LECTURE 2

International School of Physics "Enrico Fermi"

Quantum Computers, Algorithms and Chaos

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Quantum algorithms for quantum chaos

Brief idea of quantum computing...

- **A quantum computer is not only faster than classical devices, it is something else: new computer science, with new properties $\Rightarrow$ may change complexity class of problems**

- **The efficiency of quantum computation compare to classical computation depends on the problem: to benefit from the power of quantum computation, one should ask certain types of questions.**

- **Important applications: code-breaking, simulation of physical systems**

Programming instructions: quantum gates

One acts on the wave function of the quantum computer through unitary transformation. In practice, one uses elementary quantum gates which are local and compose them to build the unitary evolution needed.

- **Hadamard gate applied to one qubit** $|0\rangle \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}$; $|1\rangle \rightarrow (|0\rangle - |1\rangle)/\sqrt{2}$;
- **phase gate applied to one qubit** $|0\rangle \rightarrow |0\rangle$; $|1\rangle \rightarrow i|1\rangle$;
- **controlled not or CNOT applied to two qubits:** $|00\rangle \rightarrow |00\rangle$; $|01\rangle \rightarrow |01\rangle$;$|10\rangle \rightarrow |11\rangle$; $|11\rangle \rightarrow |10\rangle$; **the second qubit is changed if the first is in the state** $|1\rangle$;
- **controlled controlled not or Toffoli gate applied to three qubits: the third qubit is changed if the first two are both in the state** $|1\rangle$.

Universal sets of quantum gates are enough to build any unitary transformations (for example, one-qubit gates +CNOT). Different universal sets are possible, their choice depends on experimental implementations.

Classical algorithms

- **Modify strings of bits 0 and 1 through possibly irreversible transformations ($\Rightarrow$ dissipation, heat production)**
- **Input: chain of N bits $\Rightarrow n = \log_2 N$ size of the input**
- **Modifies the input in M operations $\Rightarrow$ complexity**

  M polynomial in $n = \log_2 N \Rightarrow$ polynomial algorithm (complexity class P) (example: arithmetic operations,...)

  M exponential in $n \Rightarrow$ exponential algorithm (example: factoring, traveling salesman,...)
- **Millenium problem: $P = NP$**
- **Classical complexity result: complexity class does not depend on the device**

Quantum algorithms

n **qubits** $\Rightarrow N = 2^n$ **quantum basis states such as 011001...**

Procedure to perform an algorithm:

- **Build an initial state** $|\Psi_i\rangle = \sum_{i=0}^{N-1} a_i|i\rangle$. **Example:** $1/\sqrt{N}\sum_{i=0}^{N-1}|i\rangle$ **(uniform superposition) can be build from** $|00..00\rangle$ **by application of** n **Hadamard gates.**
- **Transform it** $|\Psi_i\rangle \rightarrow |\Psi_f\rangle = \sum_{i=0}^{N-1} b_i|i\rangle$ **through a sequence of elementary (local) quantum gates. This process is reversible.**
- **Extract information by quantum measurement of** $|\Psi_f\rangle$

The result is usually probabilistic: quantum measurement gives the right result with a certain probability. The algorithm works if 1) one can recognize the right result when it comes and 2) the probability of success is significant (especially when n **increases).**

Complexity of the algorithm is measured by the number of quantum gates needed, taking into account that the process may have to be iterated since the result is probabilistic. Result: quantum computation may change the complexity class of problems.

Programming a quantum computer

Subroutines

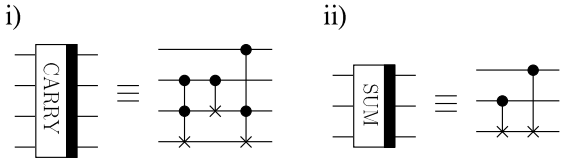
- **Quantum addition:**
- **Quantum multiplication and exponentiation**
- **Quantum Fourier transform**
- **Quantum wavelet transform**

Known algorithmic tools

- **Period finding aka hidden subgroup**
- **Grover's search of an unstrucured database**
- **Amplitude amplification**
- **Phase estimation**
- **Quantum counting**

Quantum addition

$$1/N \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle|j\rangle|0...000\rangle \Rightarrow 1/N \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |i\rangle|i+j\rangle|0...000\rangle$$



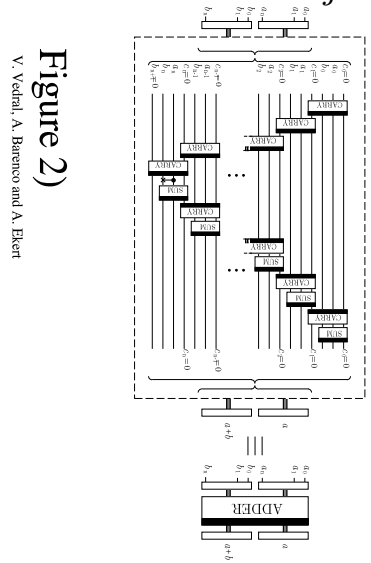


Figure 2)

V. Vedral, A. Barenco and A. Ekert

Figure 3)

