Quantum chaos, decoherence and quantum computation

Giuliano Benenti

Center for Nonlinear and Complex Systems
Università Insubria
Como, Italy

http://scienze-como.uninsubria.it/complexcomo
http://scienze-como.uninsubria.it/benenti

Motivations and purpose of these lectures

Qantum chaos has faced over the years problems of increasing complexity, from simple toy models (like the kicked rotator) to single-particle physics (hydrogen atoms in strong magnetic and/or microwave fields, cold atoms in laser fields, quantum transport in disordered lattices) and to many-body interacting systems such as nuclei, complex atoms, quantum dots, quantum spin glasses and, more recently, quantum computers

A quantum computer represents a complex system of many coupled qubits, which in general can be viewed as a many-body interacting quantum system

Typical problems in the field of quantum chaos, such as the stability of quantum motion, decoherence and the quantum to classical transition, are also essential for any realistic implementation of a quantum computer

Outline

- Remarks on classical and quantum chaos
- Effects of imperfections in the quantum computer hardware
- Quantum noise and quantum trajectories
- Quantum simulation of dissipative chaotic systems (in optical lattices)

Approach: Numerical simulation of noisy many-qubit quantum computations (mainly for the simulation of complex quantum dynamics) and random matrix theory approach

Results: time scales for reliable quantum computation under decoherence and imperfection effects, suggestions to improve the stability of quantum computation, proposal for the simulation of complex dissipative quantum dynamics in optical lattices

Remarks on classical chaos

Classical chaos is characterized by exponential local instability: two nearby trajectories separate exponentially, with rate given by the maximum Lyapunov exponent

$$\lambda = \lim_{|t|\to\infty} \frac{1}{|t|} \ln \frac{d(t)}{d(0)}$$

d length of the tangent vector

Chaotic orbits are unpredictable: in order to predict a new segment of a trajectory one needs additional information proportional to the length of the segment and independent of the previous length of the trajectory. The information associated with a segment of trajectory of length t is equal, asymptotically, to

$$\lim_{|t|\to\infty} \frac{I(t)}{|t|} = h,$$

where h is the so called KS (Kolmogorov-Sinai) entropy which is positive when $\lambda > 0$

Exponential instability $\Longrightarrow$ Continuous (frequency) Fourier spectrum of motion

The power spectrum is the Fourier transform of the autocorrelation function

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt$$

Continuous spectrum $\Longrightarrow$ Decay of correlations (mixing)

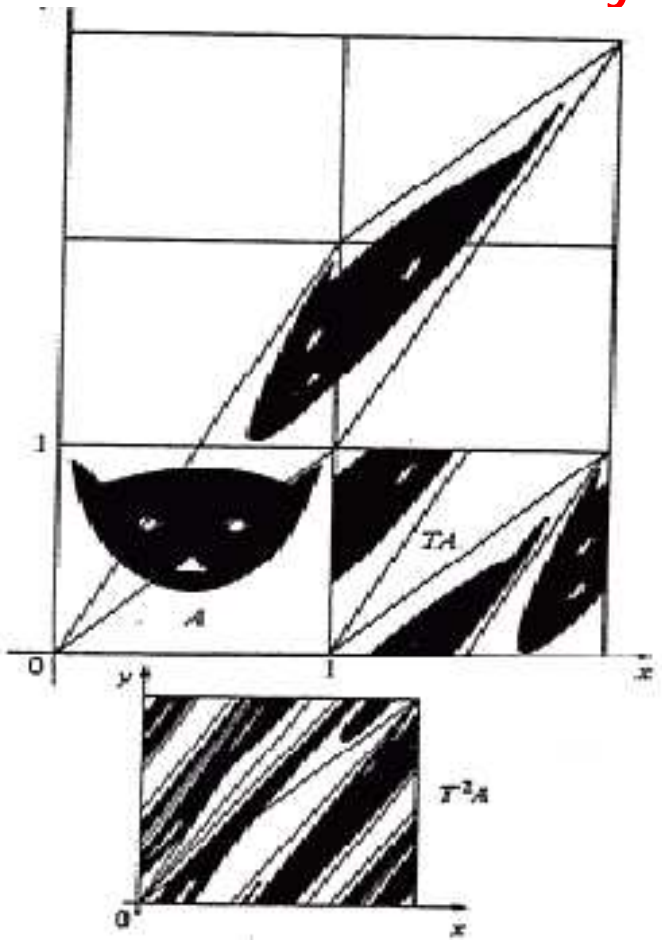
Mixing assures the statistical independence of different parts of a trajectory

Mixing $\Longrightarrow$ Statistical description of chaotic dynamics (diffusion, relaxation, ...)

Exponential instability is not necessary for a meaningful statistical description

Integrable systems $\Longrightarrow$ Nearby points separate only linearly

Loss of memory in the Arnold cat map



$$T : \begin{cases} \bar{x} = x + y \pmod 1, \\ \bar{y} = x + 2y \pmod 1 \end{cases}$$

$$h = \lambda = \ln\left(\frac{3+\sqrt{5}}{2}\right) > 0$$

Stretching and folding of the cat in phase space

Any amount of error rapidly effaces the memory of the initial distribution

Quantum chaos?

The distinction between regular and chaotic motion survives quantization, even though the distinction criteria change

The alternative of exponential or power-law divergence of trajectories disappears in quantum mechanics, Heisenberg's uncertainty principle forbidding the notion of trajectories

Conversely, genuine quantum criteria for chaos like quantum dynamical localization or level spacing statistics as in random matrix theory, have no classical meaning

The essential conditions for classical chaos are violated in quantum mechanics. Indeed the energy and the frequency spectrum of any quantum motion, bounded in phase space, are always DISCRETE. According to the existing theory of dynamical systems such motion corresponds to the limiting case of regular motion

The ultimate origin of this fundamental quantum property is the discreteness of the phase space: the uncertainty principle implies a finite size of an elementary phase space cell

On the other hand the CORRESPONDENCE PRINCIPLE requires the transition from quantum to classical mechanics for all phenomena, including dynamical chaos

Quanstion: How can the correspondence principle be reconciled with a discrete quantum energy spectrum when the limit is to be chaotic and thus characterized by a frequency continuum?

The answer to that question must lie in the existence of $\hbar_{\rm eff}$-dependent TIME SCALES and, equivalently, energy scales. For quantum features to become manifest one must resolve discrete energy levels (whose spacings vanish as $\hbar_{\rm eff} \longrightarrow 0$), that is, sustain observation times which diverge in the limit $\hbar_{\rm eff} \longrightarrow 0$

Time scales of quantum chaos in the kicked rotator model

Let us consider a rotator with angular momentum I and angle variable θ driven by a series of periodic pulses (the kicks)

$$H(I,\theta,\tau) = \frac{I^2}{2} + k\cos\theta \sum_{m=-\infty}^{+\infty} \delta(\tau - mT)$$

$$\text{CLASSICAL MAP} \begin{cases} \overline{I} = I + k\sin\theta, \\ \\ \overline{\theta} = \theta + T\overline{I} \quad (0 \le \theta < 2\pi) \end{cases}$$

$$\text{After the rescaling } I \to p = IT : \begin{cases} \overline{p} = p + K\sin\theta, \\ \\ \overline{\theta} = \theta + \overline{p} \end{cases}$$

Classical dynamics depends only on the parameter $K = kT$

For $K \gg 1$ the classical motion is ergodic, mixing and exponentially unstable with Lyapunov exponent $\lambda \approx \ln(K/2)$, apart small stability islands

The rescaled action variable $p = TI$ displays a random walk type of motion and exhibits normal diffusion:

$$\langle (\Delta p)^2 \rangle = \langle (p - \langle p \rangle)^2 \rangle \approx D_p t$$

$t = \tau/T$ time in units of map iterations, $D_p \approx K^2/2$ diffusion coefficient

QUANTUM MAP:

$$\overline{\psi} = \hat{U}\psi = e^{ik\cos\hat{\theta}} e^{-iT\hat{I}^2/2}\psi$$

$(\hat{I} = -i\partial/\partial\theta,\ \hbar = 1)$

Classical limit: $k \to \infty$, $T \to 0$, $K = kT =$ const (indeed $[\theta, p] = T[\theta, I] = iT = i\hbar_{\rm eff}$)

The Ehrenfest time scale

Ehrenfest theorem: a quantum packet follows a beam of classical orbits as long as the packet remains narrow; during this time interval the quantum wave-packet motion is exponentially unstable and random as the underlying classical trajectory

The initial size of the quantum packet is bounded from below by the elementary quantum phase space cell of order $\hbar$

Let us start from an initial minimum-uncertainty wave packet of size $\Delta\theta_0 \Delta p_0 = T \Delta\theta_0 \delta I_0 \sim T\hbar = \hbar_{\rm eff}$, with $\Delta\theta_0 \sim \Delta p_0 \sim \sqrt{\hbar_{\rm eff}}$ (least-spreading wave packet)

$\Delta\theta$ grows exponentially due to classical exponential instability:

$$\Delta\theta \sim \Delta\theta_0 \exp(\lambda t) \sim \sqrt{\hbar_{\rm eff}} \exp(\lambda t)$$

λ maximum Lyapunov exponent of the system

Complete spreading over the angle variable θ is obtained after a the so-called Ehrenfest time scale [Berman and Zaslavsky, 1978]

$$t_E \sim \frac{1}{\lambda} |\ln \hbar_{\rm eff}|$$

True dynamical chaos characterized by exponential instability is limited in quantum mechanics, for Hamiltonian systems, to the logarithmically short (in $\hbar_{\rm eff}$) Ehrenfest time scale
t_E increases indefinitely as $\hbar_{\rm eff} \longrightarrow 0$, in agreement with the correspondence principle

Cassical-like diffusion is possible, in the absence of exponential instability, also at times $t > t_E$

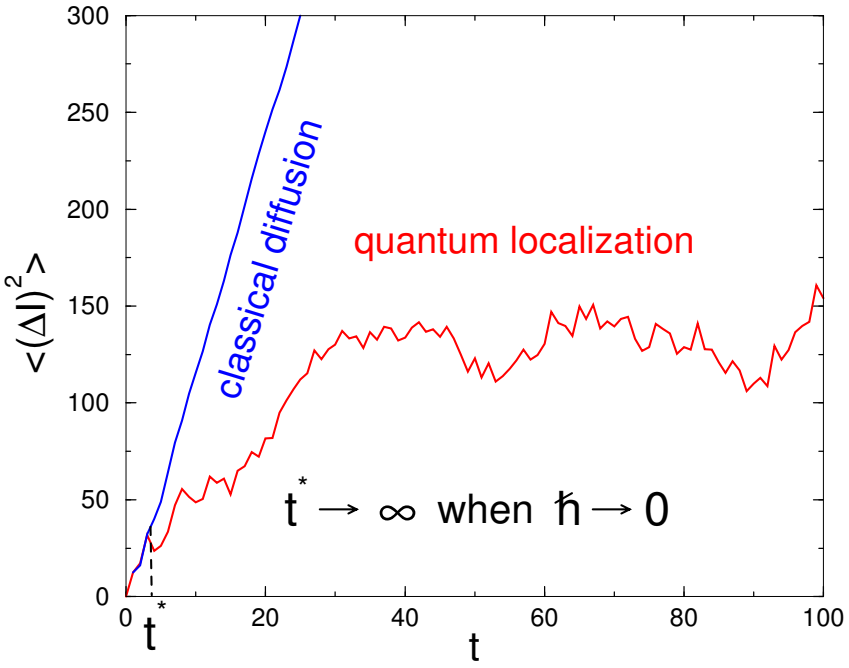
Quantum dynamical localization

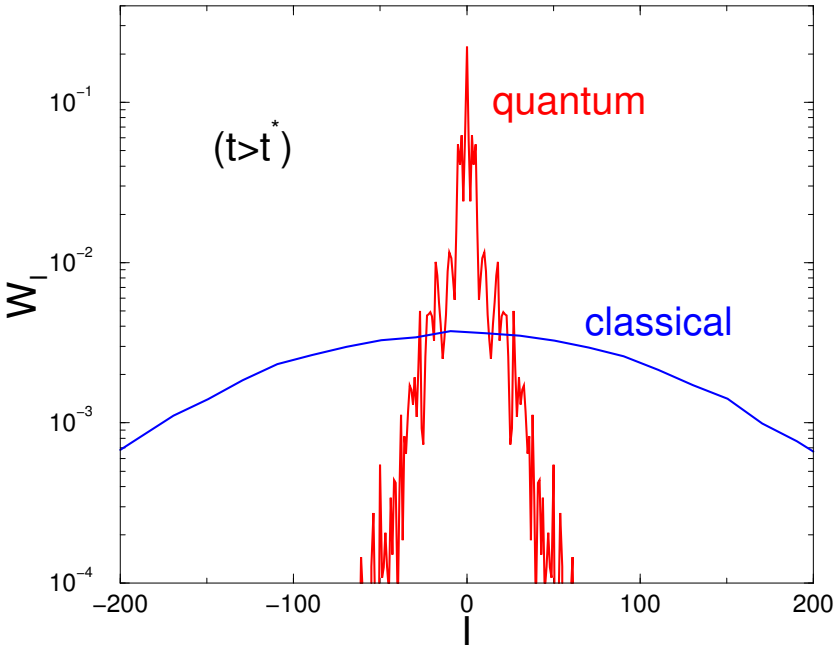
The second time scale $t^\star$, at which the quantum evolution breaks away from the classical diffusion, is related to the phenomenon of quantum dynamical localization

For $t > t^\star$, while the classical distribution goes on diffusing, the quantum distribution reaches a steady state which decays exponentially over the momentum eigenbasis

$$W_I \equiv |\langle I|\psi\rangle|^2 \approx \frac{1}{\ell} \exp\left(-\frac{2|I - I_0|}{\ell}\right)$$

I_0 initial value of the momentum
ℓ localization length, gives the width of the localized distribution


300
250
200
150
100
50
0
0
20
40
60
80
100
t
t*
<(Δl)²>
classical diffusion
quantum localization
t* → ∞ when ħ → 0


10⁻¹
10⁻²
10⁻³
10⁻⁴
W_l
(t>t*)
quantum
classical
−200
−100
0
100
200
l

Estimate $t^\star$ and ℓ (Siberian argument)

Since the number of involved levels grows $\propto \sqrt{t}$ and the discreteness of levels is resolved down to an energy spacing $\propto 1/t$, then the discreteness of spectrum eventually dominates

The localized distribution projects over ℓ eigenstates of the Floquet operator (one-period unitary evolution operator)

The mean level spacing of "significant" quasienergy eigenstates is $\Delta E \approx 2\pi/\ell$

The Heisenberg principle tells us that the minimum time (the break time $t^\star$) required to the dynamics to resolve this energy spacing is given by

$$t^\star \approx 1/\Delta E \approx \ell$$

Diffusion up to time $t^\star$ involves a number of levels given by

$$\sqrt{\langle (\Delta I)^2 \rangle} \approx \sqrt{D_I t^\star} \approx \ell$$

$D_I \approx k^2/2 \sim 1/\hbar_{\text{eff}}^2$ classical diffusion coefficient, measured in number of levels

Therefore $t^\star \approx \ell \approx D_I$

$$t^\star \sim \frac{1}{\hbar_{\text{eff}}^2} \gg t_E$$

diverges as $\hbar_{\text{eff}} \longrightarrow 0$, in accordance with the correspondence principle

Quantum dynamical localization has profound analogies with Anderson localization of electronic transport in disordered materials

