIDELITY $F_e$ is independent of the purification $\mathcal{R}$ of the quantum system $Q$.
Entropy exchange

\[ S_e = S_e(\rho, \mathcal{E}) = S(\rho^{E'}) \quad \text{von Neumann entropy} \]

The ENTROPY EXCHANGE \( S_e \) is the entropy of the final state \( \rho^{E'} \) of a “mock” environment, initially in a pure state \( |0^E\rangle \)
Coherent information

Analogous to mutual information but for quantum information

\[ I_c(\rho, \mathcal{E}) = S(\mathcal{E}(\rho)) - S_e(\rho, \mathcal{E}) \]

The COHERENT INFORMATION \( I_c = S(\rho^Q) - S(\rho^RQ') \) can never be positive for classical systems.

\( I_c \) deals with the entanglement transmission through the channel.
Ex: if the channel is noiseless, \( I_c = 0 \) is \( \rho \) pure, \( I_c \) is maximum if \( \rho \) is maximally mixed.
Quantum data-processing inequality:

$$I_c(\rho, \mathcal{E}_1) \geq I_c(\rho, \mathcal{E}_2 \circ \mathcal{E}_1)$$

We cannot increase the coherent information acting on the output

In contrast to mutual information, $I_c$ in general is not subadditive

Using entangled input states $\rho_{12} \neq \rho_1 \otimes \rho_2$ [$\rho_1 = \text{Tr}_2(\rho_{12}), \rho_2 = \text{Tr}_1(\rho_{12})$] we can obtain

$$I_c(\rho_{12}, \mathcal{E} \otimes \mathcal{E}) > I_c(\rho_1, \mathcal{E}) + I_c(\rho_2, \mathcal{E})$$
Quantum capacity

The **QUANTUM CAPACITY** $Q$ measures the maximum number of qubits (per channel use) that can be reliably transmitted down a noisy channel

For memoryless channels

$$Q = \lim_{n \to \infty} \frac{Q_n}{n}, \quad Q_n = \max_{\rho} I_c(\mathcal{E}_n, \rho)$$

$$I_c(\mathcal{E}_n, \rho) = S[\mathcal{E}_n(\rho)] - S[\tilde{\mathcal{E}}_n(\rho)], \quad \mathcal{E}_n = \mathcal{E}^\otimes n$$

$$\rho' = \mathcal{E}_n(\rho) = \text{Tr}_E[U_n(\rho \otimes |0\rangle_E \langle 0|)U_n^\dagger], \quad \rho'_E = \tilde{\mathcal{E}}_n(\rho) = \text{Tr}_S[U_n(\rho \otimes |0\rangle_E \langle 0|)U_n^\dagger]$$

The regularization $n \to \infty$ is necessary since $I_c$ in general fails to be subadditive
Degradable channels

Degradable channels: the final state $\rho'_E$ of the environment can be reconstructed from the final state $\rho'$ of the system

$$I_c = S(\rho_{Q'E'}) - S(\rho_{E'}) = S(Q'|E')$$ is subadditive

$Q = Q_1$ ("single-letter" formula)
Dephasing channels

Single-use dephasing channel:
\[ \mathcal{E} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & g\rho_{01} \\ g\rho_{10} & \rho_{11} \end{pmatrix} , \quad g \text{ dephasing factor} \]

Generalized \( n \)-uses dephasing channel

\[ U_n \ket{i} \ket{0}_E = \ket{i} \ket{\phi_i}_E, \quad \ket{i} = \ket{i_1, \ldots, i_n} \text{ preferential basis} \]

\[ \rho' = \mathcal{E}_n (\rho) = \sum_{\alpha} A_\alpha \rho A_\alpha^\dagger \quad A_\alpha = E \bra{\alpha} U_n \ket{0}_E \text{ diagonal Kraus operators} \]

\[ (A_\alpha)_{ij} = E \bra{\alpha} \phi_i \ket{E} \delta_{ij} \]
Degradability of the generalized dephasing channel

\[ \rho = \sum_{i,j} c_{ij} |i\rangle \langle j| \]  generic input state

The final environmental state depends only on the populations of \( \rho \)

\[ \rho'_E = \tilde{\mathcal{E}}_n(\rho) = \sum_i |c_i|^2 |\phi_i\rangle_E \langle \phi_i| \]

Since the dephasing channel \( \mathcal{E}_n \) does not affect populations,

\[ \tilde{\mathcal{E}}_n = \tilde{\mathcal{E}}_n \circ \mathcal{E}_n \]
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Maximization of the coherent information