V. Vedral, A. Barenco and A. Ekert

→ **Needs only $\approx 8 \log N$ quantum gates to perform N^2 additions reversibly ($N = 2^n$), with one workspace register erased at the end**

→ **Multiplications and exponentiations can be done in the same way, by using binary decomposition $\Rightarrow$ need $\sim n^2$ (multiplication) and $\sim n^3$ quantum gates (exponentiation).**

Quantum Fourier Transform

Uses n qubits to transform a vector of size $N = 2^n$ by:

$$\sum_{k=0}^{2^n-1} a_k|k\rangle \longrightarrow \sum_{l=0}^{2^n-1}(\sum_{k=0}^{2^n-1} e^{2\pi ikl/2^n} a_k)|l\rangle = \sum_{l=0}^{2^n-1} \tilde{a}_l|l\rangle \ .$$

Can be written through elementary transformations:

- H_j **: Hadamard gate applied to qubit** j
- B_{jk}**: two-qubit gate applied to the qubits** j **and** k**, characterised by** $|00\rangle \rightarrow |00\rangle$**;** $|01\rangle \rightarrow |01\rangle$**;**$|10\rangle \rightarrow |10\rangle$**;** $|11\rangle \rightarrow \exp(i\pi/2^{k-j})|11\rangle$**).**

One can verify that the sequence: $\Pi_{j=1}^{n}[(\Pi_{k=j+1}^{n} B_{jk})H_j]$

gives the Fourier transform of a vector of size 2^n **in** $n(n+1)/2$ **operations.**

Compare with $\sim N \log N$ **for the classical Fast Fourier Transform!**

Grover's search algorithm

Problem: given an unstructured list of items i, find a particular item $i = j$

Classically: best solution is going through the list $\Rightarrow \sim N/2$ on average for N items

Quantum: needs an operator S which recognizes $i = j$ ($S(|j\rangle = -|j\rangle$) ("oracle")

- **Start from $|\Psi_i\rangle = 1/\sqrt{N}\sum_{i=0}^{N-1}|i\rangle$ (uniform superposition) $= \sin\theta_0|j\rangle + \cos\theta_0/\sqrt{N-1}\sum_{i\neq j}|i\rangle$ with $\sin\theta_0 = 1/\sqrt{N}$**
- **Apply $S \Rightarrow -\sin\theta_0|j\rangle + \cos\theta_0/\sqrt{N-1}\sum_{i\neq j}|i\rangle$**
- **Apply Fourier Transform + reverse all signs but for $|0\rangle$ + Fourier transform again**
- **Result: $= \sin(\theta_0+\phi)|j\rangle + \cos(\theta_0+\phi)/\sqrt{N-1}\sum_{i\neq j}|i\rangle$**
- **Iterate $\approx \sqrt{N}$ times $\Rightarrow \sin\theta \approx 1 \Rightarrow$ quantum $\sim \sqrt{N}$. Note: gain proven ($\neq$ Shor)**

can be used to solve problems where finding solutions is hard, but testing a candidate is easy

Part I

SIMULATION OF QUANTUM MAPS ON QUANTUM COMPUTERS

Simulation of quantum physical systems

Motivation

- **Many quantum mechanical problems require large Hilbert spaces**
- **Examples: many-body systems (n particles, m orbitals $\Rightarrow m^n$ states), semiclassical limit...**
- **A lot of computer time is devoted to such simulations in modern times**

History

- **Feynman (1982): Use quantum mechanical systems to simulate quantum mechanics**
- **Lloyd (1996), Abrams and Lloyd (1997): algorithms to simulate many-body systems with local interactions.**
- **Wiesner (1996), Zalka (1998): algorithms simulating Schroedinger equation**

Quantum maps

- **Simplest systems: one-dimensional systems (phase-space of dimension 2) $\Rightarrow$ needs time-dependence to allow chaotic dynamics**
- **Example:** $H(I,\theta,t) = TI^2/2 + kV(\theta)\sum\delta(t-n)$
  The variable θ represents a phase and I corresponds to classical action. Phase space=cylinder θ in $[0,2\pi]$. Free rotation, with a "kick" every period.
- **Integration over one period $\Rightarrow$ map with discretized time $\bar{I} = I - kV'(\theta);\ \bar{\theta} = \theta + T\bar{I}$, where bars denote values of (I,θ) after one iteration; $V(\theta) = \cos(\theta) \Rightarrow$ standard map.**
- **Quantization: $\bar{\psi} = U\psi$ where ψ is the wavefunction and $\bar{\psi}$ its new value after one iteration of the evolution operator $U = e^{-ikV(\theta)}e^{-iT\hat{n}^2/2}$. $\hat{n} = -i\partial/\partial\theta$.**
- **For many choices of potential V, classical dynamics becomes chaotic when $K = kT$ increases.**
- **Many complex phenomena present in physical systems can be studied in such models, despite simplicity.**