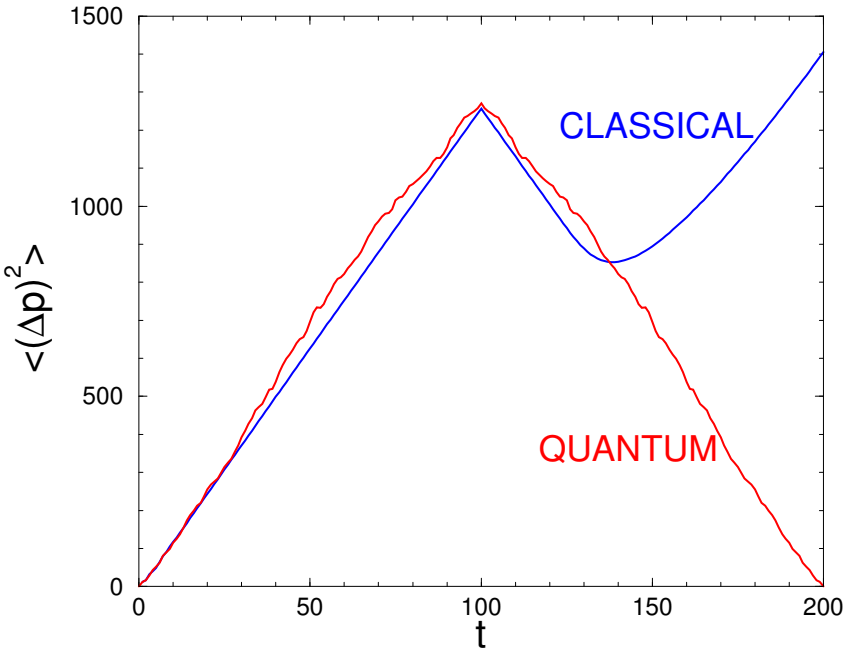
Differences: dynamical localization (i) takes place in angular-momentum space instead of coordinate space and (ii) no disorder is introduced

Dynamical localization has been observed experimentally in the microwave ionization of Rydberg atoms and is now actively studied in experiments with cold atoms in laser fields

Dynamical stability of quantum motion

Quantum evolution is very stable, in contrast to the extreme sensitivity to initial conditions and rapid loss of memory of classical chaos

Numerical simulation of velocity reversal



Practical irreversibility of classical motion as chaos magnifies computer round-off errors

The time to amplify the perturbation and significantly modify the trajectories is

$$t_\epsilon \approx \frac{1}{\lambda} |\ln \epsilon|$$

For round-off errors of order $\epsilon \sim 10^{-14}$ and the kicked rotator model at $K = 5$, $\lambda \approx \ln(K/2)$, we have $t_\epsilon \sim 35$

In the quantum case almost exact reversion is observed in numerical simulations

Therefore, quantum dynamics, even though it is diffusive, lacks dynamical instability

The physical reason of this striking difference between quantum and classical motion is rooted in the discreteness of the quantum spectrum
The reversibility of quantum motion in numerical simulations is due to the fact that computer round-off errors act on a scale much smaller than the size of the Planck's cell

The classical motion (governed by the Liuoville equation) of some phase space density, explores smaller and smaller scales exponentially fast. Correspondingly there is an exponential growth in the number of the density's Fourier harmonics that are excited. This process is limited in quantum mechanics to the size of the Planck's cell

The lack of exponential instability in quantum mechanics is relevant for the prospects of any practical implementation of quantum computation, which has to face errors due to imperfections and decoherence

The quantum Loschmidt echo

A better understanding of the stability of quantum motion can be obtained if we examine the fidelity of quantum motion, also known as the quantum Loschmidt echo

Fidelity (Loschmidt echo): Definition
Consider unitary quantum evolution

$$|\psi(t)\rangle = U^t|\psi_0\rangle$$

- $U^t = \exp(-iHt/\hbar)$ for continuous time dynamics
- $U = \text{Floquet map}$ for discrete time dynamics

Introduce a small perturbation of the Hamiltonian

$$U_\epsilon^t = \exp(-i(H + \epsilon V)t/\hbar)$$

and consider the perturbed evolution of the same initial state

$$|\psi_\epsilon(t)\rangle = U_\epsilon^t |\psi_0\rangle$$

Define fidelity as

$$f(t) = |\langle\psi_\epsilon(t)|\psi(t)\rangle|^2 = |\langle\psi_0|m_\epsilon(t)|\psi_0\rangle|^2$$

Fidelity is the expectation value of the unitary echo operator

$$m_\epsilon(t) = U_\epsilon^{-t} U^t$$

Motivations for the study of fidelity

- Quantum chaos in time domain: Stability of quantum time evolution under perturbation of the Hamiltonian

- Fingerprints of Lyapunov exponents in quantum dynamics?

- Quantum echoes: Dynamical origin of macroscopic irreversibility

- Quantum computing: Stability of quantum algorithms and reliability of quantum state manipulation

- Decoherence: Understanding decoherence with deterministic models of environment

Fidelity decay in chaotic systems

Consider for instance the sawtooth map model with N levels on the torus $-\pi \leq p < \pi$ and perturb the kicking strength: $k \to k' = k + \sigma$, corresponding to a classical perturbation $K = kT \to K' = K + \epsilon$, with $\epsilon \equiv \sigma T$

The transition matrix elements $V_{jk} = \langle u_j | V | u_k \rangle$ of the perturbation operator V between the eigenstates $\{|u_j\rangle\}$ of the Floquet operator U are of the order of

$$V_{\rm typ} \sim \frac{\sigma}{\sqrt{N}}$$

This estimate is obtained assuming that the quasienergy eigenstates $|u_j\rangle$ are ergodic: if we expand the eigenfunction over, for instance, the momentum basis, $|u_j\rangle = \sum_I c_I |I\rangle$, then the coefficients c_I have (pseudo-)random phases and, to assure the normalization of the wave function, amplitudes $\sim 1/\sqrt{N}$

$$V_{jk} = \sum_{II'} c_I c^{\star}_{I'} \langle I'|V|I\rangle$$

The matrix elements $\langle I'|V|I\rangle$ have a value $\propto \sigma$ significantly different from zero for $|I - I'| < b$ and can be neglected, as a first approximation, outside this band

V_{jk} is given by the sum of $O(bN)$ terms of amplitude σ/N and random signs. This leads to the estimate of the typical value $V_{\rm typ} \sim \sigma/\sqrt{N}$

The typical coupling strength $V_{\rm typ}$ has to be compared with the typical energy separation between the unperturbed quasienergies,

$$\Delta E \sim \frac{1}{N}$$

Regimes for the fidelity decay

1) Perturbative regime $\sigma < \sigma_p \approx 1/\sqrt{N}$ $(V_{\rm typ} < \Delta E)$
The fidelity decay can be calculated in perturbation theory:

$$f(t) \sim \exp(-V_{\rm typ}^2 t^2)$$

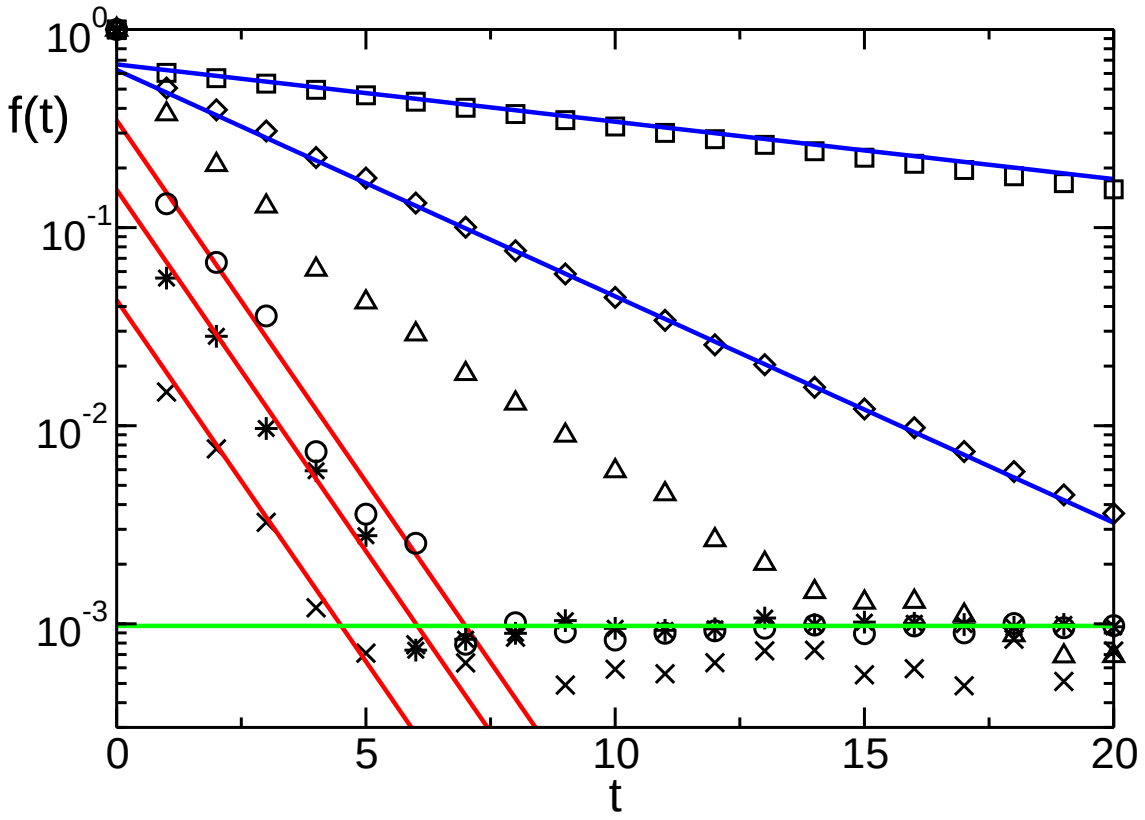
2) Fermi golden rule regime $\sigma_p < \sigma < 1$

$$f(t) \sim \exp(-\Gamma t)$$

rate $\Gamma \sim V^2/\Delta E \sim \sigma^2$ given by the Fermi golden rule

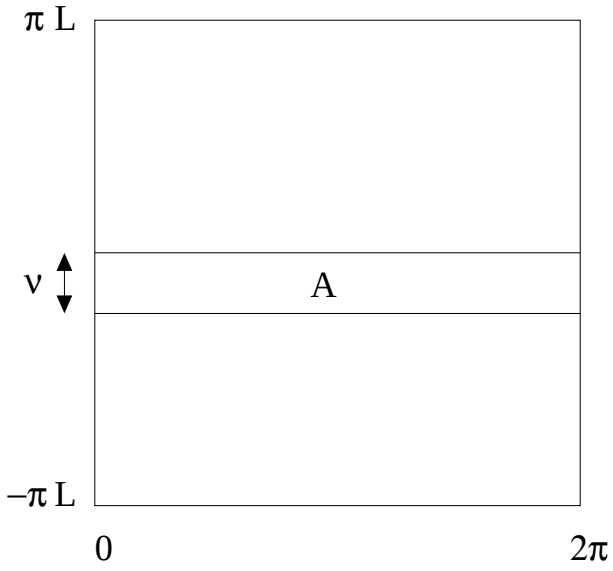
3) Semiclassical regime $\sigma > \sigma_c \sim 1$
The fidelity decay is again exponential, but with a perturbation independent rate $\Gamma = \lambda$, where λ is the Lyapunov exponent of classical chaotic dynamics

Fermi golden rule to Lyapunov crossover



Classical fidelity

Evolve an area A forward (unperturbed) for t steps and backward (perturbed) for other t steps and get A'. The overlap of A and A' gives the fidelity $f_c(t)$. In practice, follow many trajectories and count how many return back inside A.



Formal definition of classical fidelity:

$$f_c(t) = \frac{\int dqdp\rho_\epsilon(q,p,t)\rho(q,p,t)}{\sqrt{\int dqdp\rho_\epsilon^2(q,p,t)}\sqrt{\int dqdp\rho^2(q,p,t)}}$$

Note that f_c is the classical limit of the quantum fidelity for mixed states:

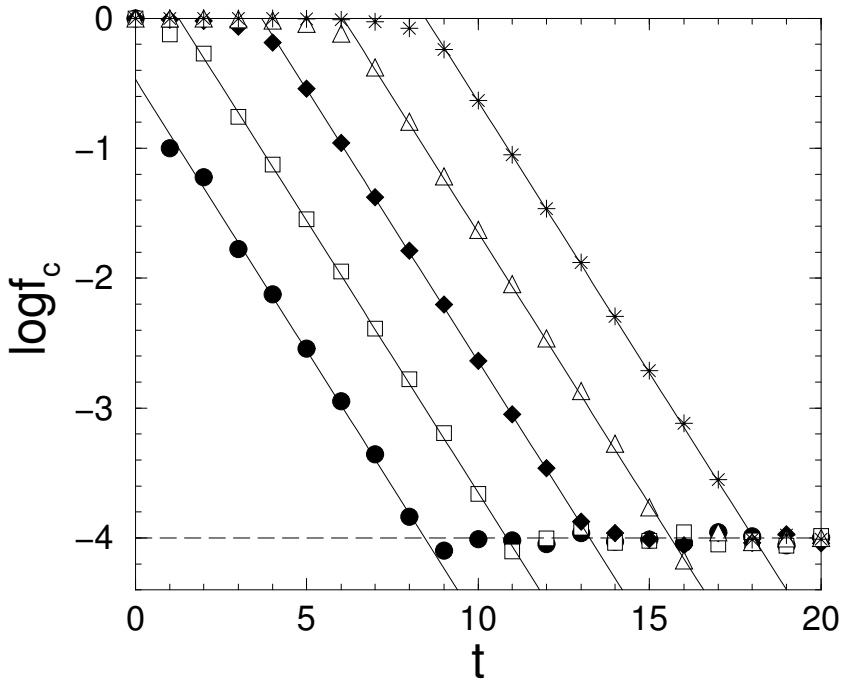
$$f(t) = \frac{\mathrm{Tr}[\rho_0(t)\rho_\epsilon(t)]}{\mathrm{Tr}(\rho_0^2)}$$

Lyapunov decay of classical fidelity

The decay rate is PERTURBATION INDEPENDENT and sets in after a transient time

$$t_\epsilon \approx \frac{1}{\lambda} \ln\left(\frac{\nu}{\epsilon}\right)$$

required to amplify the perturbation up to the scale ν of the initial distribution



Stability of numerical simulations

Round-off errors are always so small that $\sigma \ll 1$

If we model round-off errors as memoryless errors of size ϵ, then the quantum fidelity decays as $f(t) \sim \exp(-\sigma^2 t)$. For $\sigma \sim 10^{-15}$, then simulation is stable up to an enormously long time scale

$$t_f^{(q)} \sim \frac{1}{\sigma^2}$$

Classical fidelity decays after a time

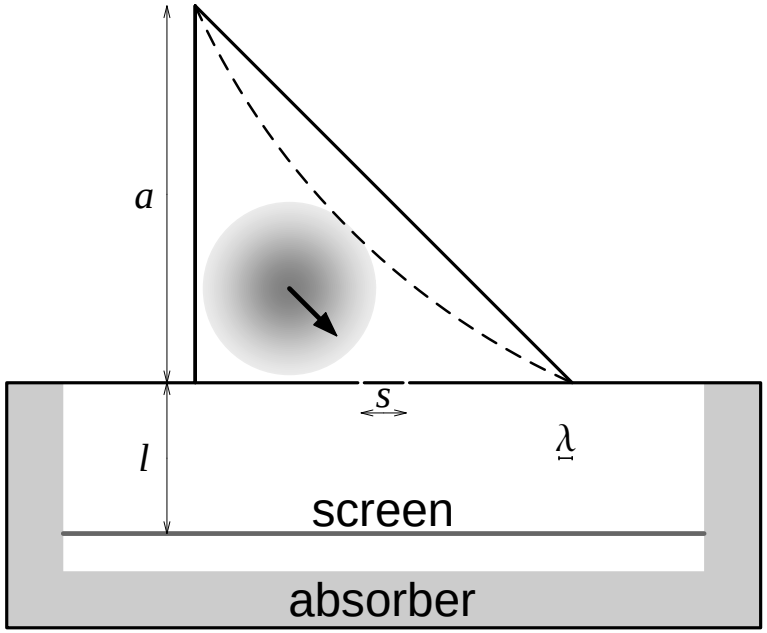
$$t_f^{(c)} \sim \frac{1}{\lambda} + t_\epsilon \sim \frac{1}{\lambda}\left[1 + \ln\left(\frac{\nu}{\epsilon}\right)\right]$$

