How complex is quantum motion?

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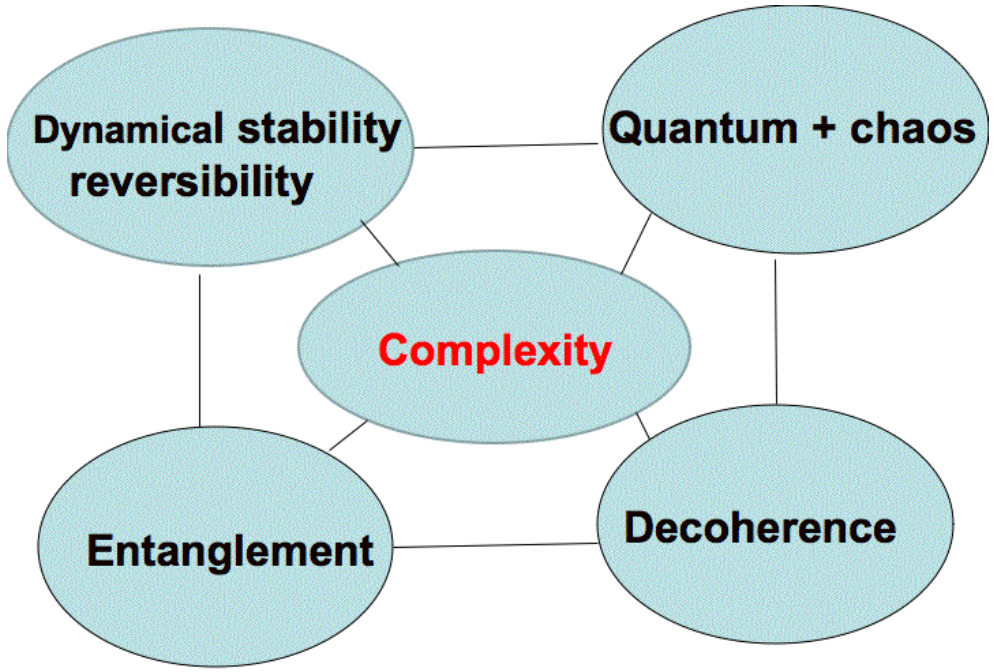


Main features QM: discrete phase space and interference

”   dynamical chaos: exponential instability and continuous spectrum

How to define complexity in quantum mechanics?

Classical chaos: Exponential instability

Classical chaos is characterized by exponential local instability: two nearby trajectories separate exponentially, with rate given by the maximum Lyapunov exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{d(t)}{d(0)}$$

d length of the tangent vector

Classical chaos: Trajectories are unpredictable

Chaotic orbits are unpredictable: in order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment and independent of the previous length of the trajectory. The information associated with a segment of trajectory of length t is equal, asymptotically, to

$$\lim_{t\to\infty} \frac{I(t)}{t} = h,$$

where h is the KS (Kolmogorov-Sinai) entropy which is positive when $\lambda > 0$

Classical chaos: Statistical description of motion

Exponential instability $\Longrightarrow$ Continuous (frequency) Fourier spectrum of motion

The power spectrum is the Fourier transform of the autocorrelation function

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

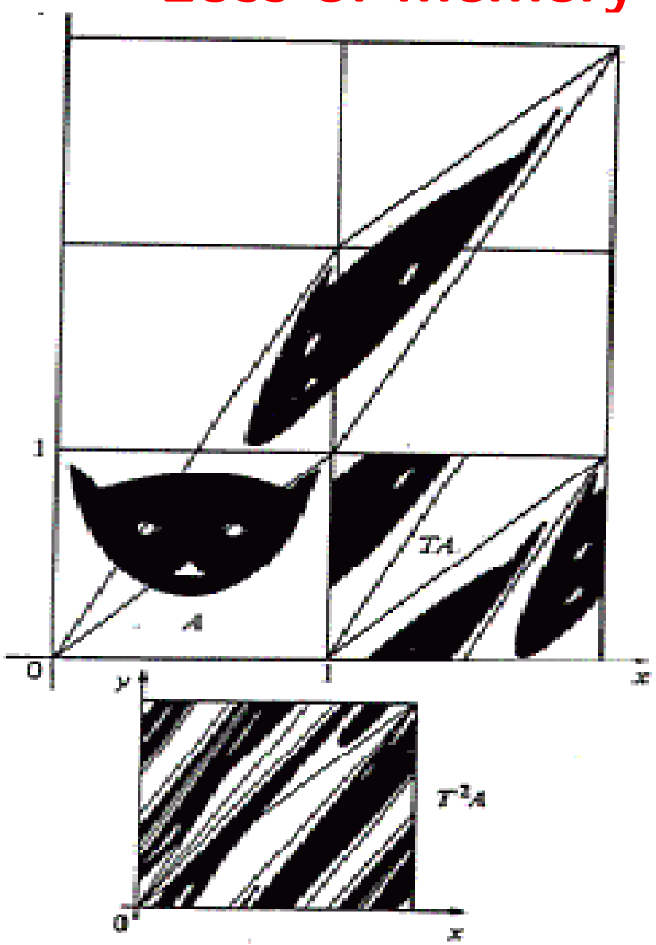
Continuous spectrum $\Longrightarrow$ Decay of correlations (mixing)