Baker's map

R. Schack Phys. Rev. A 57, 1634 (1998)

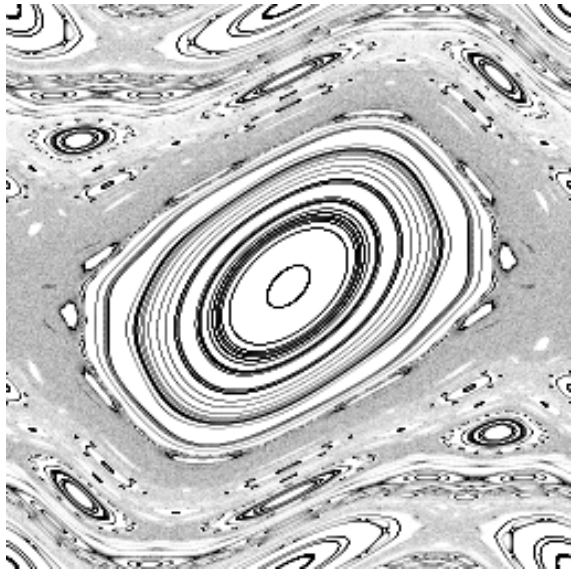
- **Not of the type discussed above**
- **Maps the unit square into itself**
- **Classical:** $(q,p) \longrightarrow (2q, \frac{p}{2})$ **for** $0 \leq q \leq \frac{1}{2}$**;** $(q,p) \longrightarrow (2q-1, \frac{p+1}{2})$ **for** $\frac{1}{2} < q \leq 1$
- **Quantum: define** F_n $N \times N$ **matrix with** $N = 2^n$ **by** $(F_n)_{kj} = \frac{e^{-\frac{2i\pi kj}{N}}}{N}$
- **Then evolution operator on a** N**-dimensional space,** $N = 2^n$ **is** $F_n^{-1} \begin{matrix} F_{n-1} & 0 \\ 0 & F_{n-1} \end{matrix}$
- F_n**=discrete Fourier transform** $\Rightarrow$ **Quantum Fourier transform** $\Rightarrow$ **can be implemented with** $n(n-1) + n(n+1)/2 = n(3n-1)/2$ **quantum gates.**
- **Implemented on a NMR quantum computer with 3 qubits (Y. S. Weinstein, S. Lloyd, J. Emerson, and D. G. Cory, Phys. Rev. Lett. 89, 157902 (2002)).**

Chirikov standard map

$$\bar{n} = n + k \sin\theta; \quad \bar{\theta} = \theta + T\bar{n} \qquad (1)$$

(n, θ) = conjugated action and angle variables, bars = variables after action of the map.

- free angle rotation + kick in momentum.
- area-preserving map
- dynamics on a cylinder (periodicity in θ), controlled by a single parameter $K = kT$.
- $K = 0 \Rightarrow$ system is integrable, all trajectories lie on one-dimensional tori (n =constant).
- $K > 0 \Rightarrow$ transition to chaos (KAM theorem).
- $K > K_g \approx 0.9716... \Rightarrow$ last torus disappears $\Rightarrow$ global chaos, classical diffusion
- Diffusion rate $D = n^2/t \approx k^2/2$



classical phase space at $K = K_g = 0.9716...$

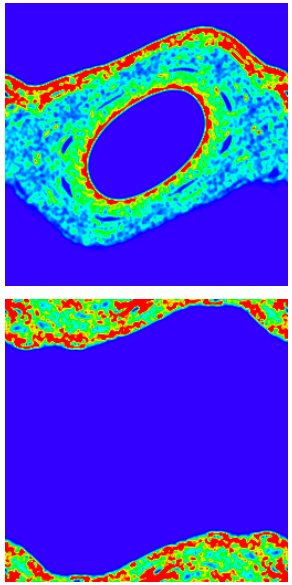
Quantum kicked rotator

Quantization of Chirikov standard map

$\bar{\psi} = \hat{U}\psi = e^{-ik\cos\hat{\theta}} e^{-iT\hat{n}^2/2}\psi$

where $\hat{n} = -i\partial/\partial\theta$, **and** $\psi(\theta + 2\pi) = \psi(\theta)$.

- **Quantum dynamics depends on two parameters** k **and** T.
- **Classical limit corresponds to** $k \to \infty$, $T \to 0$ **while keeping** $K = kT =$**constant**
- T **plays the role of an effective** $\hbar$
- $K < K_g$: **quantum diffusion limited by KAM tori**
- $K \gg K_g$: **classical diffusion replaced by quantum localization**
- **If localization length is greater than system size** $\Rightarrow$ **Random Matrix Theory**



wave packet evolution,
$K = K_g$

Simulating the kicked rotator

B. G. and D. Shepelyansky, Phys. Rev. Lett. 86, 2890 (2001)

- **paradigmatic model of quantum chaos**
- **modelizes Rydberg atoms in microwave fields**
- **enables to study Anderson localization of electrons**
- **experimentally realized with cold atoms**

On a Hilbert space of dimension $N = 2^{nq}$:

classical computation: one iteration $\Rightarrow O(N \log N)$ operations

quantum computation: one iteration $\Rightarrow O((\log N)^3)$ quantum gates

Two regimes:

a) k and T constant, N increases $\Rightarrow$ number of cells increases, with fixed number of states per cell (localization visible)

b) $T = 1/N$ and $K = kT$ constant, N increases $\Rightarrow$ number of cells fixed, number of states per cell increase (semiclassical limit visible)