The coherent information $I_c(\mathcal{E}_n, \rho)$ of a generalized dephasing channel is maximized by SEPARABLE input states DIAGONAL in the preferential basis $\{|i\rangle\}$

$$\rho_k = \frac{\rho_{k-1} + \sigma_z^{(k)} \rho_{k-1} \sigma_z^{(k)}}{2}, \quad (k = 1, \ldots, n)$$

- the Kraus operators commute with $\sigma_z^{(k)}$
- the coherent information is concave for degradable channels

$$I_c(\mathcal{E}_n, \rho_n) \geq I_c(\mathcal{E}_n, \rho_{n-1}) \geq \cdots \geq I_c(\mathcal{E}_n, \rho_0)$$

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Forgetful channels

Memory effects vanish exponentially fast with time

DOUBLE-BLOCKING strategy:

- consider blocks of $n + l$ uses of the channel
- do the actual coding and decoding for the first $n$ uses
- let $n \to \infty$
- quantum capacity $Q = \lim_{n \to \infty} \frac{Q_n}{n}$
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A phenomenological noise model

\[ \rho' = \mathcal{E}_n(\rho) = \sum_{i_1,\ldots,i_n = 0,z} A_{i_1\ldots i_n} \rho A_{i_1\ldots i_n}^\dagger, \quad A_{i_1\ldots i_n} = \sqrt{p_{i_1\ldots i_n}} \sigma_{i_1}^{(1)} \otimes \cdots \otimes \sigma_{i_n}^{(n)}, \]

\( p_{i_1\ldots i_n} \) probability that the ordered sequence \( \sigma_{i_1}^{(1)}, \ldots, \sigma_{i_n}^{(n)} \) of Pauli operators (\( I \) or \( \sigma_z \)) is applied to the \( n \) qubits crossing the channel

- **Dephasing probability stationary:** \( p_{i_k=z} = p_z [p_{i_k=0} = p_0 = 1 - p_z] \) for all \( k \)
  \( (p_{i_k} = \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n} p_{i_1\ldots i_n}) \)

- ** Forgetful channel:** \( |p_{i_{k'}i_k} - p_{i_{k'}i_k}p_{i_k}| \) decays exponentially with \( |k' - k| \)

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The maximum of coherent information in this model is obtained for the maximally mixed input state $\rho_I \equiv \frac{1}{2^n} I^{\otimes n}$

$$\rho_k = \frac{\rho_{k-1} + \sigma_x^{(k)} \rho_{k-1} \sigma_x^{(k)}}{2}, \quad (k = 1, \ldots, n)$$

Starting from a diagonal $\rho_0$ we can prove

$$I_c(\mathcal{E}_n, \rho_n = \rho_I) \geq I_c(\mathcal{E}_n, \rho_0)$$
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**Markov-chain model**

\[ p_{i_1,\ldots,i_n} = p_{i_1}p_{i_2|i_1}\cdots p_{i_n|i_{n-1}}, \quad p_{i_k|i_{k-1}} = (1-\mu)p_{i_k} + \mu\delta_{i_k,i_{k-1}} \]

\( \mu \) measures the partial memory of the channel
\( \mu = 0 \) memoryless channel
\( \mu = 1 \) perfect memory

\( \mu \) might depend on the time interval \( \tau \) between two consecutive channel uses, compared with the memory time scale \( \tau_c \)

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Quantum capacity of a Markov chain

\[ S[\mathcal{E}_n(\rho_I)] = S(\rho_I) = n \]

\[ S_e = - \sum_{i_1, \ldots, i_n} p_{i_1 \ldots i_n} \log_2 p_{i_1 \ldots i_n} \equiv H(X_1, \ldots, X_n) \]

\( H(X_1, \ldots, X_n) \) Shannon entropy of the collection of random variables \( X_1, \ldots, X_n \) (characterized by the joint probabilities \( p_{i_1 \ldots i_n} \))

For a stationary Markov chain

\[ \lim_{n \to \infty} \frac{1}{n} H(X_1, \ldots, X_n) = H(X_2|X_1) = p_0H(q_0)+p_zH(q_z), \quad q_i \equiv p(i|i) = (1-\mu)p_i+\mu \]