Mixing assures the statistical independence of different parts of a trajectory

Mixing $\Longrightarrow$ Statistical description of chaotic dynamics (diffusion, relaxation, ...)

Integrable systems $\Longrightarrow$ Nearby points separate only linearly

Loss of memory in the Arnold cat map



$$T : \begin{cases} \bar{x} = x + y \pmod 1, \\ \bar{y} = x + 2y \pmod 1 \end{cases}$$

$$h = \lambda = \ln\left(\frac{3+\sqrt{5}}{2}\right) > 0$$

Stretching and folding of the cat in phase space

Any amount of error rapidly effaces the memory of the initial distribution

Quantum chaos?

The alternative of exponential or power-law divergence of trajectories disappears in quantum mechanics, Heisenberg's uncertainty principle forbidding the notion of trajectories

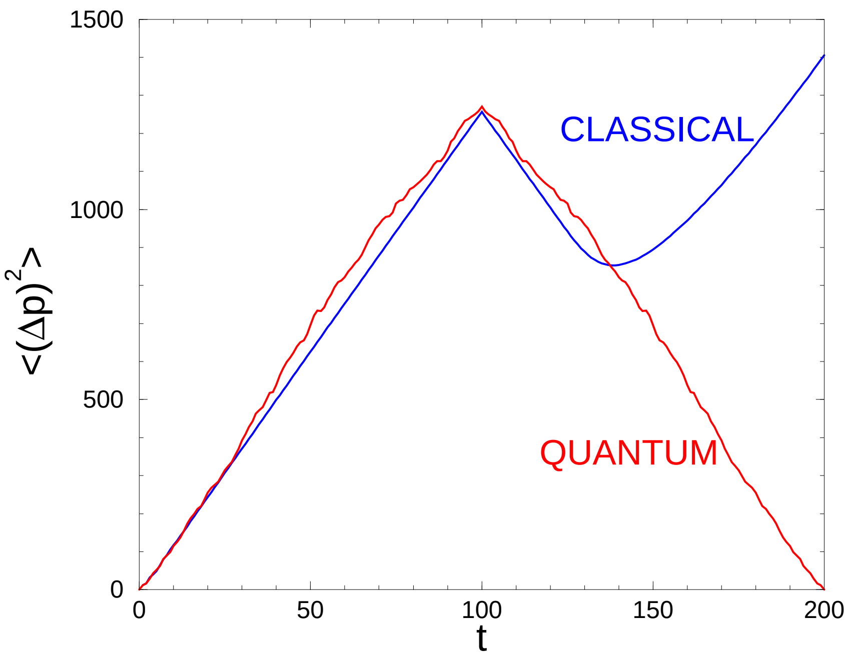
The essential conditions for classical chaos are violated in quantum mechanics. Indeed the energy and the frequency spectrum of any quantum motion, bounded in phase space, are always DISCRETE. $\Longrightarrow$ regular motion

The ultimate origin of this fundamental quantum property is the discreteness of the phase space: the uncertainty principle implies a finite size of an elementary phase space cell

Dynamical stability of quantum motion

Quantum evolution is very stable, in contrast to the extreme sensitivity to initial conditions and rapid loss of memory of classical chaos

Numerical simulation of velocity reversal



Practical irreversibility of classical motion as chaos magnifies computer round-off errors