Classical algorithm

on a Hilbert space of dimension N:

- **Step I action of $e^{-iT\hat{n}^2/2}$; diagonal in momentum basis**
  $\sim N$ **operations**
- **Step II Fast Fourier transform**
  $\Rightarrow$ **change from n to θ representation**
  $\sim N \log N$ **operations**
- **Step III action of $e^{-ik\cos\hat{\theta}}$; diagonal in position basis**
  $\sim N$ **operations**
- **Step IV Fast Fourier transform**
  $\Rightarrow$ **back to n representation**
  $\sim N \log N$ **operations**

Total complexity of one classical iteration $\sim N \log N$ operations; also needs $\sim N \log N$ classical bits

Quantum algorithm

- **Step I preparation of initial state:** $\psi(0) = \sum_{p=0}^{N-1} a_n |n\rangle$**; For example** $\psi(0) = |N/2\rangle$
- **Step II action of** $e^{-iT\hat{n}^2/2}$**:** $\sum_{n=0}^{N-1} a_n |n\rangle \Rightarrow \sum_{n=0}^{N-1} a_n e^{-iTn^2/2} |n\rangle$
  (by $n_q^2 = (\mathbf{Log}_2(N))^2$ **applications of two-qubit gates)**
- **Step III quantum Fourier transform:** $\sum_{p=0}^{N-1} a'_n |n\rangle \Rightarrow \sum_{i=0}^{N-1} b_i |\theta_i\rangle$ **with** $b_j = \frac{1}{N} \sum_{n=0}^{N-1} a'_n e^{2i\pi jn/N} \Rightarrow$ **change from** n **to** θ **representation**
- **Step IV construction of the cosines:** $\sum_{i=0}^{N-1} b_i |\theta_i\rangle |0\rangle \Rightarrow \sum_{i=0}^{N-1} b_i |\theta_i\rangle |\cos\theta_i\rangle$
  (needs $\sim n_q^3$ **gates;)**
- **Step V action of** $e^{-ik\cos\hat{\theta}}$**:** $\sum b_i |\theta_i\rangle |\cos\theta_i\rangle \Rightarrow \sum b_i e^{-ik\cos\theta_i} |\theta_i\rangle |\cos\theta_i\rangle$
  (by $n_q = (\mathbf{Log}_2(N))$ **applications of one-qubit gates)**
  Then $\sum_{i=0}^{N-1} b'_i |\theta_i\rangle |\cos\theta_i\rangle \Rightarrow \sum_{i=0}^{N-1} b'_i |\theta_i\rangle |0\rangle$ **(erasing the cosines)**
- **Step VI quantum Fourier transform:** $\Rightarrow$ **back to** n **representation**

Total complexity: $O((\log N)^3)$ **quantum gates; needs only** $\sim \log N$ **qubits**

Sawtooth map

G. Benenti, G. Casati, S. Montangero and D. L. Shepelyansky, Phys. Rev. Lett. 87, 227901 (2001): $\bar{\psi} = U\psi$ **with the evolution operator** $U = e^{-iT\hat{n}^2/2}e^{ik(\hat{\theta}-\pi)^2/2}$**. on a Hilbert space of dimension** $N = 2^{nq}$

- **Step I preparation of initial wavefunction:** $\psi(0) = \sum_{n=0}^{N-1} a_n|n\rangle$
  For example $\psi(0) = |N/2\rangle$
- **Step II action of** $e^{-iT\hat{n}^2/2}$**:** $\sum_{n=0}^{N-1} a_n|n\rangle \Rightarrow \sum_{n=0}^{N-1} a_n e^{-iTn^2/2}|n\rangle$
  (by $n_q^2 = (\text{Log}_2(N))^2$ **applications of two-qubit gates)**
- **Step III quantum Fourier transform:** $\Rightarrow$ **change from** n **to** θ **representation**
- **Step IV action of** $e^{ik(\hat{\theta}-\pi)^2/2}$**:** $\sum b_i|\theta_i\rangle \Rightarrow \sum b_i e^{ik(\hat{\theta}-\pi)^2/2}|\theta_i\rangle$
  (by $n_q^2 = (\text{Log}_2(N))^2$ **applications of two-qubit gates)**
- **Step IV quantum Fourier transform:** $\Rightarrow$ **back to** n **representation**

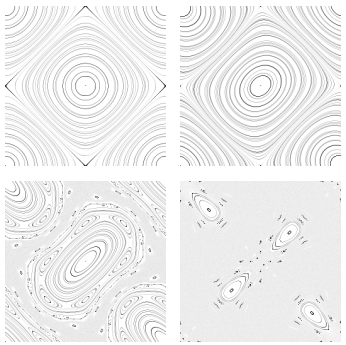
In total needs only $3n_q^2 + n_q$ **quantum gates per iteration and uses no workspace qubit.**