Due to exponential instability, in this case the scaling is only logarithmic in ϵ. Thus the classical motion is in practice irreversible after the logarithmically short time scale $t_f^{(c)}$

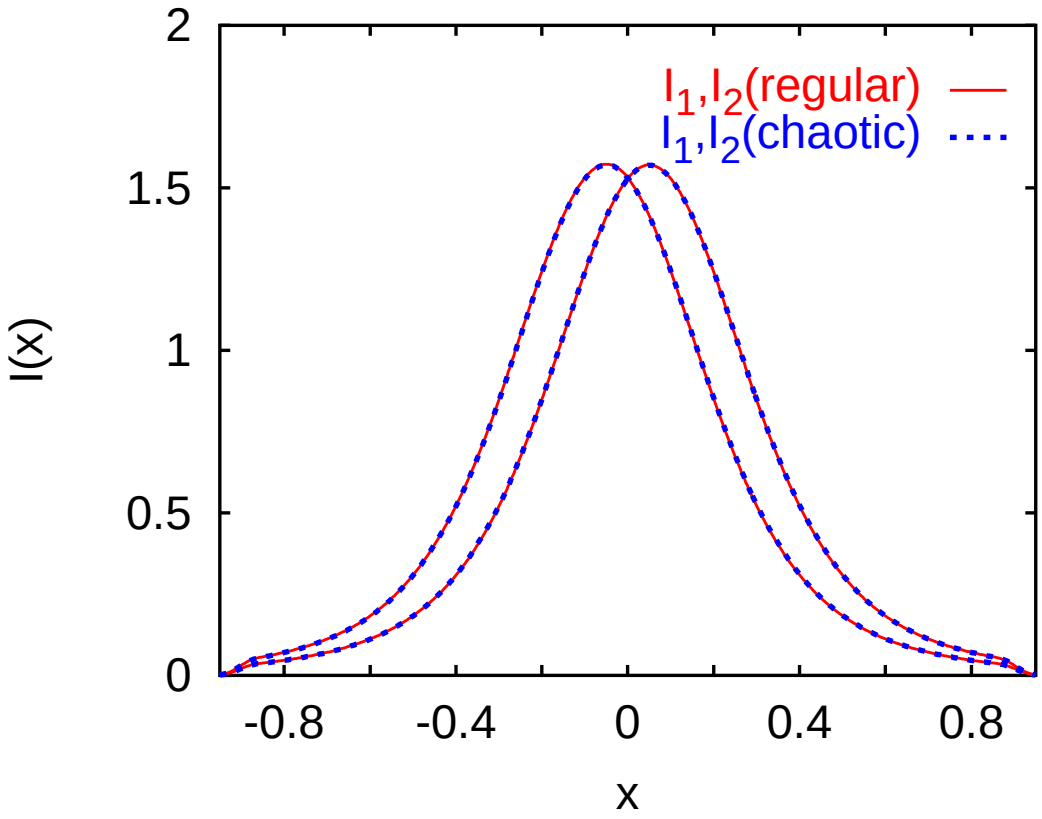
Note that by itself an exponential decay of the fidelity does not imply exponential instability: This is not the case in the Fermi golden rule regime

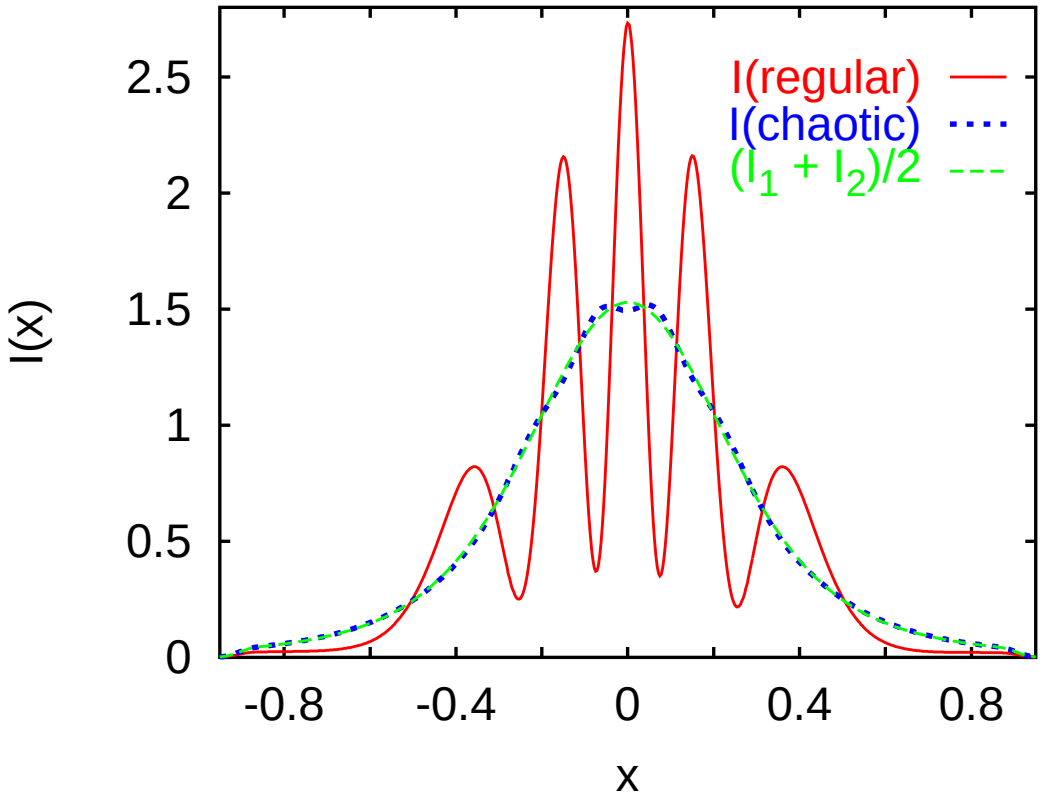
Chaos and decoherence

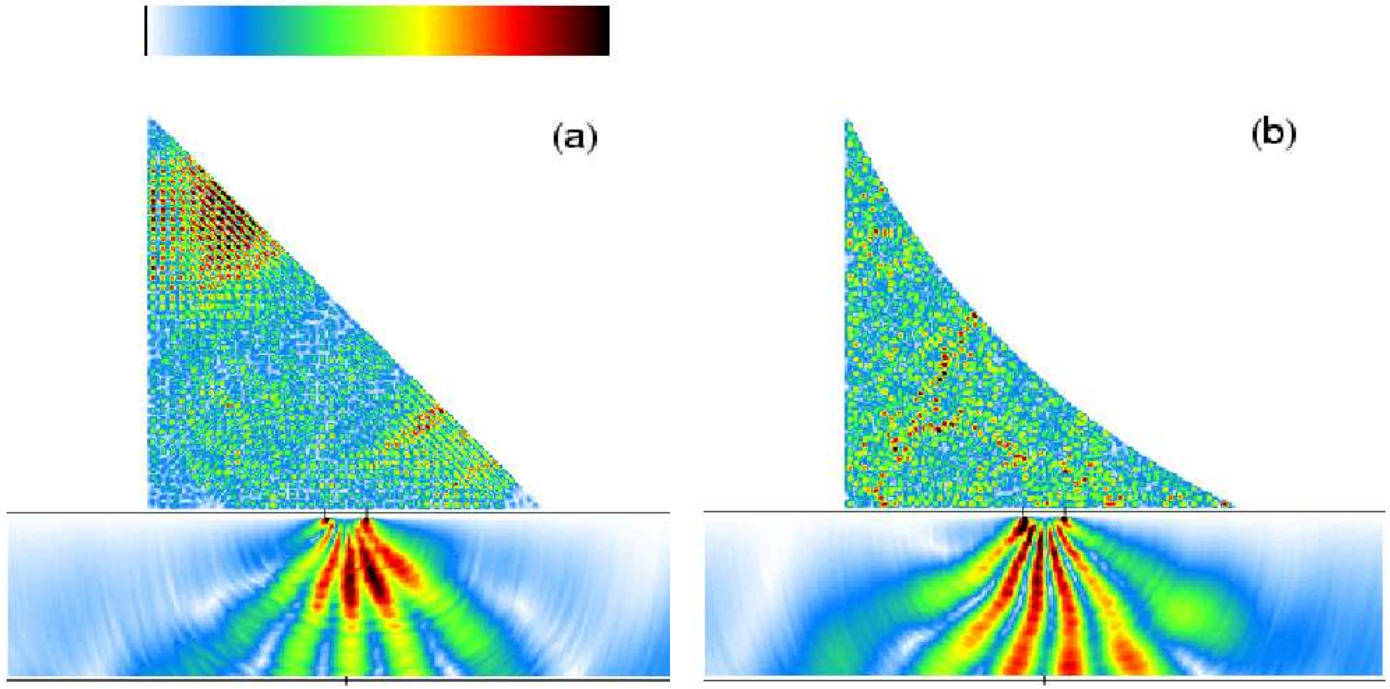
According to Feynman, the double-slit experiment *"... is impossible absolutely impossible, to explain in any classical way, and has in it the heart of quantum mechanics. In reality, it contains the only mystery."*



(Casati and Prosen, 2004)


2
1.5
1
0.5
0
I(x)
-0.8
-0.4
0
0.4
0.8
x
I1,I2(regular)
I1,I2(chaotic)


I(regular)
I(chaotic)
(I1 + I2)/2
I(x)
x
2.5
2
1.5
1
0.5
0
-0.8
-0.4
0
0.4
0.8



"Decoherence" due to tracing over time instead that over environmental freedoms

Limits to quantum computation due to chaos and decoherence

Accuracy of quantum computation measured by fidelity: $f(t) = \langle\psi(t)|\rho_\epsilon(t)|\psi(t)\rangle$

Quantum algorithm: $|\psi_f\rangle = U\,|\psi(0)\rangle, \qquad U = \underbrace{U_{N_g}\cdot\ldots\cdot U_1}_{\text{elementary gates}}$

Errors: $U_j \;\longrightarrow\; e^{-i\epsilon V(j)}U_j$

(i) Memoryless unitary errors:

V random and different at each j,

e.g.: random phase fluctuations: $\delta\phi \in [-\epsilon, \epsilon]$ in phase-shift gates

(ii) Static imperfections in the quantum computer itself:

V (random but) constant at each j,

e.g.:

$$V = H_s\tau_g = \sum_{j=0}^{n-1}\delta_j\,\sigma_j^{(z)} + 2\sum_{j=0}^{n-2}J_j\,\sigma_j^{(x)}\,\sigma_{j+1}^{(x)} \quad , \quad J_j, \delta_j \in [-\epsilon, \epsilon]$$

(iii) Non-unitary errors in quantum computation:

$e^{-i\epsilon V}$ is non-unitary ($V \neq V^\dagger$, density matrix and quantum trajectories approach)

Effects of imperfections in the quantum computer hardware

- Objectives: determine time scales for reliable quantum computation in the presence of quantum computer hardware imperfections, device strategies to improve the stability of quantum computation

- Approaches: numerical simulations and random matrix theory

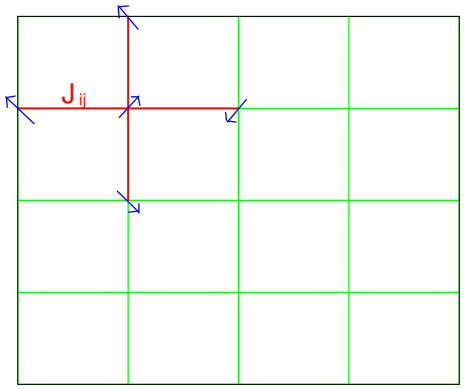
- Main test bed: quantum algorithms simulating complex dynamics

- Main results: simulated quantum protocols with more than 25 qubits, found time scales for reliable quantum computation under realistic imperfection models, suggested strategies to reduce the impact of static imperfections

A model of quantum computer hardware

Even if the quantum computer is (ideally) decoupled from the environment, internal imperfections can disturb quantum computation

$$H_s = \sum_{i=1}^{n} \Gamma_i \sigma_i^z + \sum_{\langle i<j \rangle} J_{ij} \sigma_i^x \sigma_j^x$$



(Georgeot, Shepelyansky, 2000)

$\Gamma_i = \Delta_0 + \delta_i$

$\delta_i \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$ energy fluctuations

$J_{ij} \in [-J, J]$ residual short-range interaction

The non-interacting eigenstates ($J = 0$) are the quantum register states used for computation

H_s quantum computer hardware

$H_s \rightarrow H(\tau) = H_s + H_g(\tau) = H_s + \sum_k \delta(\tau - k\tau_g) h_k$ software (gate operations added)

A testing ground: the quantum sawtooth map model

Kicked Hamiltonian $H(I,\theta,\tau) = \frac{I^2}{2} - \frac{k}{2}(\theta-\pi)^2 \sum_{m=-\infty}^{+\infty} \delta(\tau - mT)$

CLASSICAL SAWTOOTH MAP
$$\begin{cases} \overline{I} = I + k(\theta - \pi), \\ \\ \overline{\theta} = \theta + T\overline{I} \quad (0 \le \theta < 2\pi) \end{cases}$$

After the rescaling $I \to p = IT$:
$$\begin{cases} \overline{p} = p + K(\theta - \pi), \\ \\ \overline{\theta} = \theta + \overline{p} \end{cases}$$

Classical dynamics depens only on $K = kT$

* $-4 < K < 0$ stable motion
* $K < -4$ and $K > 0$ chaotic dynamics

KAM theorem does not apply, for any $K \neq 0$ the motion is not bounded by invariant KAM tori:

* $K > 0$ diffusive motion, $< (\Delta p)^2 > \approx D(K)t$:
(i) $K > 1$ random phase approximation, $D(K) \propto K^2$
(ii) $0 < K < 1$ cantori diffusion, $D(K) \propto K^{5/2}$

* $-4 < K < 0$ anomalous diffusion, $< (\Delta p)^2 > \approx t^\alpha$ ($\alpha = 0.57$ when $K = -0.1$)

* $K = -1, -2, -3$ integrable regime

QUANTUM SAWTOOTH MAP:

$$\overline{\psi} = \hat{U}\psi = e^{ik(\hat{\theta}-\pi)^2/2} e^{-iT\hat{I}^2/2}\psi, \quad (\hat{I} = -i\partial/\partial\theta)$$

Classical limit: $k \to \infty$, $T \to 0$, $K = kT =$ const ($[\theta, p] = T[\theta, I] = iT = i\hbar_{\rm eff}$)

Quantum localization effects: dynamical localization and cantori localization

One-period unitary evolution operator $\hat{U} = \hat{U}_k \hat{U}_T$:
* $\hat{U}_k = \exp(ik(\hat{\theta} - \pi)^2/2)$ (kick)
diagonal in the θ- representation
* $\hat{U}_T = \exp(-iT\hat{I}^2/2)$ (free rotation)
diagonal in the I- representation

On a classical computer, the time evolution is simulated via forward/backward Fast Fourier Transforms, in $O(N \log N)$ operations (N number of levels)

Quantum algorithm for the sawtooth map

(i) Free rotation $|\psi\rangle = \sum_{I=0}^{N-1} a_I |I\rangle \rightarrow |\psi\rangle = \hat{U}_T |\psi\rangle = \sum_I a_I \exp(-iTI^2/2)|I\rangle$
$I = \sum_{j=0}^{n-1} \alpha_j 2^j$ (binary code, $\alpha_j = 0, 1$, $n = \log_2 N$ number of qubits)
$\exp(-iTI^2/2) = \prod_{j_1,j_2} \exp(-iT\alpha_{j_1}\alpha_{j_2} 2^{j_1+j_2-1})$

This step can be performed in n^2 controlled-phase shift gates:

$$\begin{cases} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |10\rangle \\ |11\rangle \rightarrow \exp(-iT2^{j_1+j_2-1})|11\rangle \end{cases}$$

(ii) Quantum Fourier Transform
QFT can be performed in n Hadamard gates and $n(n-1)/2$ controlled phase-shift gates

(iii) Kick This step is similar to (i) since now $|\psi\rangle$ is given in the θ representation, where $\hat{U}_k = \exp(ik(\hat{\theta} - \pi)^2/2)$ is diagonal and for the sawtooth map analogous to $\hat{U}_T$.