The time to amplify the perturbation and significantly modify the trajectories is

$$t_\epsilon \approx \frac{1}{\lambda} |\ln \epsilon|$$

For round-off errors of order $\epsilon \sim 10^{-14}$ and the kicked rotator model at $K = 5$, $\lambda \approx \ln(K/2)$, we have $t_\epsilon \sim 35$

In the quantum case almost exact reversion is observed in numerical simulations

Therefore, quantum dynamics, even though it is diffusive, lacks dynamical instability

Reversibility vs. discreteness of quantum spectrum

The physical reason of this striking difference between quantum and classical motion is rooted in the discreteness of the quantum spectrum

The reversibility of quantum motion in numerical simulations is due to the fact that computer round-off errors act on a scale much smaller than the size of the Planck's cell

The classical motion (governed by the Liouville equation) of some phase space density, explores smaller and smaller scales exponentially fast.
Correspondingly there is an exponential growth in the number of the density's Fourier harmonics that are excited.
This process is limited in quantum mechanics to the size of the Planck's cell

Applications: Quantum information

The lack of exponential instability in quantum mechanics is in principle relevant for the prospects of any practical implementation of quantum computation, which has to face errors due to imperfections and decoherence

Examples:

Stability of quantum algorithms under imperfection effects [G.B., G. Casati, S. Montangero, D.L. Shelepyansky, PRL **87**, 227901 (2001)]

Robustness of multipartite entanglement [D. Rossini and G.B., PRL **100**, 060501 (2008)]

Peres' proposal: Measure quantum complexity via fidelity

The quantum fidelity (Loschmidt echo):

Overlap of two states starting with the same initial condition and evolving under two slightly different Hamiltonians

H_0 and $H_\epsilon = H_0 + \epsilon V$

$$F(t) = |\langle\psi_\epsilon(t)|\psi(t)\rangle|^2 = \left|\langle\psi_0|e^{iH\epsilon t}\hbar e^{-iH_0 t/\hbar}|\psi_0\rangle\right|^2$$

Measures stability of quantum motion

Measures stability of quantum computation under hardware imperfections or noisy gate operations

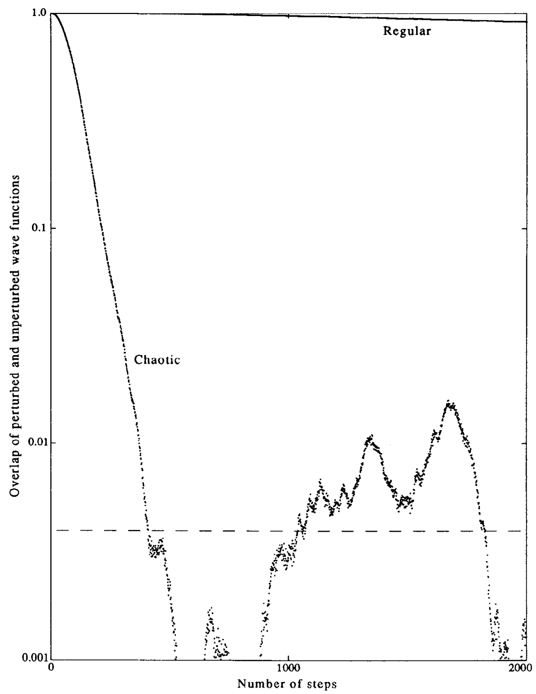


Fig. 11.12. The first 2000 steps. The overlap of the two chaotic states starts to decay quadratically with time, then as an exponential, and finally it fluctuates around its time average 0.003923 (shown by a broken line).

Exercise 11.11 *Explain why the initial decay stages are quadratic.* ⋆

Example from Peres' book:

Fidelity remains close to 1 if $|\psi\rangle$ lies in the regular region of classical phase space

Fidelity decays exponentially if $|\psi\rangle$ lies in the chaotic region

Fidelity is not a good measure of complexity

For classically chaotic quantum systems, the fidelity decay, depending on the perturbation strength, can be Gaussian or exponential
Power-law decay also possible in the quantum diffusive regime [G.B. and G. Casati, PRE **65**, 066205 (2002)]

For integrable systems, the fidelity decay can be faster than for chaotic systems [T. ProsenT. Gorin, T. Prosen, T.H. Seligman, and M. Žnidarič, Phys. Rep. **435**, 33 (2006)]