Anderson transition map

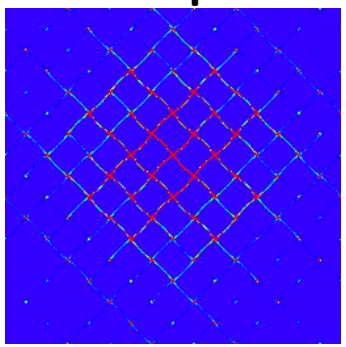
- **Kicked rotator and sawtooth map have localized states for K large enough**
- **Ergodicity of wavefunctions is realized if localization length is larger than system size**
- **Physical system can display an Anderson metal-insulator transition**
- **Can be realized in a generalized kicked rotator model described by**
  $\bar{\psi} = \hat{U}\psi = e^{-iV(\theta,t)}e^{-iH_0(n)}\psi$
  with $V(\theta,t) = k[1 + 0.75\cos(2\pi t/\lambda)\cos(2\pi t/\lambda^2)]\cos\theta$ **with** $\lambda = 1.3247...$
  and $H_0(n)$ **random phases distributed in** $(0, 2\pi)$
  $\Rightarrow$ **Anderson transition at** $k_c \approx 1.8$ **between localized and extended states**
- **Simulation of** N**-dimensional wave function can be realized in** $O((\log N)^2)$ **operations (A. Pomeransky and D. Shepelyansky, Phys. Rev. A 69, 014302 (2004)).**
- $e^{-iV(\theta,t)}$ **implemented by approximate time slice algorithm (see later)**
- $e^{-iH_0(n)}$ **implemented by "quantum random phase generator"**
  $\prod_{j=1}^{nq} e^{i\phi_j\sigma_j^z} \prod_{k=1}^{M} CNOT(i_{M-k}, j_{M-k}) \prod_{k=1}^{M} e^{i\phi'_{j_k}\sigma_{j_k}^z} CNOT(i_k, j_k)$ **where** ϕ **are classically chosen random phases. Increasing** M **increases the number of independent random phases.**

Harper and kicked Harper models

B. Lévi and B. G. Phys. Rev. E 70, 056218 (2004)
Harper model: $H_0(p,q) = \cos(p) + \cos(q)$
fractal spectrum "Hofstadter butterfly" but integrable system
Kicked Harper model:
$H(p,q,t) = L\cos(p) + K\cos(q)\sum_n \delta(t-n)$
$\Rightarrow$ **Transition to chaos as** K, L **increase**
integration over one period:
$\bar{n} = n + K\sin\theta\ ,\ \bar{\theta} = \theta - L\sin\bar{n}$
quantization:
$\bar{\psi} = \hat{U}\psi = e^{-iL\cos(\hbar\hat{n})/\hbar} e^{-iK\cos(\hat{\theta})/\hbar}\psi$
where $\hat{n} = -iq\partial/\partial\theta$ **and** $\psi(\theta + 2q\pi) = \psi(\theta)$.
$\Rightarrow$ **Motion of electrons in EM fields**
$\Rightarrow$ **Stochastic heating of plasma**



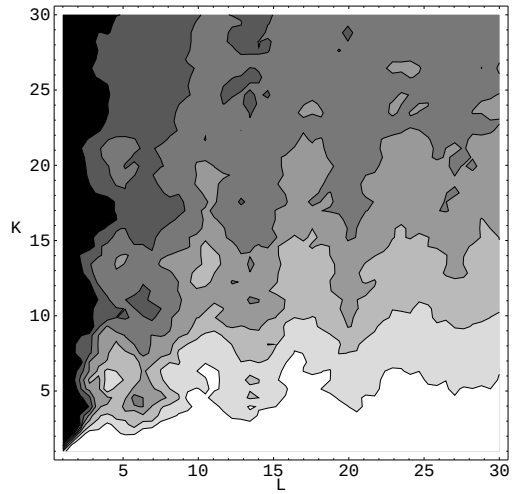
classical phase space



stochastic web

Kicked Harper: localization properties

dynamical localization $\longrightarrow$ similar to Anderson localization of electrons in solids
transition to a partially delocalized regime, with coexistence of localized and delocalized states
Figure: Inverse Participation Ratio
$\xi = 1/\Sigma_n |\psi(n)|^4$
$\xi \approx l$ **(localized state)** $\longrightarrow$ $\xi \approx N$ **(delocalized)**



On a N dimensional Hilbert space with $N = 2^{nq} \Rightarrow$ Exact algorithm (cf kicked rotator): needs $O(\log N^3)$ quantum gates for evolution of the wave function + workspace registers

Kicked Harper: time-slice algorithm

(cf A. A. Pomeransky and D. L. Shepelyansky, Phys. Rev. A 69, 014302 (2004).)

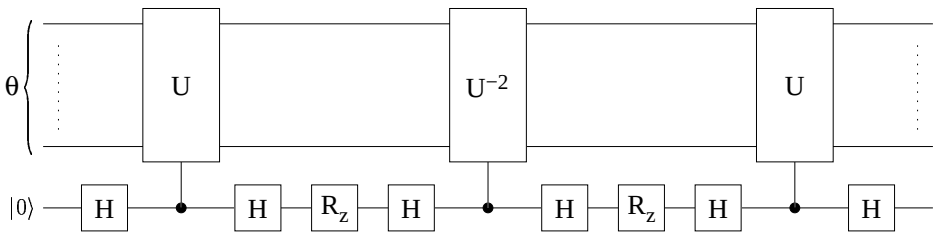
$M(\alpha, U) = HC_U He^{i\frac{\alpha}{2}\sigma_z} HC_{U^{-2}} He^{i\frac{\alpha}{2}\sigma_z} HC_U H$ **with** $C_U = U$ **controlled by ancilla**