(iv) Inverse QFT $\rightarrow$ back to the momentum basis

Advantages of this quantum algorithm

- The simulation of time evolution is exponentially faster than classical computation: it requires $n_g = O(n^2 = (\log_2 N)^2)$ quantum gates per map iteration instead of $O(N \log_2 N)$ elementary operations

- Optimum use of qubits: no extra work space qubits

- Complex dynamics can be simulated with less than 10 qubits (less than 40 qubits would be sufficient to make simulations inaccesible to present-day supercomputers)

We can reach exponentially fast with the number of qubits two distinct limits:

- Classical limit:

$$T = \frac{2\pi L}{N} = \frac{2\pi L}{2^n}, \quad K = \text{const}$$

The number of levels inside the interval $-\pi L \leq p < \pi L$ grows exponentially
The effective Planck constant $\hbar_{\text{eff}} = T \sim 1/N = 1/2^n \to 0$ when $N \to \infty$

- Thermodynamic limit:

$$k, K \ \text{ constant}, \quad L = \frac{TN}{2\pi} = \frac{T2^n}{2\pi}$$

The thermodynamic limit corresponds to the system size (number of cells) $L \to \infty$
The effective Planck constant is fixed

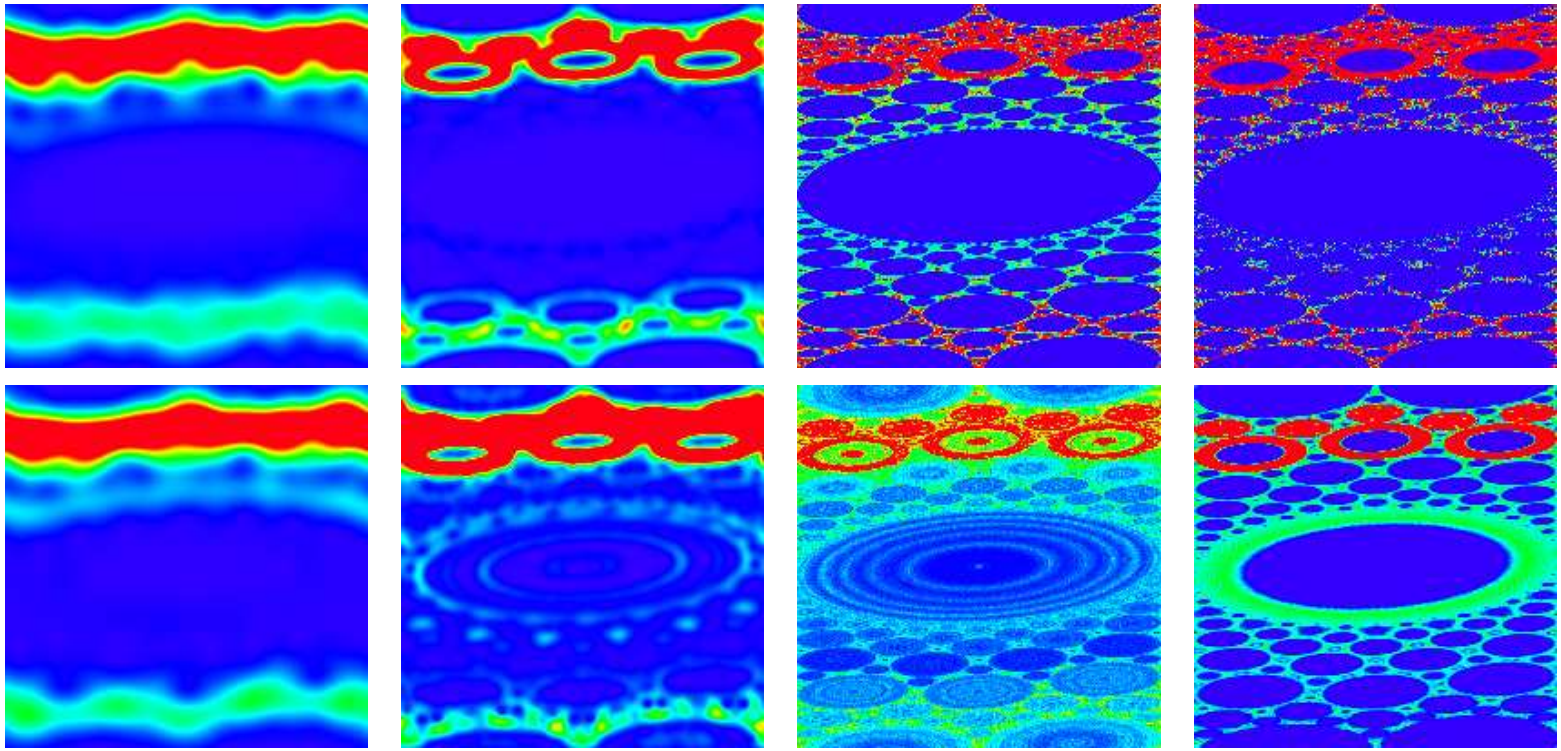
Study of static imperfections

Numerical study of the effect of static imperfections for a many-body/qubit quantum computer simulating the quantum algorithm for the sawtooth map

We assume that:
1) The quantum computer is decoupled from the environment
2) Short-range, instantaneous and perfect one- and two-qubit gates separated by a time interval τ_g
3) The hardware Hamiltonian contains static imperfections, giving unwanted phase rotations and qubit couplings

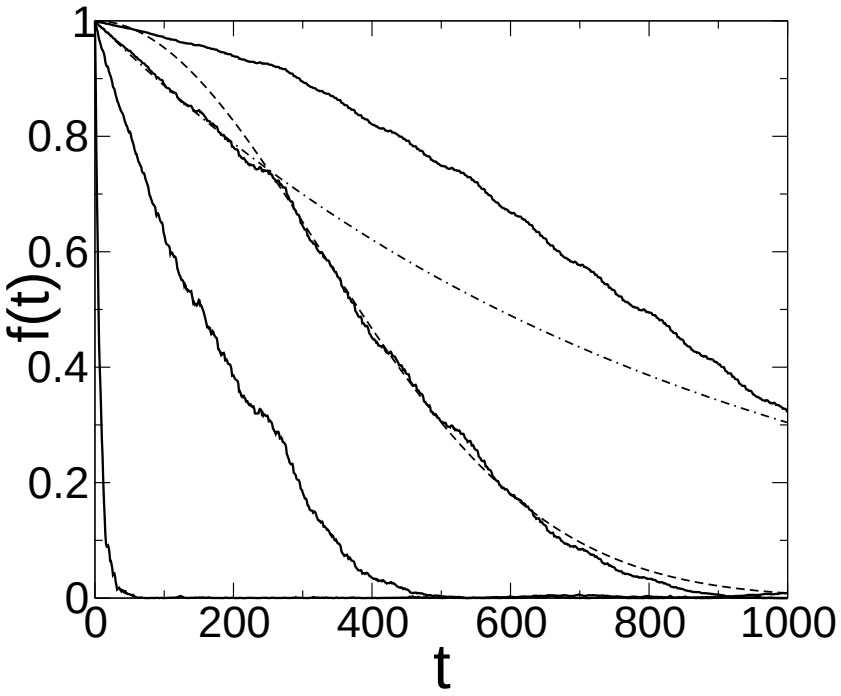
$$H_s \to H(\tau) = H_s + H_g(\tau) = H_s + \sum_k \delta(\tau - k\tau_g) h_k$$

Quantum versus classical errors



- Quantum errors are "nonlocal" in phase space: they can induce direct transfer of probability on a large distance in phase space: injection of quantum probabilities inside integrable islands
- Classical round-off errors: slow diffusive spreading inside integrable islands

Static imperfections vs. noisy gates



Static imperfections: from exponential fidelity decay (Fermi golden rule, $f(t) \approx \exp(-A\epsilon^2 t)$) to Gaussian fidelity decay ($f(t) \approx \exp(-B\epsilon^2 t^2)$)

The exponential to Gaussian crossover takes place at the Heisenberg time $t_H \sim N$, given by the inverse mean level spacing: Before t_H the system does not resolve the discreteness of the spectrum. Therefore, the density of states can be treated as a continuous and the Fermi golden rule can be applied.

"Noisy gates": modeled by

$$H = \sum_i (\Delta_0 + \delta_i(t))\sigma_i^z + \sum_{<i,j>} J_{ij}(t)\sigma_i^x \sigma_j^x$$

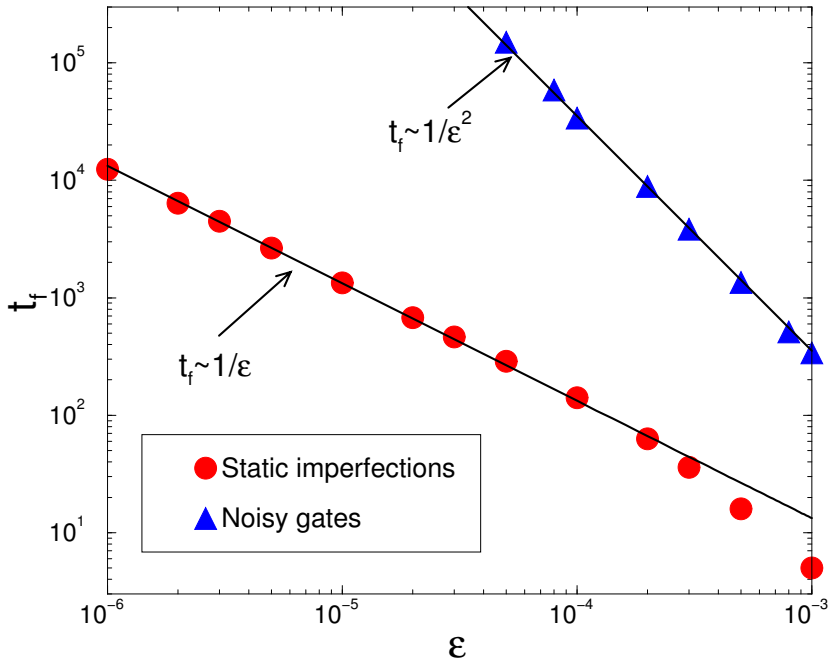
with δ_i and J_{ij} fluctuating without memory from gate to gate

In the case of noisy gates the fidelity decay is exponential. This corresponds to the Fermi golden rule regime, where at each gate operation a probability of order ϵ^2 is transferred from the ideal state to other states. Since there are no correlations between consecutive noisy gates, the population of the ideal noiseless state decays exponentially. We can write

$$f(t) \approx \exp(-C\epsilon^2 N_g)$$

where $N_g = n_g t$ is the total nomber of gates required to evolve t steps of the sawtooth map

Fidelity time scale



t_f obtained from the condition $f(t_f) = 0.9$

The static imperfections give shorter time scales t_f and therefore can be considered more dangerous for quantum computation

Random matrix theory approach

Assuming that the Floquet operator U can be modeled by a random matrix one obtains [Frahm et al., EPJD **29**, 139 (2004)]

$$-\ln f(t) \approx \frac{t}{t_c} + \frac{t^2}{t_c t_H}$$

$t_c \approx 1/(\epsilon^2 n n_g^2)$ characterizes the effective strength of the perturbation
$t_H = N = 2^n$ Heisenberg time

This relation is valid as long as ϵ and t are sufficiently small so that $1 - F \ll 1$

Perturbative regime for $\epsilon < \epsilon_c \approx 1/(\sqrt{n2^n}n_g)$, that is, for $t_c > t_H$, the fidelity decay is dominated by the quadratic term in the above expression: The decay essentially takes place after the Heisenberg time and is Gaussian, $f(t) \approx \exp(-t^2/t_c t_H)$

Fidelity time scale $t_f \sim \sqrt{t_c t_H} \approx \frac{\sqrt{2^n}}{\epsilon\sqrt{n}n_g}$

Number of gates inside this time $(N_g)_f = t_f n_g \sim \frac{\sqrt{2^n}}{\epsilon\sqrt{n}}$

Quantum chaos regime for $\epsilon > \epsilon_c \approx 1/(\sqrt{n2^n}n_g)$, that is, for $t_c < t_H$, the fidelity decay is dominated by the linear term in the above expression: The decay is exponential, $f(t) \approx \exp(-t/t_c)$, and occurs before the Heisenberg time

Fidelity time scale $t_f \sim t_c \approx \frac{1}{\epsilon^2 n n_g^2}$

Number of gates inside this time $(N_g)_f = t_f n_g \sim \frac{1}{\epsilon^2 n n_g}$

The threshold ϵ_c is the chaos border above which static imperfections mix the Floquet eigenstates

How to reduce the impact of static imperfections?

Since random imperfections changing from gate to gate always lead to an exponential decay of the fidelity, it is tempting to try to randomize the static imperfections to slow down the fidelity decay from Gaussian to exponential
This idea can be formalized if one observes that the fidelity can be expressed in terms of a correlation function of the perturbation [Prosen and Žnidarič, J. Phis. A **35**, 1455 (2002)]

$U = U(T)U(T-1)\cdots U(1)$ sequence of ideal quantum gates
$U_\epsilon = e^{-i\epsilon V(T)}U(T)e^{-i\epsilon V(T-1)}U(T-1)\cdots e^{-i\epsilon V(1)}U(1)$ perturbed sequence
$V(t)$ Hermitian operator
ϵ perturbation strength

Fidelity of this quantum algorithm:

$$f(T) = \left| \frac{1}{N} \mathrm{Tr}[U_\delta(T,0) U(T,0)] \right|^2$$

the trace average the result over a complete set of initial states (for instance the quantum register states)
$U(t,t') \equiv U(t)U(t-1)\cdots U(t'+1)$ evolution operator from t' to $t > t'$
$U_\delta(t,t')$ is defined in the same way for the perturbed evolution

After defining the Heisenberg evolution of the perturbation as $V(t,t') = U^\dagger(t,t')V(t)U(t,t')$, we obtain

$$f(T) = \left| \frac{1}{N} \mathrm{Tr} \left(e^{i\epsilon V(1,0)} e^{i\epsilon V(2,0)} \cdots e^{i\epsilon V(T,0)} \right) \right|^2$$

As we are interested in the case in which the fidelity is close to unity, we can expand it up to the second order in ϵ:

$$f(t) \approx 1 - \epsilon^2 \sum_{t,t'=1}^{T} C(t,t'), \quad C(t,t') = \frac{1}{N}\mathrm{Tr}[V(t',0)V(t,0)]$$

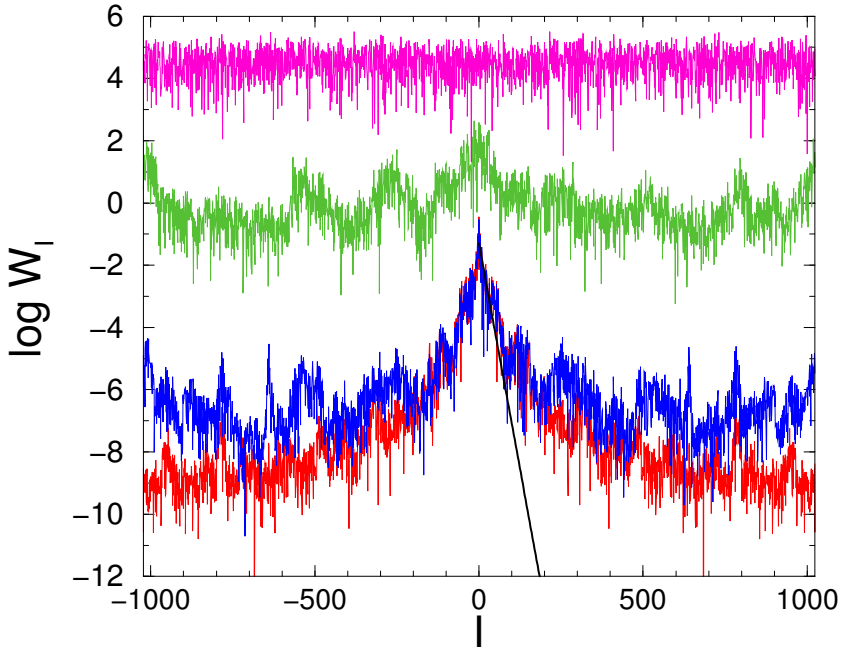
$C(t,t')$ is a two-point time correlation of the perturbation