Fidelity is not a good quantity to characterize the complexity of motion, neither in quantum nor in classical mechanics

How complex is quantum motion?,
Phys. Rev. E **79**, 025201(R) (2009)

Propose a new measure of quantum complexity

In classical mechanics the number of harmonics of the classical distribution function in phase space grows linearly for integrable systems and exponentially for chaotic systems, with the growth rate determined by the rate of local exponential instability of classical motion [J. Gong and P. Brumer, PRA **68**, 062103 (2003)]

The (growth rate of the) number of harmonics is a measure of classical complexity

Since the phase space approach can be equally used for both classical and quantum mechanics, we propose the number of harmonics of the Wigner function as a suitable measure of the complexity of a quantum state

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Quantum dynamics in phase space

The phase space representation of QM allows one to compare quantum evolution of the Wigner function with classical evolution of the phase space density

$$\hat{H} \equiv H(\hat{a}^\dagger, \hat{a}; t) = H^{(0)}(\hat{n} = \hat{a}^\dagger \hat{a}) + H^{(1)}(\hat{a}^\dagger, \hat{a}; t), \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad \hat{a}|\alpha\rangle = \frac{\alpha}{\sqrt{\hbar}}|\alpha\rangle$$

Harmonic's amplitudes W_m of the Wigner function:

$$W(\alpha^*, \alpha; t) = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} W_m(I; t)\, e^{im\theta}, \quad \alpha = \sqrt{I}\, e^{-i\theta} \quad (I, \theta)\ \text{action-angle variables}$$

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Reversibility of quantum motion vs. quantum phase space complexity

Forward evolution up to time T: $\hat{\rho}(T) = \hat{U}(T)\hat{\rho}(0)\hat{U}^\dagger(T)$

Perturbation at time T: $\hat{\tilde{\rho}}(T,\xi) = \hat{P}(\xi)\hat{\rho}(T)\hat{P}^\dagger(\xi), \quad \hat{P}(\xi) = e^{-i\xi\hat{V}}$

Backward evolution: $\hat{\tilde{\rho}}(0|T,\xi) = \hat{U}^\dagger(T)\hat{\tilde{\rho}}(T,\xi)\hat{U}(T)$

Peres distance (fidelity) between the reversed $\hat{\tilde{\rho}}(0|T,\xi)$ and the initial $\hat{\rho}(0)$ state:

$$F(\xi;T) = \frac{\mathrm{Tr}[\hat{\tilde{\rho}}(0|T,\xi)\hat{\rho}(0)]}{\mathrm{Tr}[\hat{\rho}^2(0)]} = \frac{\mathrm{Tr}[\hat{\tilde{\rho}}(T,\xi)\hat{\rho}(T)]}{\mathrm{Tr}[\hat{\rho}^2(T)]}$$

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Fidelity in the phase space representation

$$F(\xi;T) = \frac{\int d^2\alpha\, W\left(\alpha^*,\alpha;0\right)\tilde{W}\left(\alpha^*,\alpha;0\middle|T,\xi\right)}{\int d^2\alpha\, W^2\left(\alpha^*,\alpha;0\right)} = \frac{\int d^2\alpha\, W\left(\alpha^*,\alpha;T\right)\tilde{W}\left(\alpha^*,\alpha;T,\xi\right)}{\int d^2\alpha\, W^2\left(\alpha^*,\alpha;T\right)}$$

Advantage of this representation: it remains valid in the classical case when the Wigner function reduces to the classical phase-space distribution function $W_c(\alpha^*,\alpha;t)$

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Growth of the number of harmonics

Let us call $\mathcal{M}(t)$ the number of harmonics (Fourier components) with appreciable large amplitudes W_m in the expansion

$$W(\alpha^*, \alpha; t) = \frac{1}{\pi} \sum_{m=-\infty}^{\infty} W_m(I; t)\, e^{im\theta}$$

For classically chaotic motion: $\mathcal{M}(t) \propto e^{\Lambda t}$

Such exponential growth cannot take place in quantum mechanics: the Fourier components of the Wigner function are related to expectation values of physical observables [Chirikov, Izrailev, Shepelyansky, Sov. Sci. Rev. C **2**, 209 (1981)]

Exponential growth possible only up to Ehrenfest time scale: $t_E \propto \frac{1}{\lambda} \ln \frac{1}{\hbar}$

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Fidelity vs. number of harmonics