For $U = e^{ip\theta}$ **then** $M(\alpha, U) = 1 + i\alpha\cos(p\,\theta)\sigma_z + O(\alpha^2) \approx e^{i\alpha\cos(p\,\hat{\theta})}$

Thus $e^{-ik\cos(p\,\hat{\theta})} \approx M(\alpha, U)^{n_s}$ **with** $\alpha = \frac{-k}{n_s}$**and error** $O(\alpha^2)$

More accurate with symmetrization: $\widetilde{M}(\alpha, U) = M\ \ \frac{\alpha}{2}, U\ \ M\ \ \frac{\alpha}{2}, U^{-1}$ **,**

then $e^{-ik\cos(p\,\hat{\theta})} \approx \widetilde{M}(\alpha, U)^{n_s}$ **with error** $O(\alpha^3)$



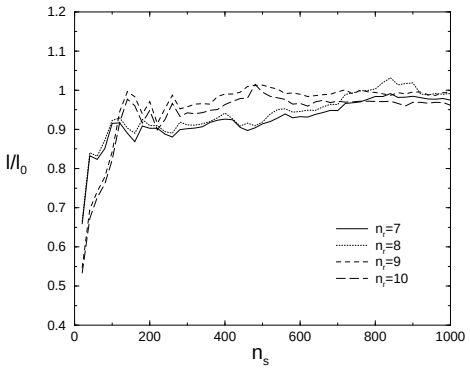
Time-slice algorithm II

For kicked Harper model on $N = 2^{n_q}$ dimensional space:

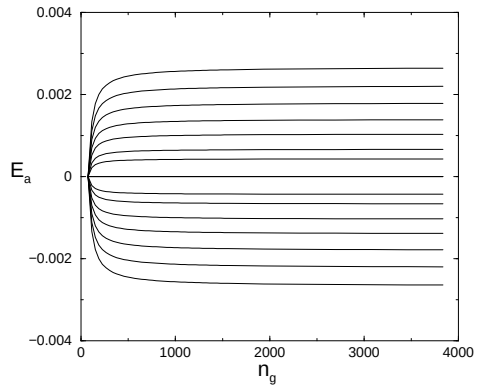
$e^{-iK\cos(\hat{\theta})/\hbar}$ **and** $e^{-iL\cos(\hbar\hat{n})/\hbar} \Rightarrow 4 + 2(n_q - a) + (n_s - 1)(7 + 2(n_q - a))$ **gates**

QFT $\Rightarrow n_q^2$ **gates**

Only one ancilla qubit! $O(\log N)^2$ **quantum gates**



localization length



eigenphases

Kicked Harper: Chebychev polynomials algorithm

$T_0(x) = 1, T_1(x) = x$, $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$

If $f(x)$ **is a function on** $[-1, 1]$**, and** $c_j = \frac{2}{M}\sum_{k=0}^{M-1} f \quad \cos \quad \frac{\pi\left(k+\frac{1}{2}\right)}{M} \quad \cos \quad \frac{\pi j\left(k+\frac{1}{2}\right)}{M}$

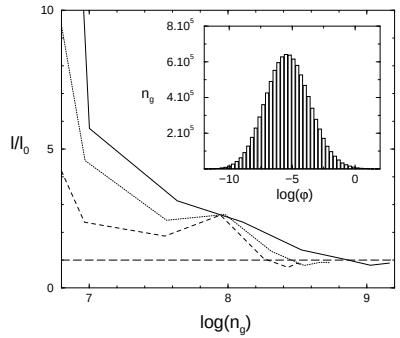
Then, for large M**,** $\sum_{j=0}^{M-1} c_j T_j(x) - \frac{1}{2}c_0$ **is a very good approximation of** $f(x)$ **on** $[-1, 1]$**.**

$P(x) \approx \cos\left(\pi(x+1)\right) \Rightarrow e^{-ik\cos\left(p\,\hat{\theta}\right)} \approx e^{-ikP \quad \frac{\hat{\theta}}{\pi}-1}$

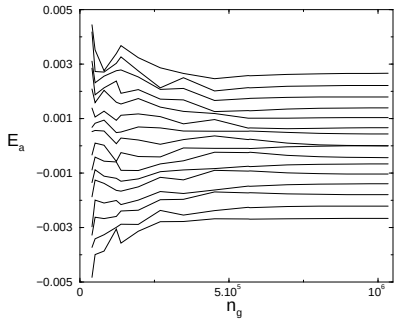
Chebyshev polynomial approximation of degree $d \Rightarrow$ **complexity is** $O({n_q}^d)$**. Numerics:** $d = 6 \Rightarrow$ **very good approximation of the wave function. (**$N = 2^{nq}$**)**

Dropping the gates with the smallest phases shortens the computation keeping a reasonable accuracy.