A quantum algorithm is therefore more stable when the correlation time of the perturbation is smaller. This can be done by devising a "less regular" sequence of gates that realize the transformation U required by the algorithm [Prosen and Znidarič, J. Phis. A **34**, L681 (2001)]

For instance, using the Pauli operators one can change the computational basis repeatedly and randomly during a quantum computation [Kern et al., EPJD **32**, 153 (2005)]

Stability of local and nonlocal characteristics

Let us study the stability of physical quantities computed by a quantum algorithm simulating a dynamical system



- diffusion coefficient $D_I(\epsilon)$, obtained from the relation

$$\langle (\Delta I)^2 \rangle \approx D_I(\epsilon) t$$

This is an important characteristic related to transport properties of the system

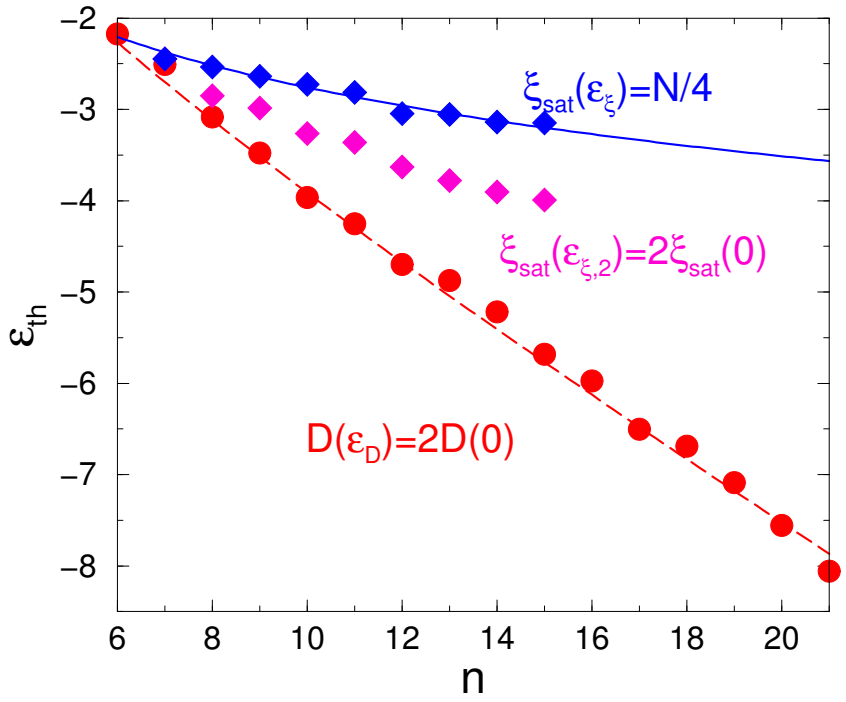
- inverse participation ratio

$$\xi = \frac{1}{\sum_I W_I^2}$$

This quantity determines the number of basis states significantly populated by the wave function and gives an estimate of the localization length of the system

Differently from the diffusion coefficient quantity, ξ is local in the localized regime, i.e. it is insensitive to the behavior of exponentially small tails

Stability of local quantities, like the inverse participation ratio vs. exponential sensitivity of non-local characteristics, like the diffusion coefficient



$\epsilon_\xi \propto n^{-5/2}$ algebraic drop

$\epsilon_D \propto n^{-2} N^{-1}$ exponential drop

Quantum noise

- Objectives: simulate noisy quantum protocols and determine time scales for reliable quantum computation in a noisy environment

- Approach: numerical simulations using quantum trajectories

- Main test bed: quantum algorithms simulating complex dynamics

- Main results: simulated quantum protocols with up to 24 qubits, found time scales for reliable quantum computation under realistic noise models

Quantum trajectories approach

As a consequence of the unwanted environmental coupling, a quantum processor becomes, in general, entangled with its environment, and therefore its state is mixed and described by a density matrix. Its evolution is ruled, under the assumption that the environment is Markovian, by a master equation. Solving this equation for a state of several qubits is a prohibitive task in terms of memory cost

Quantum Trajectories allow us to store only a stochastically evolving state vector, instead of a density matrix

This has an enormous advantage in memory requirements: if the Hilbert space has size N, we store only a state vector of size N instead of a density matrix of size $N \times N$

By averaging over many runs we get the same probabilities (within statistical errors) as the ones obtained by solving the density matrix directly

The theory of quantum trajectories is of widespread use in quantum optics to investigate physical phenomena such as spontaneous emission, resonance fluorescence and Doppler cooling

Quantum trajectories in the Markov approximation

If a system interacts with the environment, its state is described by a density operator ρ. Under the Markov assumption, the dynamics of the system is described by a (Lindblad) master equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H_s, \rho] - \frac{1}{2}\sum_k \{L_k^\dagger L_k, \rho\} + \sum_k L_k \rho L_k^\dagger,$$

H_s is the system's Hamiltonian, $\{\ ,\ \}$ denotes the anticommutator and L_k are the Lindblad operators, with $k \in [1, \ldots, M]$ (the number M depending on the particular model of interaction with the environment)

- The first two terms of the above equation can be regarded as the evolution performed by an effective non-hermitian Hamiltonian, $H_{\text{eff}} = H_s + iK$, with $K = -\hbar/2 \sum_k L_k^\dagger L_k$:

$$-\frac{i}{\hbar}[H_s, \rho] - \frac{1}{2}\sum_k \{L_k^\dagger L_k, \rho\} = -\frac{i}{\hbar}[H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger].$$

- The last term is the one responsible for the so called quantum jumps

If the initial density matrix describes a pure state ($\rho(t_0) = |\phi(t_0)\rangle\langle\phi(t_0)|$), then, after an infinitesimal time dt, it evolves into the statistical mixture

$$\begin{aligned}\rho(t_0+dt) &= \rho(t_0) - \tfrac{i}{\hbar}[H_{\text{eff}}\rho(t_0) - \rho(t_0)H_{\text{eff}}^\dagger]dt + \textstyle\sum_k L_k\rho(t_0)L_k^\dagger dt \\ &\approx (I - \tfrac{i}{\hbar}H_{\text{eff}}dt)\rho(t_0)(I + \tfrac{i}{\hbar}H_{\text{eff}}^\dagger dt) + \textstyle\sum_k L_k\rho(t_0)L_k^\dagger dt \\ &= (1 - \textstyle\sum_k dp_k)\ |\phi_0\rangle\langle\phi_0|\ + \textstyle\sum_k dp_k|\phi_k\rangle\langle\phi_k|,\end{aligned}$$

where the probabilities $dp_k = dt\langle\phi(t_0)|L_k^\dagger L_k|\phi(t_0)\rangle$, and the (normalized) new states are defined by

$$|\phi_0\rangle = \frac{(I - iH_{\text{eff}}dt/\hbar)|\phi(t_0)\rangle}{\sqrt{1 - \sum_k dp_k}}, \quad |\phi_k\rangle = \frac{L_k|\phi(t_0)\rangle}{||L_k|\phi(t_0)\rangle||}.$$

Therefore with probability dp_k a jump occurs and the system is prepared in the state $|\phi_k\rangle$. With probability $1 - \sum_k dp_k$ there are no jumps and the system evolves according to the effective Hamiltonian H_{eff}. (normalization is included because the evolution is non-hermitian)

Numerical method (Monte Carlo wave function approach)

- Start the time evolution from a pure state $|\phi(t_0)\rangle$

- At intervals dt much smaller than the time scales relevant for the evolution of the system, choose a random number ϵ from a uniform distribution in the unit interval $[0, 1]$

- 1) If $\epsilon \leq dp$, where $dp = \sum_k dp_k$, the state of the system jumps to one of the states $|\phi_k\rangle$ (to $|\phi_1\rangle$ if $0 \leq \epsilon \leq dp_1$, to $|\phi_2\rangle$ if $dp_1 < \epsilon \leq dp_1 + dp_2$, and so on)
2) if $\epsilon > dp$ the evolution with the non-hermitian Hamiltonian H_{eff} takes place and we end up in the state $|\phi_0\rangle$

- Repeat this process as many times as $n_{\text{steps}} = \Delta t/dt$, where Δt is the total evolution time

This procedure describes a stochastically evolving wave vector, and we say that a single evolution is a quantum trajectory

• Average over different runs to recover, up to statistical errors, the probabilities obtained using the density operator. Given an operator A, we can write the mean value $\langle A \rangle_t = \mathrm{Tr}[A\rho(t)]$ as the average over $\mathcal{N}$ trajectories:

$$\langle A \rangle_t = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \langle \phi_i(t)|A|\phi_i(t)\rangle$$

There is also an advantage in computation time with respect to density matrix direct calculations: it is indeed generally found that $\mathcal{N} \approx 100 - 500$ trajectories are needed in order to obtain a satisfactory statisticalconvergence, so that there is an advantage in computer time provided that $N > \mathcal{N}$

Quantum trajectories and stochastic Schrödinger equation

A quantum trajectory represents a single member of an ensemble whose density operator satisfies the corresponding master equation. This picture can be formalized by means of the nonlinear stochastic Schrödinger equation

$$
\begin{aligned}
|d\phi\rangle \quad &= \quad -iH|\phi\rangle dt - \frac{1}{2}\sum_k (L_k^\dagger L_k - \langle\phi|L_k^\dagger L_k|\phi\rangle)|\phi\rangle dt \\
&\quad + \sum_k \left(\frac{L_k}{\sqrt{\langle\phi|L_k^\dagger L_k|\phi\rangle}} - I \right) |\phi\rangle \, dN_k .
\end{aligned}
$$

The stochasticity is due to the measurement results: we think that the environment is actually measured (as it is the case in indirect measurement models) or simpler, that the contact of the system with the environment produces an effect similar to a continuous (weak) measurement

The nonlinearity appears due to the renormalization of the state vector after each measurement process

The stochastic differential variables dN_k are statistically independent and represent measurement outcomes. Their ensemble mean is given by $M[dN_k] = \langle\phi|L_k^\dagger L_k|\phi\rangle dt$. The probability that the variable dN_k is equal to 1 during a given time step dt is $\langle\phi|L_k^\dagger L_k|\phi\rangle dt = dp_k$. Therefore, most of the time the variables dN_k are 0 and as a consequence the system evolves continuously by means of the non Hermitian effective Hamiltonian $H_{\rm eff}$. However, when a variable dN_k is equal to 1, a quantum jump occurs

Differently from the master equation for the density operator, the stochastic Schrödinger equation represents the evolution of an individual quantum system, as exemplified by a single run of a laboratory experiment

Quantum trajectories and the Kraus formalism

To see the connection between the quantum jump picture and the Kraus operators formalism, let us write the solution to the master equation as a completely positive map:

$$\rho(t_0 + dt) = \mathcal{S}(t_0; t_0 + dt)\rho(t_0) = \sum_{k=0}^{\mathcal{M}} M_k(dt)\rho(t)M_k^\dagger(dt),$$

$$M_0 = I - iH_{\text{eff}}dt/\hbar, \quad M_k = L_k\sqrt{dt}, \ (k > 0)$$

The Kraus operators M_k satisfy the condition $\sum_{k=0}^{\mathcal{M}} M_k^\dagger M_k = I$ to first order in dt

The action of the superoperator $\mathcal{S}$ in can be interpreted as ρ being randomly replaced by $M_k\rho M_k^\dagger/\text{Tr}(M_k\rho M_k^\dagger)$, with probability $\text{Tr}(M_k\rho M_k^\dagger)$

The set $\{M_k\}$ defines a Positive Operator-Valued Measurement (POVM) with POVM elements $E_k = M_k^\dagger M_k$, that satisfy $\sum_{k=0}^{\mathcal{M}} E_k = I$.
The outlined process is equivalent to performing a continuous (weak) measurement on the system, that can be seen as an indirect measurement if the environment is actually measured

An example from quantum optics: spontaneous emission

Let us consider the simplest, zero temperature instance of the quantum optics master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] - \frac{\gamma}{2}(\sigma_-\sigma_+\rho + \rho\sigma_-\sigma_+) + \gamma\sigma_+\rho\sigma_-,$$

where the Hamiltonian $H = \frac{1}{2}\hbar\omega_0\sigma_z$ describes the free evolution of a two-level atom, γ is the atom-field coupling constant and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \sigma_y)$

In this case there is a single Lindblad operator $L_1 = \sqrt{\gamma}\sigma_+$ and a jump is a transition from the excited state ($|1\rangle$) to the ground state ($|0\rangle$) of the atom

Starting from an initial pure state $|\phi(t_0)\rangle = \alpha|0\rangle + \beta|1\rangle$ and evolving it for an infinitesimal time dt, the probability of a jump in a time dt is given by

$$dp = \langle\phi(t_0)|L_1^\dagger L_1|\phi(t_0)\rangle dt = \gamma\langle\phi(t_0)|\sigma_-\sigma_+|\phi(t_0)\rangle dt = \gamma p_e(t_0)dt,$$

where $p_e(t_0) = |\beta|^2$ is the population of the excited state $|1\rangle$ at time t_0

If a jump occurs, the new state of the atom is

$$|\phi_1\rangle = \frac{L_1|\phi(t_0)\rangle}{||L_1|\phi(t_0)\rangle||} = \frac{\sqrt{\gamma}\,\sigma_+(\alpha|0\rangle + \beta|1\rangle)\sqrt{dt}}{\sqrt{dp}} = \frac{\beta}{|\beta|}|0\rangle.$$

In this case, the transition $|1\rangle \rightarrow |0\rangle$ takes place and the emitted photon is detected. As a consequence, the atomic state vector collapses onto the ground state $|0\rangle$

If instead there are no jumps, the system's evolution is ruled by the non-Hermitian effective Hamiltonian $H_{\text{eff}} = H - i\frac{\hbar}{2}L_1^\dagger L_1 = H - i\frac{\hbar}{2}\gamma\sigma_-\sigma_+$, so that the state of the atom at time $t_0 + dt$ is

$$|\phi_0\rangle = \frac{\left(I - \frac{i}{\hbar}H_{\text{eff}}dt\right)|\phi(t_0)\rangle}{\sqrt{1-dp}} = \frac{\left(1 - i\frac{\omega_0}{2}dt\right)\alpha|0\rangle + \left(1 + i\frac{\omega_0}{2}dt - \frac{\gamma}{2}\right)\beta|1\rangle}{\sqrt{1-\gamma|\beta|^2dt}}$$

The normalization factor $\frac{1}{\sqrt{1-dp}}$ is due to the fact that, if no counts are registered by the photodetector, then we consider more probable that the system is unexcited

To see this, let us consider the evolution without jumps in a finite time interval, from t_0 to $t_0 + t$. We obtain

$$|\phi_0(t_0+t)\rangle = \frac{\alpha \exp\left[-i\frac{\omega_0}{2}(t-t_0)\right]|0\rangle + \beta \exp\left[\left(i\frac{\omega_0}{2} - \frac{\gamma}{2}\right)(t-t_0)\right]|1\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 \exp[-\gamma(t-t_0)]}}$$