For the sake of simplicity, consider as a perturbation a phase space rotation of angle ξ

$$F(\xi;t) = 1 - 2\frac{\sum_{m=-\infty}^{+\infty} \sin^2(\xi m/2) \int_0^\infty dI |W_m(I;t)|^2}{\sum_{m=-\infty}^{+\infty} \int_0^\infty dI |W_m(I;t)|^2}$$

To the lowest order in the perturbation strength:

$$F(\xi;t) \approx 1 - \frac{1}{2}\xi^2 \langle m^2 \rangle_t, \quad \langle m^2 \rangle_t = \frac{\sum_{m=-\infty}^{+\infty} m^2 \int_0^\infty dI |W_m(I;t)|^2}{\sum_{m=-\infty}^{+\infty} \int_0^\infty dI |W_m(I;t)|^2}$$

$\langle m^2 \rangle_t$ estimates the number of harmonics

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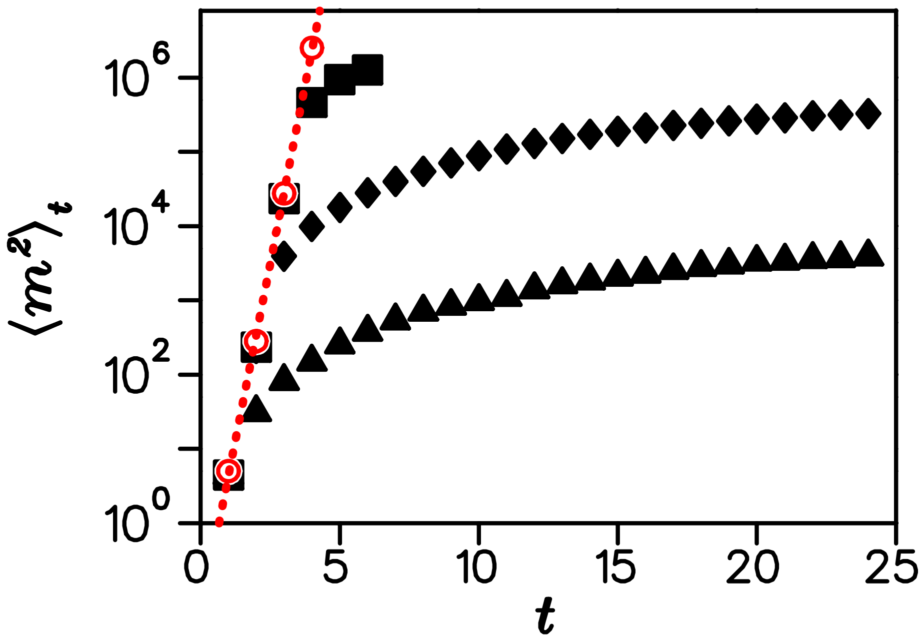
Numerical illustration

Kicked quartic oscillator model

$$\hat{H} = \hbar\,\omega_0\hat{n} + \hbar^2\,\hat{n}^2 - \sqrt{\hbar}\,g(t)(\hat{a} + \hat{a}^\dagger),$$

$$g(t) = g_0 \sum_s \delta(t-s),$$

$$\hat{n} = \hat{a}^\dagger\hat{a},\ [\hat{a}, \hat{a}^\dagger] = 1$$



After the Ehrenfest time, the number of harmonics grows linearly for pure states and slower than linearly for mixtures