No ancilla qubit! $O(\log N)^d$ **quantum gates**



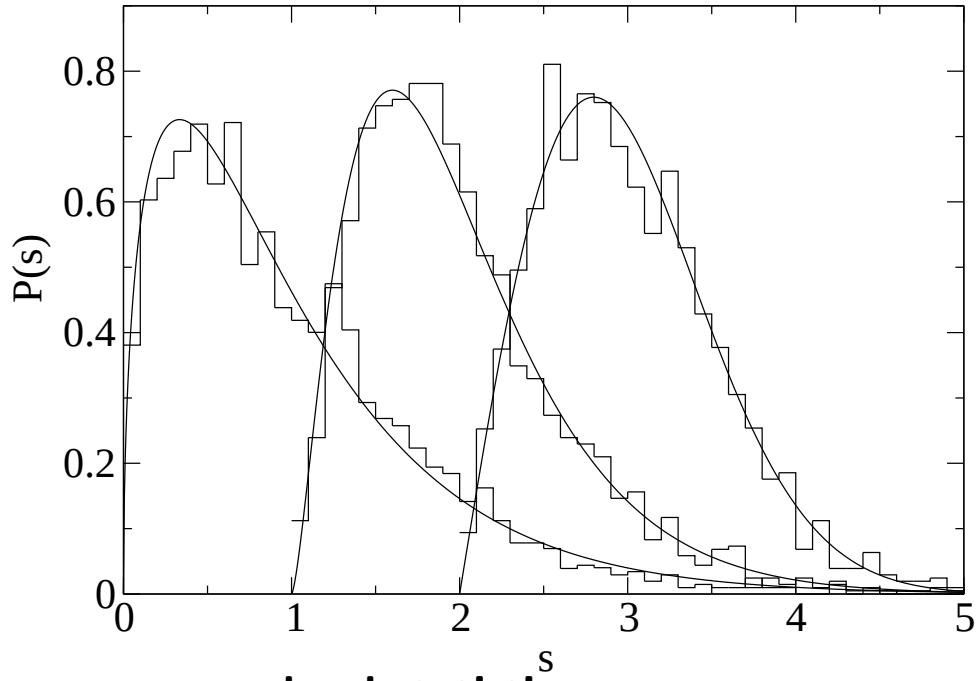
localization length



eigenphases

Intermediate statistics quantum map

(O.Giraud and B. G., quant-ph/0504230)
intermediate statistics: at Anderson metal-insulator transition, pseudointegrable systems, diffractive billiards
$\Rightarrow$ **Semi-Poisson distribution:** $P_\beta(s) \propto s^\beta e^{-(\beta+1)s}$.
Quantum map: $\hat{U} = e^{-2i\pi\hat{p}^2/N} e^{2i\pi\alpha\hat{q}}$.
α **irrational: Wigner;**
$\alpha = a/b$**: semi-Poisson with** $\beta = b/2 - 1$.
On a quantum computer, for $N = 2^{n_q}$**, requires** $2n_q^2 + 2n_q$ **gates and** n_q **qubits**
$\Rightarrow$ **very economical**



level statistics

Part II

EXTRACTING INFORMATION

Simple approach: coarse grained measurements

- **Measure only the first n_f qubits $\longrightarrow$ polynomial number of measurements gives coarse grained image of the wave function**

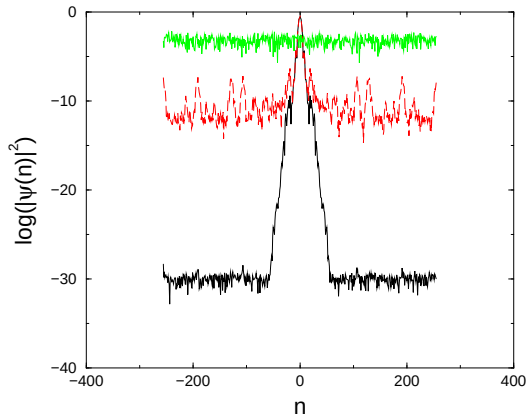
- **Possibility of exponential gain**

- **But polynomial approximation of wavefunction $\Rightarrow$ should be compared to other approximations**

- **Also, quantum wave packets have to spread enough time to get useful information $\Rightarrow$ may make the gain much smaller**

Localization length

Kicked rotator, sawtooth map and other models present localized states
$\longrightarrow$ Localization length l measured directly by fitting an exponential function around maximal values of ψ, obtained by coarse grained measurements $\Rightarrow$ effective, no extra cost (G. Benenti, G. Casati, S. Montangero and D. L. Shepelyansky, Phys. Rev. A 67, 052312 (2003))
$\longrightarrow$ But needs to evolve wave function until size $\approx l \approx D$ (D is diffusion constant)
Short time $\Rightarrow$ diffusive spreading $\langle n(t)^2 \rangle \approx Dt$
wave packet needs to be evolved until time $t^* \approx l^2/D \approx l$
Classically: to evolve a vector of dimension $\sim l$ until time $t^* \Rightarrow \sim l^2$ classical operations.
Quantum computer: total number of gates $\sim l$.
$\Rightarrow$ quadratic improvement for the quantum algorithm.



in black: example of localized wave function

Transport properties

Kicked Harper: partially delocalized regime

Coexistence of localized and delocalized wave functions; wave packet = localization peak + delocalization plateau