Note that as $t \to +\infty$ the state $|\phi_0(t)\rangle \to |0\rangle$ (up to an overall phase factor). That is, if after a long time we never see a count, we conclude that we have been in the ground state $|0\rangle$ from the beginning

Generalized many-qubit noise channels

In the single-qubit case the dissipative environment can be conveniently modeled by means of the amplitude damping channel, defined by the following Kraus operators:

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

The operator M_1 is responsible for the jumps $|1\rangle \longrightarrow |0\rangle$, the jump probability being p

Many different generalizations to the many-qubit case are possible. For instance:

1) Collective interaction: a single damping probability describes the action of the environment, irrespective of the internal many-body state of the system
Example: start from the four-qubit pure state $\rho(t_0) = |1011\rangle\langle 1011|$; after a time dt we obtain

the statistical mixture

$$\rho(t_0+dt) = \left(1 - \frac{\Gamma dt}{\hbar}\right) |1011\rangle\langle 1011| + \frac{\Gamma dt}{3\hbar}(|0011\rangle\langle 0011| + |1001\rangle\langle 1001| + |1010\rangle\langle 1010|)$$

2) Single-qubit interactions: each qubit has its own interaction with the environment, independently of the other qubits

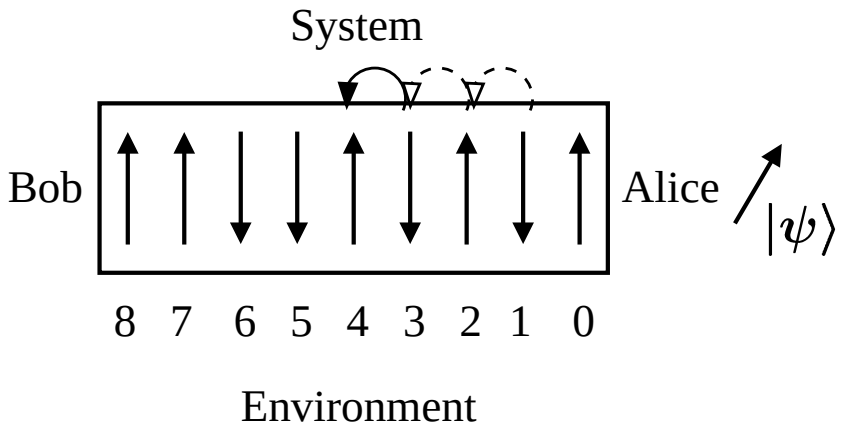
$$M_\mu = I \otimes \cdots \otimes I \otimes M_1 \otimes I \otimes \cdots \otimes I, \quad (\mu = 1, ..., n)$$

In the previous example,

$$\rho(t_0+dt) = \left(1 - \frac{3\Gamma dt}{\hbar}\right) |1011\rangle\langle 1011| + \frac{\Gamma dt}{\hbar}(|0011\rangle\langle 0011| + |1001\rangle\langle 1001| + |1010\rangle\langle 1010|$$

Teleportation in a noisy environment

We assume that the delivery of one of the qubits of the EPR pair required in the teleportation protocol is done by means of SWAP gates along a noisy chain of qubits



- Assume that the initial state of the chain is given by

$$\sum_{i_{n-1},\ldots,i_2} c_{i_{n-1},\ldots,i_2} |i_{n-1}\ldots i_2\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

where $i_k = 0, 1$ denotes the down or up state of qubit k

- In order to deliver one of the qubits of the EPR pair to Bob, we implement a protocol consisting of $n-2$ SWAP gates, each one exchanging the states of a pair of qubits:

$$\sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}}(|i_{n-1}\ldots i_2 00\rangle + |i_{n-1}\ldots i_2 11\rangle)$$

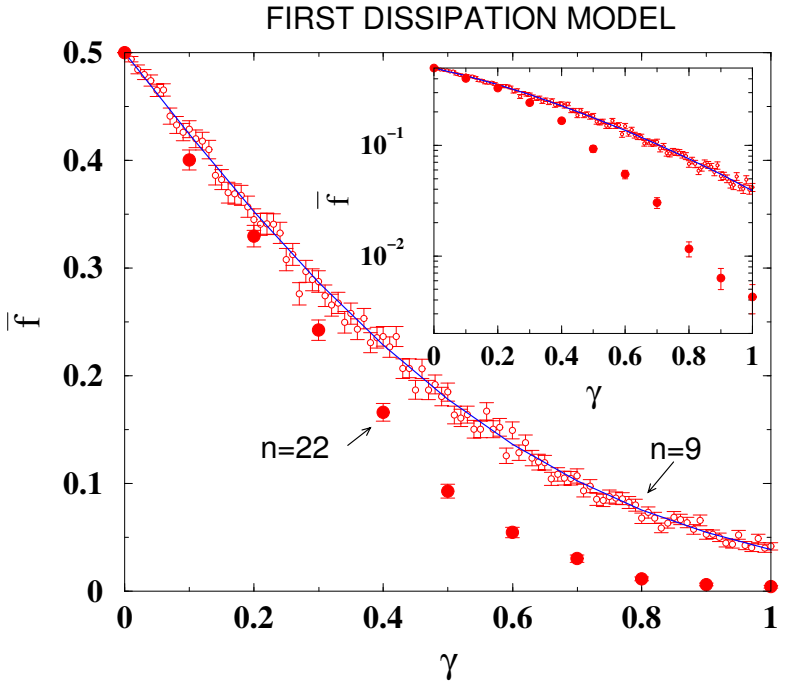
$$\rightarrow \sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}}(|i_{n-1}\ldots 0 i_2 0\rangle + |i_{n-1}\ldots 1 i_2 1\rangle) \rightarrow$$

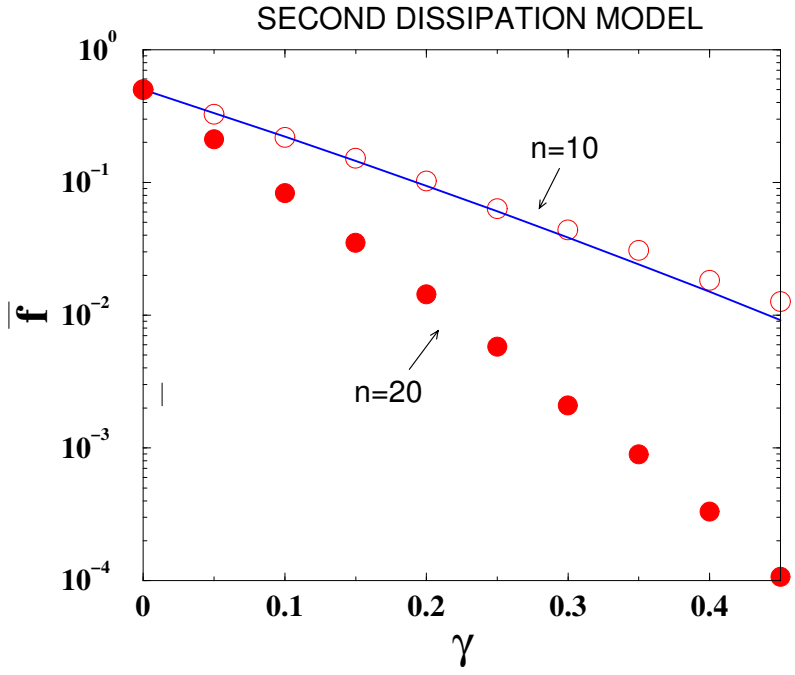
$$\ldots \rightarrow \sum_{i_{n-1},\ldots,i_2} \frac{c_{i_{n-1},\ldots,i_2}}{\sqrt{2}}(|0 i_{n-1}\ldots i_2 0\rangle + |1 i_{n-1}\ldots i_2 1\rangle).$$

- To model the transmission of the qubit through a chaotic quantum chain we take random coefficients $c_{i_{n-1},\ldots,i_2}$, that is they have amplitudes of the order of $1/\sqrt{2^{n-2}}$ (to assure wave function normalization) and random phases

Numerical results with up to 24 qubits

We have computed the fidelity $\bar{f} = f - f_\infty$ of teleportation in the presence of a dissipative environment, as a function of the dimensionless damping rate γ and for up to $n = 24$ qubits





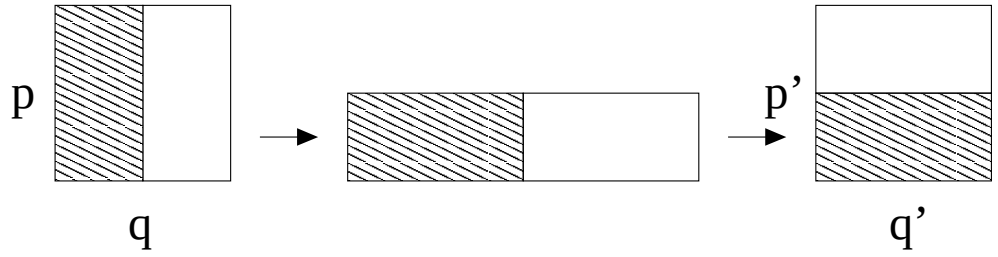
Fidelity time scale for quantum computation in a noisy environment

To be concrete, let us consider the quantum baker's map, a prototypical map for theoretical studies of quantum chaos, which can be simulated efficiently on a quantum computer [Schack, PRA **57**, 1634 (1998)] and has already been implemented on a 3-qubit NMR-based quantum processor [Weinstein, Lloyd, Emerson, Cory, PRL **89**, 157902 (2002)]

The classical baker's transformation maps the unit square $0 \leq q, p < 1$ onto itself according to

$$(q,p) \to (q',p') = \begin{cases} (2q, \frac{1}{2}p), & \text{if } 0 \leq q \leq \frac{1}{2}, \\ (2q-1, \frac{1}{2}p + \frac{1}{2}), & \text{if } \frac{1}{2} < q < 1. \end{cases}$$

This corresponds to compressing the unit square in the p direction and stretching it in the q direction, then cutting it along the p direction, and finally stacking one part on top of the other (similarly to the way a baker kneads dough).



The baker's map is a paradigmatic model of classical chaos. It exhibits sensitive dependence on initial conditions, which is the distinctive feature of classical chaos: any small error in determining the initial conditions amplifies exponentially in time. In other words, two nearby trajectories separate exponentially, with a rate given by the maximum Lyapunov exponent $\lambda = \ln 2$

The baker's map can be quantized (following Balazs, Voros and Saraceno): we introduce the position (q) and momentum (p) operators, and denote the eigenstates of these operators by $|q_j\rangle$ and $|p_k\rangle$, respectively. The corresponding eigenvalues are given by $q_j = j/N$ and $p_k = k/N$, with $j, k = 0, ..., N-1$, where N is the dimension of the Hilbert space. Note that, to fit N levels onto the unit square, we must set $2\pi\hbar = 1/N$.

It can be shown that the quantized baker's map can be defined in the position basis by the transformation

$$|\psi\rangle \rightarrow |\psi'\rangle = B\,|\psi\rangle = F_n^{-1} \begin{pmatrix} F_{n-1} & 0 \\ 0 & F_{n-1} \end{pmatrix} |\psi\rangle,$$

where F is the discrete Fourier transform and n is the number of qubits ($N = 2^n$). Therefore, the quantum baker's map can be implemented efficiently on a quantum computer [Schack, PRA **57**, 1634 (1998); Brun and Schack, PRA **59**, 2649 (1999)]

The environment is modeled as a phase shift channel for each qubit, corresponding to the Kraus operators

$$M_0 = \sqrt{1-p}\, I, \quad M_1 = \sqrt{p}\, \sigma_z$$

We perform t steps of the noisy evolution of the baker's map considering a random initial state and measure the fidelity

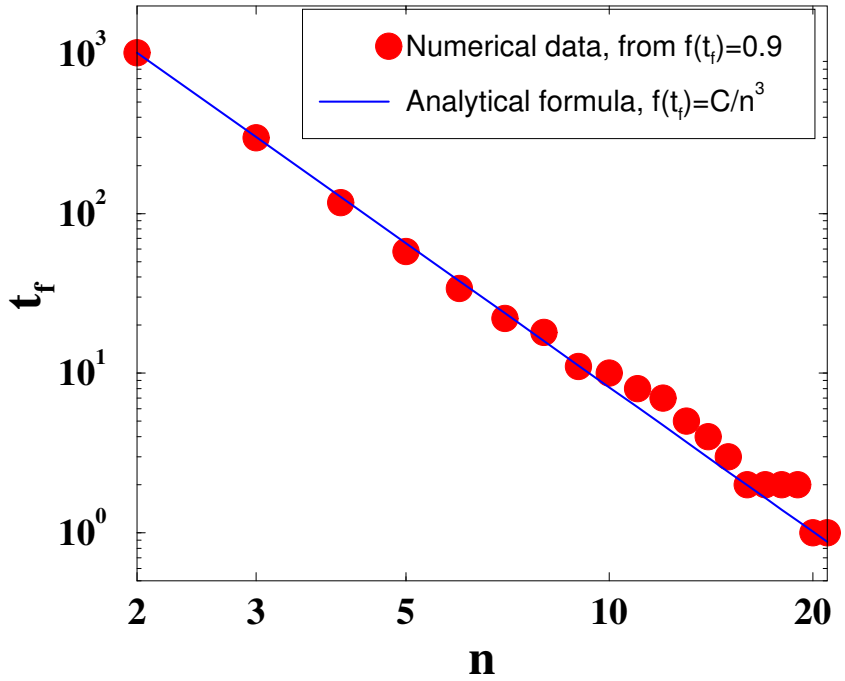
$$f(t) = \langle \psi_{\text{exact}}(t) | \rho_{\text{noise}}(t) | \psi_{\text{exact}}(t) \rangle$$

Theoretical expectations:

$$f(t) = \exp\left(-n\gamma N_g\right) = \exp(-2\gamma n^3 t),$$

t number of map steps
γ dimensionless decay rate
N_g total number of quantum gates

The number of gates that can be reliably implemented without quantum error correction drops only polynomially with the number of qubits: $(N_g)_f \propto 1/n$



The theoretical formula for fidelity decay assumes that the decay of correlations induced by chaotic dynamics takes place in a time scale shorter than the time scale for fidelity decay

The law $(N_g)_f \propto 1/n$ should remain valid also for other environment models that allow only one qubit at a time to perform a transition, like the simple amplitude damping channel

Furthermore, we note that this time scale for reliable quantum computation is expected to be valid, beyond the baker's map model, for any quantum algorithm simulating complex dynamics, such as an evolution in the regime of quantum chaos [see also J.W. Lee and D.L. Shepelyansky, PRE 71, 056202 (2005) about dissipative decoherence in the quantum sawtooth algorithm]

Final remarks

Decoherence and imperfections appear to be the ultimate obstacle to the practical realization of a large-scale quantum computer

Numerical simulations including realistic noise parameters for specific experimental implementations can be done following the approaches outlined in these lectures. Such simulations promise to become a valuable tool for quantum hardware design and to determine optimal regimes for the operability of quantum computers

We can expect that a deeper understanding of quantum chaos in many-body quantum computers will suggest strategies suitable for a better working of quantum processors

Quantum Simulation of Dissipative Chaotic Systems: Quantum Ratchets and Ehrenfest Explosion

Quantum simulation is a special instance of quantum computation

In particular, optical lattices allowed the observation of the superfluid to Mott insulator and could be used for the study of a wide range of phenomena, from strongly correlated condensed matter physics to spin glasses

We investigate the possibilities opened by optical lattices for the quantum simulation of complex dissipative systems

Outline

QUANTUM RATCHETS

- A model for quantum directed transport (ratchet effect) in a periodic chaotic system with dissipation, in presence of lattice asymmetry and unbiased driving

- Discuss possible experimental implementation with cold atoms in optical lattices

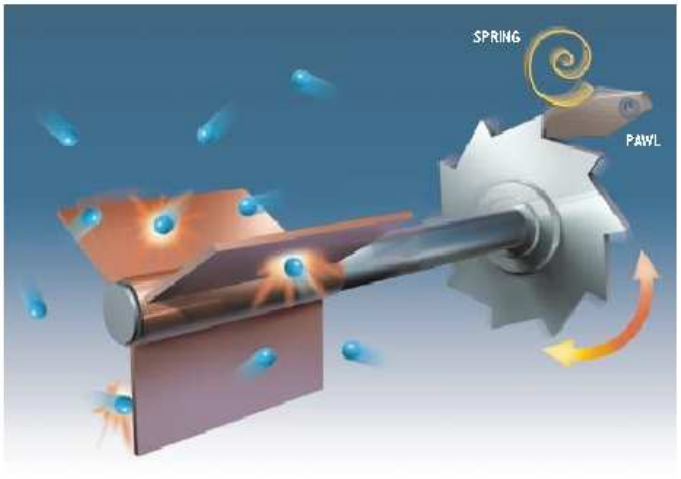
EHRENFEST EXPLOSION

- It is possible to recover classical-like chaotic dynamics (positive Lyapunov exponent of quantum trajectories) in a dissipative system

- Interplay between the wave packet collapse due to dissipation and the wave packet explosion induced by exponential instability within the Ehrenfest time scale: collapse to explosion transition

The Feynman ratchet

Can useful work be extracted out of unbiased microscopic random fluctuations if all acting forces and temperatures gradients average out to zero?