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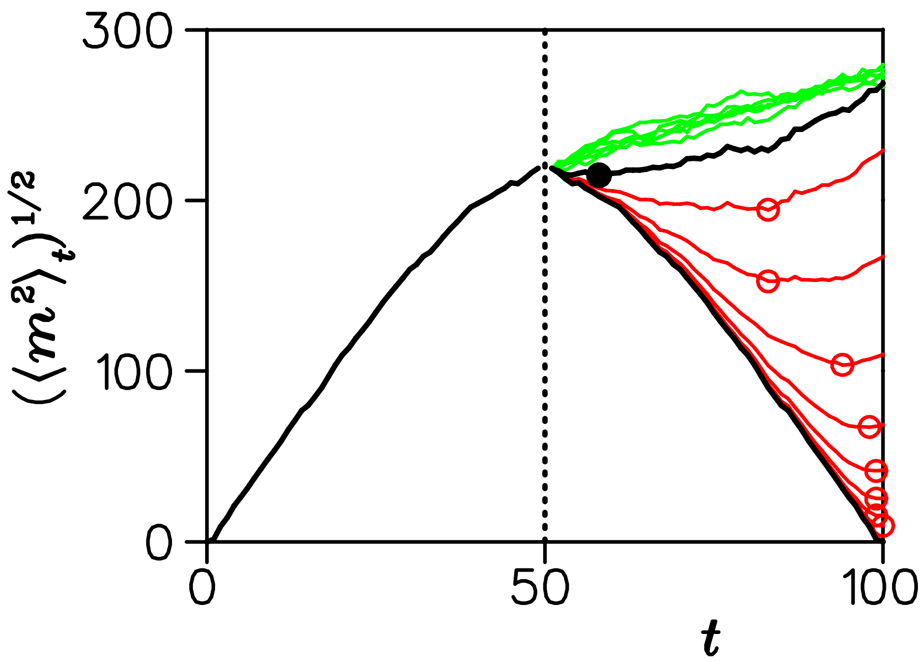
Reversibility border

$$F(\xi;t) \approx 1 - \frac{1}{2}\xi^2 \langle m^2 \rangle_t$$
$$\rightarrow \xi_c(T) \approx \sqrt{2/\langle m^2 \rangle_T}$$

$\xi_c(T)$ drops exponentially with T in the classical case and at most linearly in the quantum case

Clasical diffusion: fidelity drops as a power law, while the number of harmonics grows exponentially



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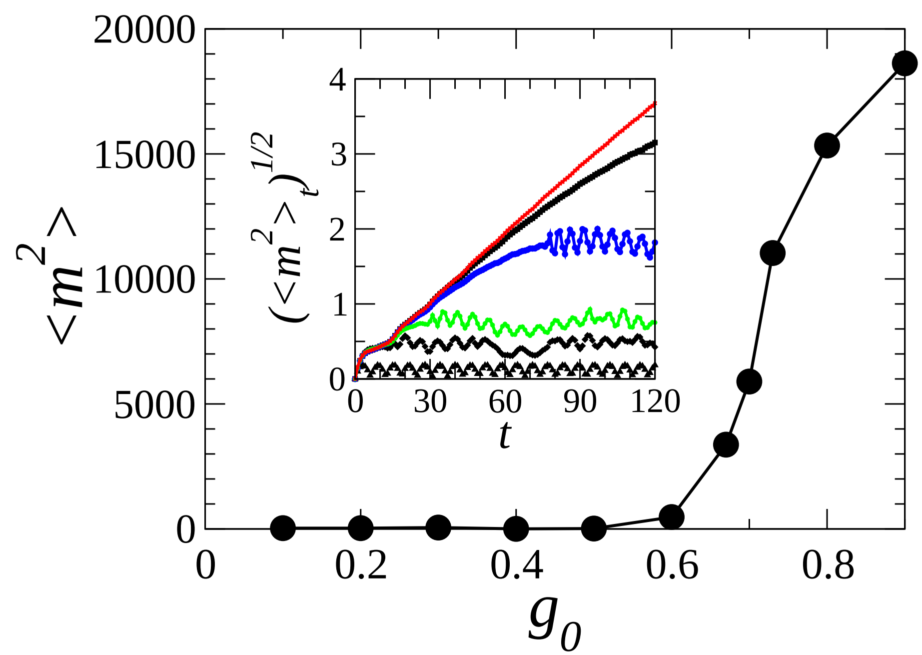
Crossover from integrability to quantum chaos

The number of harmonics may be used to detect, in the time domain, the crossover from integrability to quantum chaos

This proposal differs from previous studies based on on the statistical distribution of energy levels



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Conclusions

• The number of harmonics of the Wigner function is a suitable measure of complexity of a quantum state:
(1) it is directly related to the reversibility properties of quantum motion,
(2) at the classical limit, it reproduces the notion of complexity based on local exponential instability

• In relation to other proposed measures of quantum complexity, such as quantum dynamical entropies, our quantity is very convenient for numerical investigations

• Our phase-space approach can be extended to many-body systems, including qubit systems, whose Hamiltonian can be expressed in terms of bosonic operators

• In many-body systems the (number of) harmonics of the Wigner function could shed some light on the connection between complexity and entanglement and could be used to detect quantum phase transitions