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\[ Q = 1 - p_0 H(q_0) - p_z H(q_z) \]

\[ Q = 1 - H(p_0) \] memoryless limit

\[ Q = 1 \] with perfect memory (noiseless channel)
Convergence of $Q_n/n$ to $Q$

$$Q_n = n - (n - 1)[p_0H(q_0) + p_zH(q_z)] - H(p_0)$$

$\epsilon_n \equiv Q - \frac{Q_n}{n}$ growing function of $\mu$

$\epsilon_n(\mu = 0) = 0$

$\epsilon_n(\mu = 1) = H(p_0)/n$

$\epsilon_n(\mu) \approx \frac{1}{2\ln 2} \frac{\mu^2}{n}$ for $\mu \ll 1$

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**Bosonic bath environment**

\[
H(t) = H_E - \frac{1}{2} X_E F(t) + H_C, \quad H_E = \sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha,
\]

\[
X_E = \sum_\alpha (b_\alpha^\dagger + b_\alpha), \quad F(t) = \lambda \sum_{j=1}^n \sigma_z^{(j)} f_j(t), \quad H_C = \sum_\alpha \frac{\lambda^2}{4\omega_\alpha} \sum_{j=1}^n \sigma_z^{(j)}
\]

Diagram:

- \( f(t) \)
  - \( f(t)_1 \)
    - \( 0 \) to \( \tau_p \)
  - \( f(t)_2 \)
    - \( 0 \) to \( \tau \) and \( \tau_p \)

- \( t \)
  - \( t_1 \) to \( t_2 \) to \( t \)

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\[ \rho(t) = \text{Tr}_E[U(t)(\rho \otimes \rho_E)U^\dagger(t)], \quad U(t) = Te^{-\frac{i}{\hbar} \int_0^t ds H(s)} \]

\[ U(t|i) = \langle i|U(t)|i \rangle \] conditional evolution operator for the environment alone

\[ (\rho')_{ij} = (\rho)_{ij} \sum_\alpha E \langle \alpha|U(t|i)\rho_EU^\dagger(t|j)|\alpha \rangle_E \]

Multimode environment of oscillators initially at thermal equilibrium, \( \rho_E = \exp(-\beta H_E) \):

\[ \sum_\alpha E \langle \alpha|U(t|i)\rho_EU^\dagger(t|j)|\alpha \rangle_E = e^{\left\{-\lambda^2 \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{1-\cos(\omega \tau)}{\omega^2} \right\} \left| \sum_{k=1}^n (i_k-j_k)e^{i\omega(k-1)\tau} \right|^2} \]

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Assume that the bath correlation function

\[ C(t) \equiv \frac{1}{2} \langle X_E(t)X_E(0) + X_E(0)X_E(t) \rangle \]

decays exponentially with time (forgetful channel)

This is the case, e.g., for a Lorentian power spectrum

\[ S(\omega) = \frac{2\tau_c}{[1 + (\omega \tau_c)^2]} \]

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Decoherence-protected subspace

In the limit of perfect memory \((\tau_c \rightarrow \infty)\) there exists for any \(n\) a decoherence-free subspace corresponding to a qubit train with an equal number of \(|0\rangle\) and \(|1\rangle\) states.

Since the dimension \(d\) of this subspace is such that \(\log_2 d \approx n - \frac{1}{2} \log_2 n\), then the channel is asymptotically noiseless \((Q = 1)\).

If \(\bar{n} \gg 1\) qubits can be sent within the memory time scale \(\tau_c\) and the quantum information is encoded in the decoherence-protected subspace, then

\[
1 - \log_2 \frac{\bar{n}}{(2\bar{n})} \text{ lower bound for } Q_{\bar{n}}/\bar{n}
\]
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**Lower bound for** $Q_n/n$

maximally mixed input state

Lorentzian power spectrum

$\xi \equiv \tau_c/(\tau + \tau_c)$

Numerical data suggest that $I_c/n$ converges for $n \to \infty$: it is possible to increase the transmission rate if quantum information is encoded in long blocks, separated by time intervals larger than $\tau_c$

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Conclusions

The coherent information in a dephasing channel with memory is maximized by separable input states.

Computed the quantum capacity $Q$ for a Markov chain noise model and provided numerical evidence of a lower bound for $Q$ in the case of a bosonic bath.

- Find realistic coding strategies for few-qubit trains.

- The results of this study could be adapted to environments with algebraically decaying memory effects, e.g. low-frequency noise in the solid state?

- Effects of integrable/chaotic environments or of quantum phase transitions on quantum channel capacity.