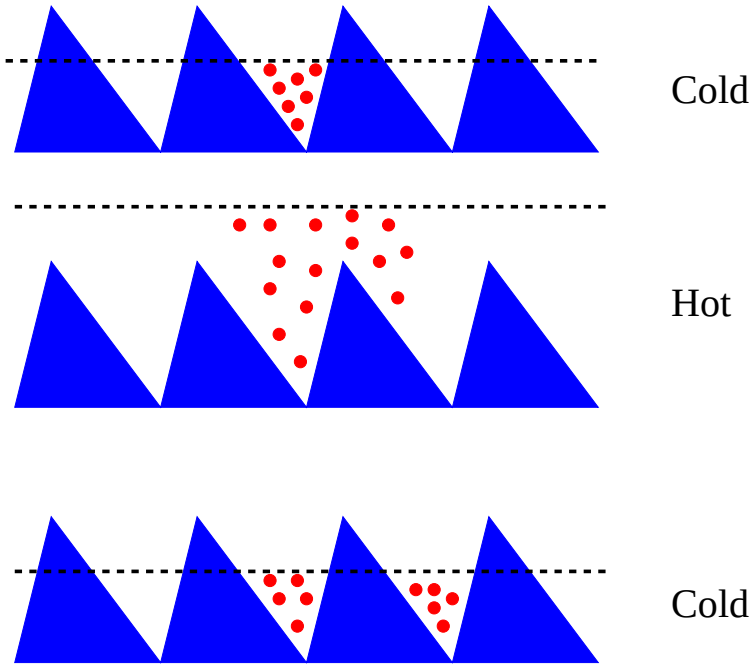
(taken from D.Astumian, Scientific American, July 2001)

Thermal equilibrium: the gas surrounding the paddles and the ratchet (plus the pawl) are at the same temperature

In spite of the built asymmetry no preferential direction of motion is possible. Otherwise, we could implement a perpetuum mobile, in contradiction with the second law of thermodynamics

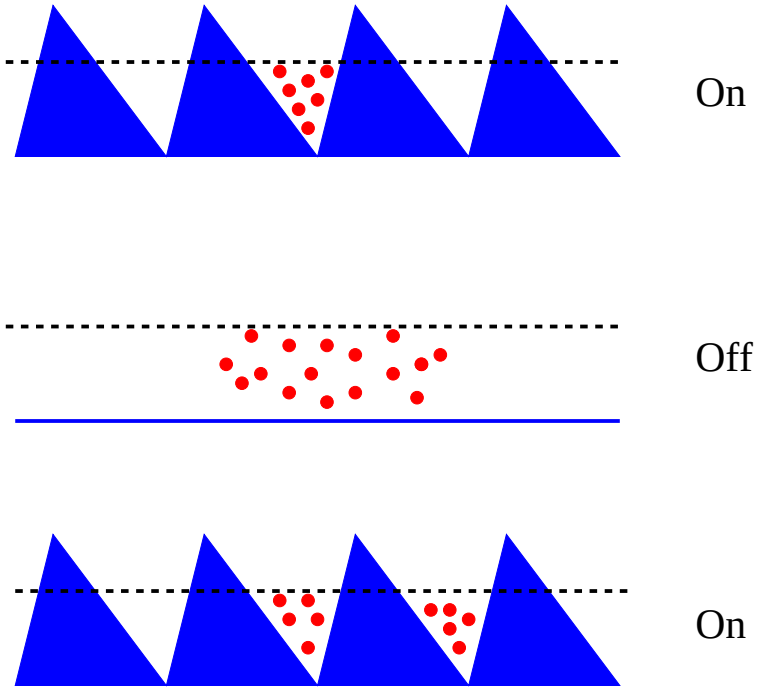
Brownian motors

To build a Brownian motor drive the system out of equilibrium



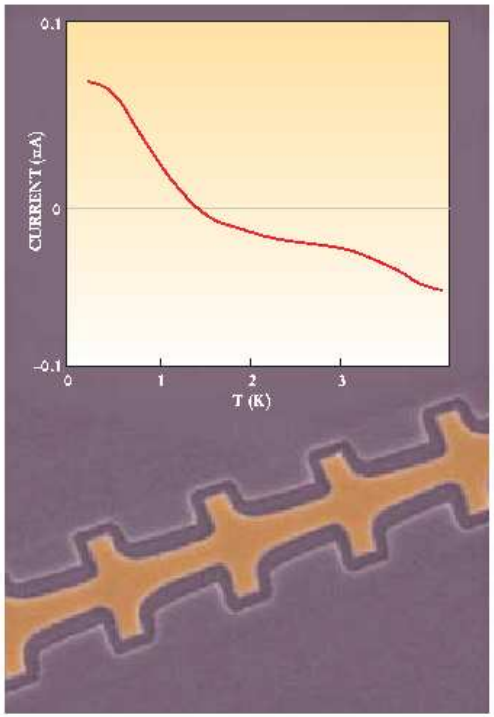
Working principle of a Brownian motor driven by temperature oscillation

Another model of Brownian motor: a pulsating (flashing) ratchet



Quantum ratchets

Quantum tunneling provides a second mechanism (the first being the thermal activation) to overcome energy barriers and lead to directed motion



(String of triangular quantum dots, Linke *et al* experiments, Science, 1999)

Quantum ratchets in dissipative chaotic systems,
Phys. Rev. Lett. **94**, 164101 (2005)

A deterministic model of quantum chaotic dissipative ratchet

Particle moving in a kicked periodic asymmetric potential

$$V(x,\tau) = k\left[\cos(x) + \frac{a}{2}\cos(2x+\phi)\right]\sum_{m=-\infty}^{+\infty}\delta(\tau - mT),$$

Classical evolution in one period described by the map

$$\begin{cases} \overline{I} = (1-\gamma)I + k(\sin(x) + a\sin(2x+\phi)), \\ \overline{x} = x + T\overline{I}, \end{cases}$$

$0 < \gamma < 1$ dissipation parameter (velocity proportional damping):
$\gamma = 1$ overdamping – $\gamma = 0$ Hamiltonian evolution

G.G. Carlo, G.Benenti, G.Casati, D.L.Shepelyansky

Study of the quantized model

Quantization rules: $x \longrightarrow \hat{x}$, $I \longrightarrow \hat{I} = -i(d/dx)$ (we set $\hbar = 1$)

Since $[\hat{x}, \hat{p}] = iT$, the effective Planck constant is $\hbar_{\text{eff}} = T$

In order to simulate a dissipative environment in the quantum model we consider a master equation in the Lindblad form for the density operator $\hat{\rho}$ of the system:

$$\dot{\hat{\rho}} = -i[\hat{H}_s, \hat{\rho}] - \frac{1}{2}\sum_{\mu=1}^{2}\{\hat{L}_\mu^\dagger \hat{L}_\mu, \hat{\rho}\} + \sum_{\mu=1}^{2}\hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$$

$\hat{H}_s = \hat{n}^2/2 + V(\hat{x}, \tau)$ system Hamiltonian
$\hat{L}_\mu$ Lindblad operators
$\{\ ,\ \}$ denotes the anticommutator

The dissipation model

We assume that dissipation is described by the lowering operators

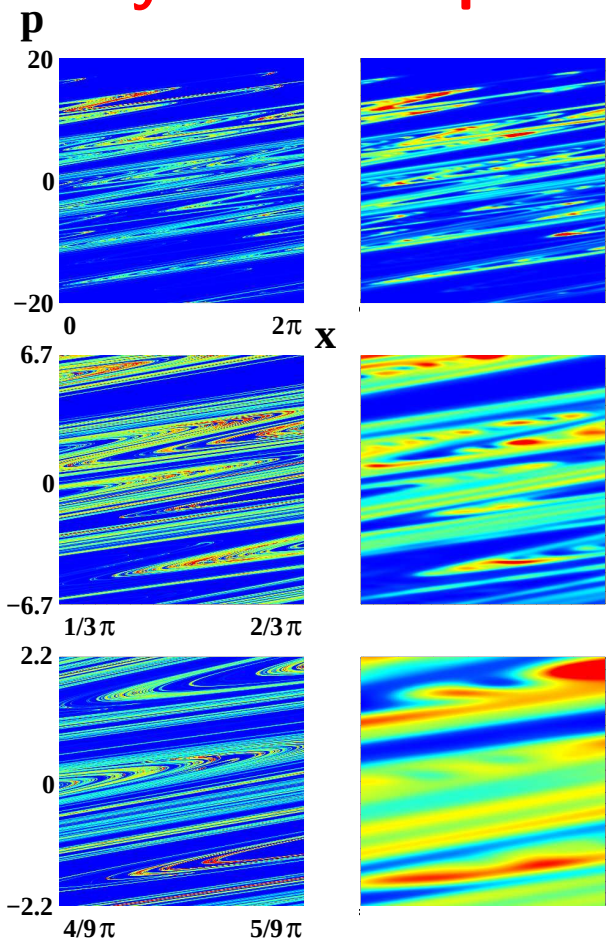
$$\hat{L}_1 = g \sum_I \sqrt{I+1}\, |I\rangle \langle I+1|,$$
$$\hat{L}_2 = g \sum_I \sqrt{I+1}\, |-I\rangle \langle -I-1|, \quad n = 0, 1, ...$$

These Lindblad operators can be obtained by considering the interaction between the system and a bosonic bath. The master equation is then derived, at zero temperature, in the usual weak coupling and Markov approximations

Requiring that at short times $\langle p \rangle$ evolves like in the classical case, as it should be according to the Ehrenfest theorem, we obtain $g = \sqrt{-\ln(1-\gamma)}$

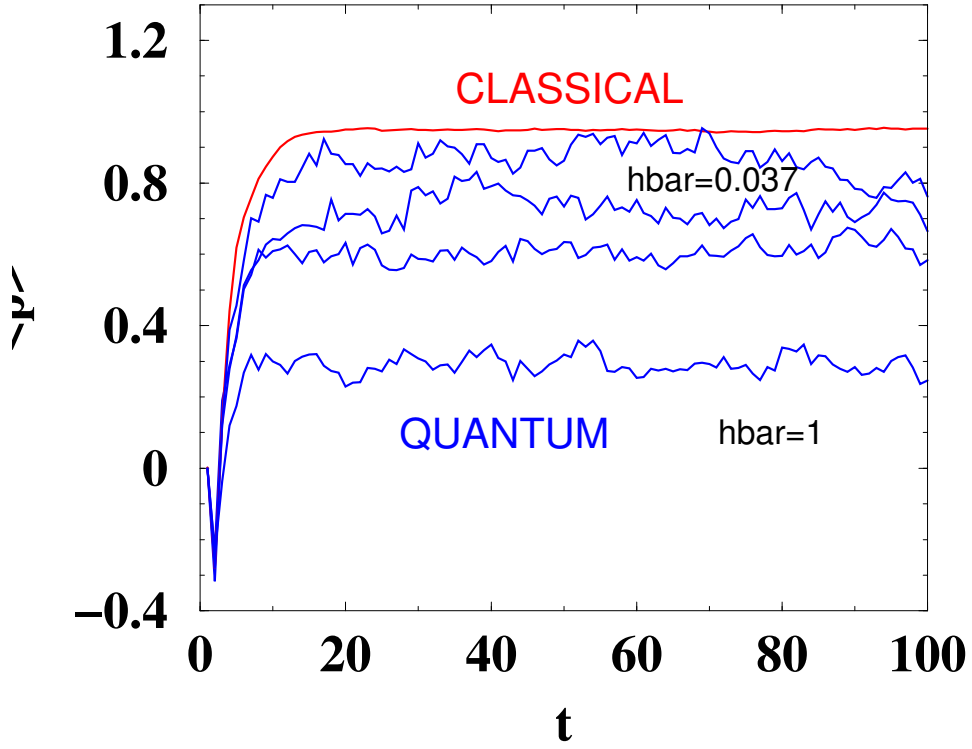
Simulation of quantum dissipation with quantum trajectories

Asymmetric quantum strange attractor



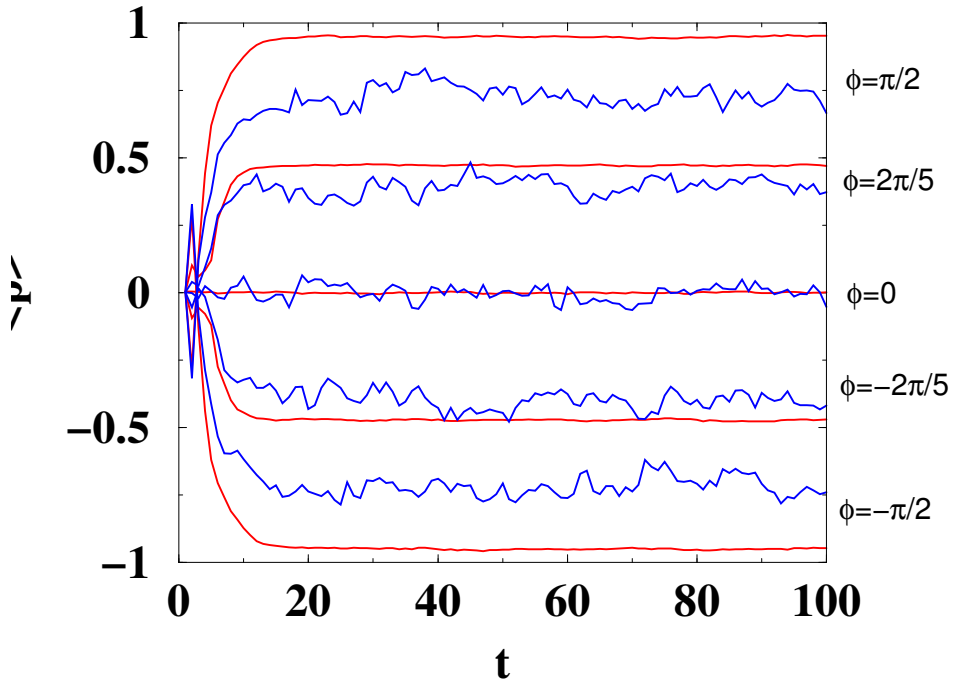
Phase space pictures for $K = 7$, $\gamma = 0.3$, $\phi = \pi/2$, $a = 0.7$, after 100 kicks: classical Poincaré sections (left) and quantum Husimi functions at $\hbar_{\rm eff} = 0.012$ (right)

Ratchet effect



Average momentum $\langle p \rangle$ as a function of time t (measured in number of kicks)

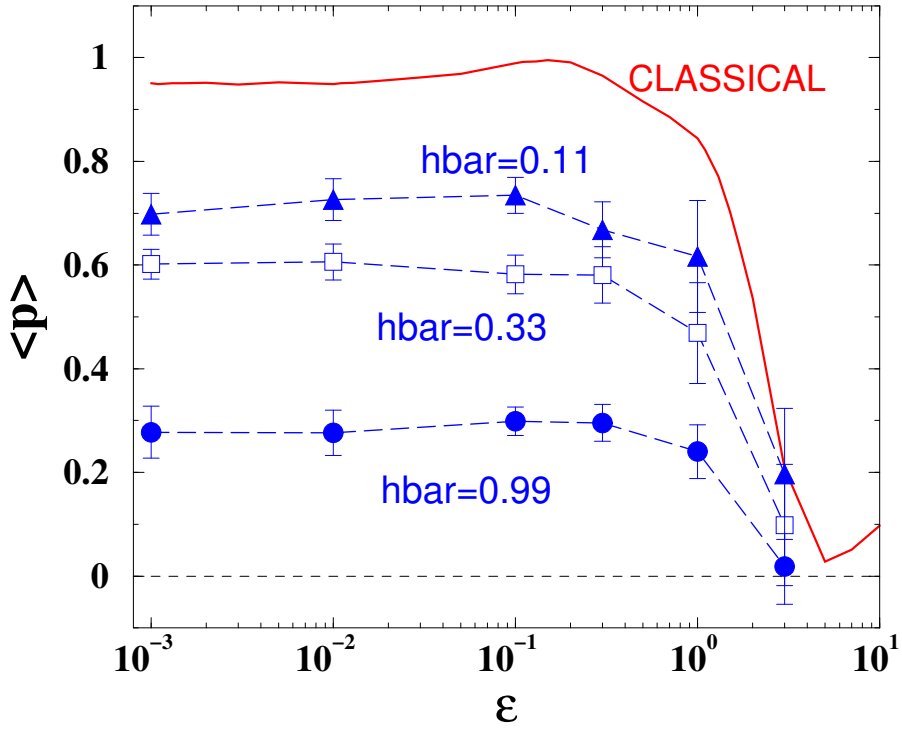
Control the direction of transport



Zero net current for $\phi = n\pi$, due to the space symmetry $V(x,\tau) = V(-x,\tau)$

In general $\langle p\rangle_{-\phi} = -\langle p\rangle_{\phi}$, due to the symmetry $V_{\phi}(x,\tau) = V_{-\phi}(-x,\tau)$

Stability under noise effects



Memoryless fluctuations in the kicking strength: $K \to K_\epsilon(t) = K + \epsilon(t)$, $\epsilon(t) \in [-\epsilon, +\epsilon]$

The ratchet effect survives up to a noise strength ϵ of the order of the kicking strength K

Possible experimental implementation

Possible experimental implementations with cold atoms in a periodic standing wave of light

Values $K = 7$, $\hbar_{\rm eff} \sim 1$ were used in the experimental implementations of the kicked rotor model

Friction force can be implemented by means of Doppler cooling techniques

State reconstruction techniques could in principle allow the experimental observation of a quantum strange ratchet attractor

The ratchet effect is robust when noise is added; due to the presence of a strange attractor, the stationary current is independent of the initial conditions

Dissipative quantum chaos: transition from wave packet collapse to explosion,
preprint quant-ph/0503081

Dissipative quantum chaos: transition from wave packet collapse to wave packet explosion

The instability of classical dynamics leads to exponentially fast spreading of the quantum wave packet on the logarithmically short Ehrenfest time scale

$$t_E \sim \frac{|\ln \hbar|}{\lambda}$$

λ Lyapunov exponent, $\hbar$ effective Planck constant

After the logarithmically short Ehrenfest time a description based on classical trajectories is meaningless for a closed quantum system

What is the interplay between wave packet explosion (delocalization) induced by chaotic dynamics and wave packet collapse (localization) caused by dissipation?

G.G. Carlo, G.Benenti, D.L.Shepelyansky

A model of dissipative chaotic dynamics

Markovian master equation $\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] - \frac{1}{2}\sum_{\mu}\{\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}, \hat{\rho}\} + \sum_{\mu}\hat{L}_{\mu}\hat{\rho}\hat{L}_{\mu}^{\dagger}$

Kicked rotator Hamiltonian $\hat{H} = \frac{\hat{I}^2}{2} + k\cos(\hat{x})\sum_{m=-\infty}^{+\infty}\delta(\tau - mT)$

Dissipation described by the Lindblad operators

$$\hat{L}_1 = g\sum_{I}\sqrt{I+1}\,|I\rangle\langle I+1|,\ \hat{L}_2 = g\sum_{I}\sqrt{I+1}\,|-I\rangle\langle -I-1|$$

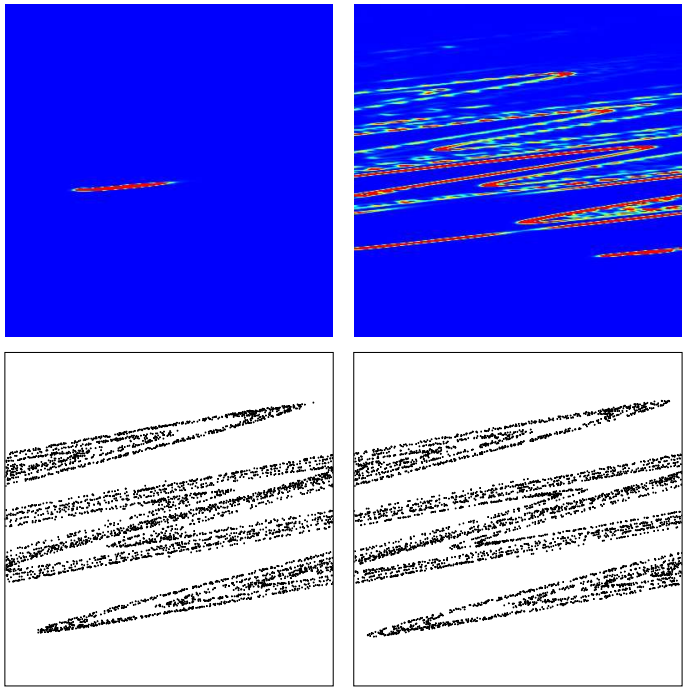
At the classical limit, the evolution of the system in one period is described by the Zaslavsky map

$$\begin{cases} I_{t+1} = (1-\gamma)I_t + k\sin x_t, \\ x_{t+1} = x_t + TI_{t+1}, \end{cases}$$

Quantum trajectories

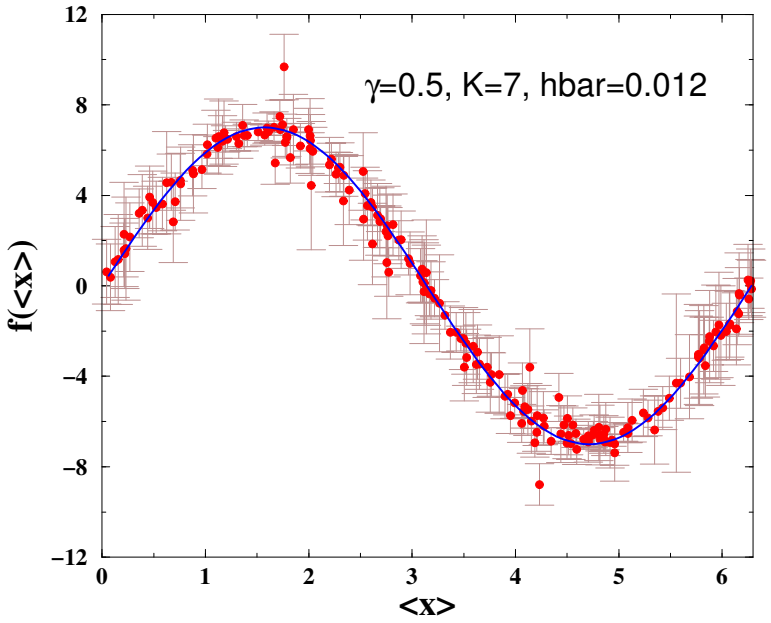
The changing state of a single open quantum system is represented directly by a stochastically evolving quantum wave function, as for a single run of a laboratory experiment - a single evolution is termed a quantum trajectory

Collapse to explosion transition (going from stron to weak dissipation)



$K = 7$, $\hbar = 0.012$, $\gamma = 0.5$ and $\gamma = 0.01$

Classical-like evolution of quantum trajectories

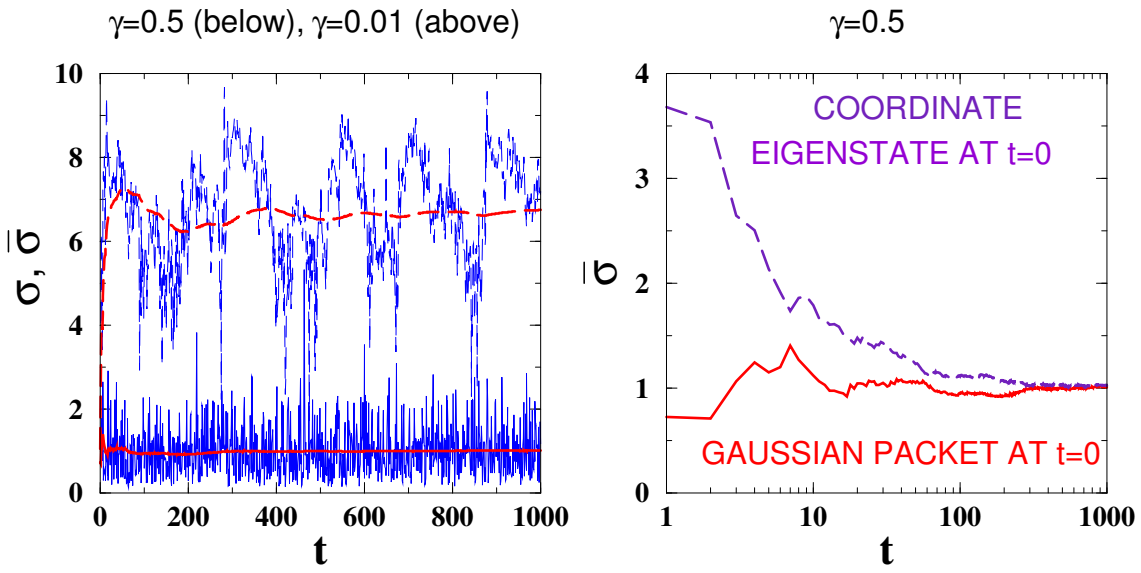


$$f \equiv \langle p \rangle_{t+1} - (1-\gamma)\langle p \rangle_t, \quad \langle p \rangle_t = \langle x \rangle_t - \langle x \rangle_{t-1}$$

From classical dynamics we expect $f(x) = K \sin x$ - Quantum fluctuations $\propto \sqrt{\hbar}$

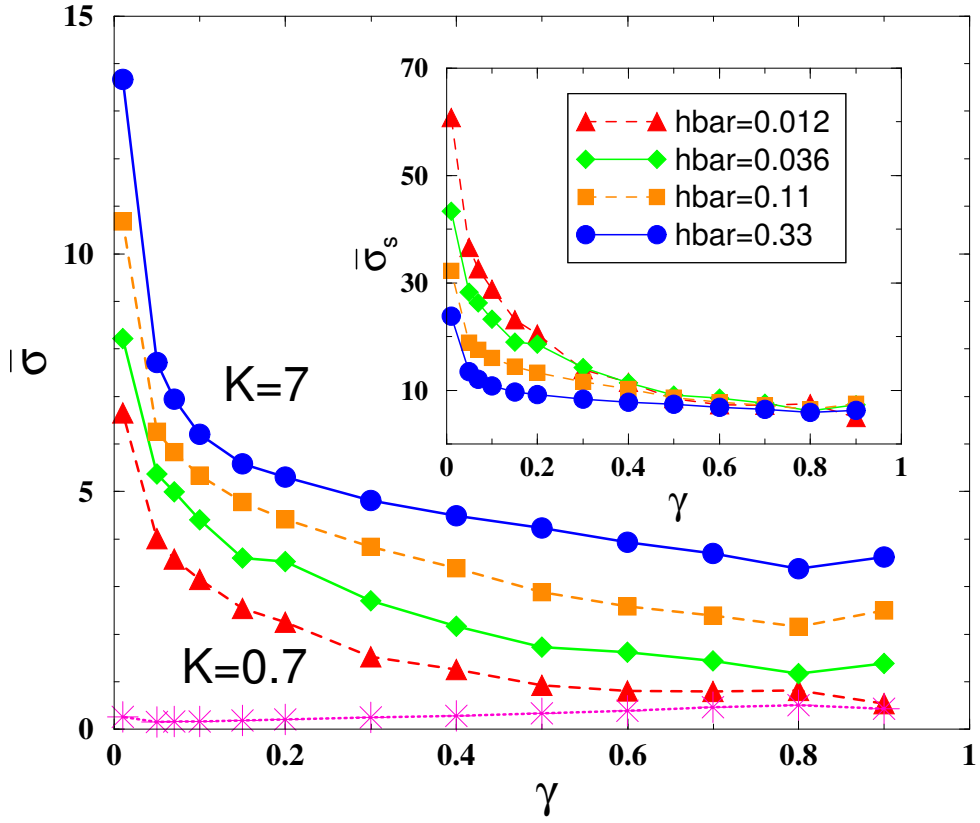
Wave packet dispersion

$$\sigma_t = \sqrt{(\Delta x)_t^2 + (\Delta p)_t^2}, \quad \text{cumulative average } \overline{\sigma}_t \equiv \frac{1}{t}\sum_{j=1}^{t} \sigma_j$$



$(K = 7,\ \hbar = 0.012)$

Localization - delocalization crossover



$\overline{\sigma}_s \equiv \overline{\sigma}/\sqrt{\hbar}$ scaled dispersion

Ehrenfest explosion

Due to the exponential instability of chaotic dynamics the wave packet spreads exponentially and for times shorter than the Ehrenfest time we have $\sigma_t \sim \sqrt{\hbar}\exp(\lambda t)$

The dissipation localizes the wave packet on a time scale of the order of $1/\gamma$

Therefore, for $1/\gamma \ll t_E \sim |\ln\hbar|/\lambda$, we obtain $\overline{\sigma} \sim \sqrt{\hbar}\exp(\lambda/\gamma) \ll 1$

In contrast, for $1/\gamma > t_E$ the chaotic wave packet explosion dominates over dissipation and we have complete delocalization over the angle variable

In this case, the wave packet spreads algebraically due to diffusion for $t > t_E$: for $t \gg t_E$ we have $\sigma_t \sim \sqrt{D(K)t}$, $D(K) \approx K^2/2$ being the diffusion coefficient; this regime continues up to the dissipation time $1/\gamma$, so that $\overline{\sigma} \sim \sqrt{D(K)/\gamma}$

Tthe transition from collapse to explosion (Ehrenfest explosion) takes place at

$$t_E \sim \frac{|\ln \hbar|}{\lambda} \sim \frac{1}{\gamma}$$

Therefore, even for infinitesimal dissipation strengths the quantum wave packet is eventually localized when $\hbar \to 0$: we have $\lim_{\hbar \to 0} \overline{\sigma} = 0$; in contrast, in the Hamiltonian case $(\gamma = 0)$ $\lim_{\hbar \to 0} \overline{\sigma} = \infty$

Only for open quantum systems the classical concept of trajectory is meaningful for arbitrarily long times; on the contrary, for Hamiltonian systems a description based on wave packet trajectories is possible only up to the Ehrenfest time scale

Conclusions

• Cold atoms and Bose-Einstein condensates exposed to time-dependent standing waves of light may provide an ideal test bed to explore dissipative quantum chaos

• Quantum ratchets: explore thermal effects on the ratchet current; adapt the proposed ratchet model to condensates (including Gross-Pitaevsky nonlinearity effects); study the impact of dynamical effects such as bifurcations on the ratchet phenomenon

• Ehrenfest explosion: investigate the dynamical stability of condensates subjected to chaotic dynamics and dissipation