Diffusion constant of the plateau

a) away from $K = L$ line: anomalous diffusion (ballistic)

$\Rightarrow$ evolve wave packet up to time t^*: $\sim (t^*)^2$ operations classically, $\sim t^*$ quantum

b) on the $K = L$ line: normal diffusion

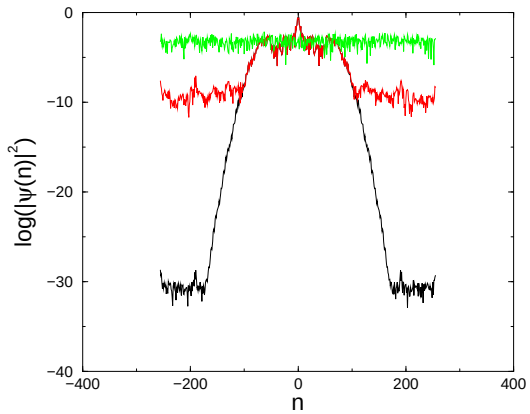
$\Rightarrow$ evolve wave packet up to time t^*: $\sim (t^*)^{3/2}$ operations classically, $\sim t^*$ quantum

$\Rightarrow$ polynomial improvement for the quantum algorithm.

Metal-insulator transition

Quadratic gain for critical point

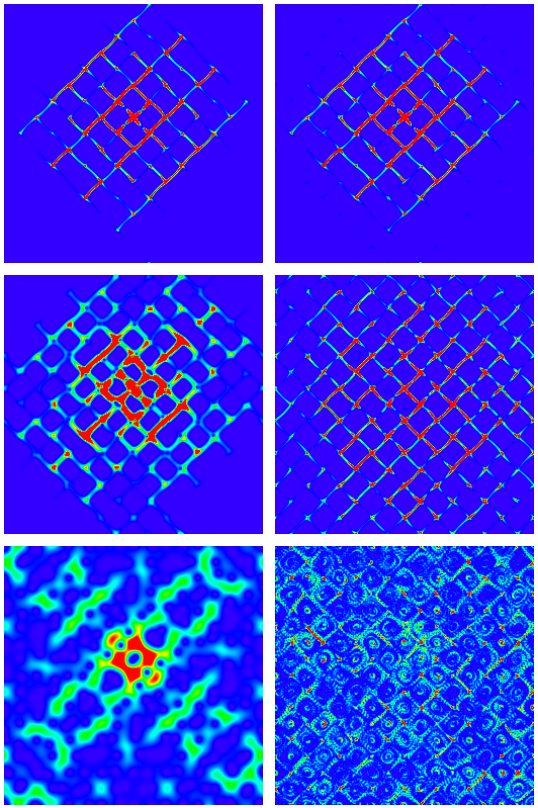
Larger polynomial gain above critical point



in black: example of wave function (kicked Harper)

Kicked Harper: quantum stochastic web

K, L very small $\Rightarrow$ small chaotic layer surrounding large integrable islands "stochastic web"
$\Rightarrow$ transport=diffusion through layer+tunneling
$\Rightarrow$ much faster than classical
Measuring diffusion constant:
$\Rightarrow$ Quantum computer: time evolution up to time t^*: $\sim t^*$ operations
$\Rightarrow$ Classical computer: time evolution up to a time t^*: $\sim t^*\sqrt{t^*}$ operations
$\Rightarrow$ Quantum gain (polynomial)
Figure: Husimi distribution for spreading of a wave packet on a stochastic web for different n_q values (left) or imperfection levels (right).



Measuring the fidelity decay

J. Emerson, Y. S. Weinstein, S. Lloyd and D. Cory, Phys. Rev. Lett. 89, 284102 (2002)

- **Needs a quantum map with evolution operator U efficiently implementable on QC + a perturbed evolution operator U_p also efficiently implementable**
- **Fidelity is $F(t) = |\langle U^t \psi_0 | U_p^t \psi_0 \rangle|^2$ where $\psi_0 = U_0|0\rangle$=initial state**
- **Then $\langle U^t \psi_0 | U_p^t \psi_0 \rangle = \langle 0 | U_0^+ (U^+)^t U_p^t U_0 | 0 \rangle$**
- **Simulating $U_0^+ (U^+)^t U_p^t U_0$ and sampling population of the state $|0\rangle$ gives the fidelity**
- **Possible exponential gain**
- **Can be used with one qubit of quantum information (D. Poulin, R. Blume-Kohout, R. Laflamme, and H. Ollivier Phys. Rev. Lett. 92, 177906 (2004))**

Measuring the form factor

D. Poulin, R. Laflamme, G. J. Milburn and J. P. Paz, Phys. Rev. A 68, 022302 (2003)

- **Needs an evolution operator U efficiently simulated on quantum computer**
- **Uses one ancilla qubit on which the value of the traces are transfered "scattering circuit"**
- **TrU^p/N correspond to expectation values of Pauli operators for the probe qubit**
- **Form factor $K(t)$= Fourier transform of two-point correlation function**
  For quantum maps, $<$ Tr$U^p >_p$ gives $K(t)$
- **Value of form factor near $t = 0 \Rightarrow$ characterizes RMT, Poisson and intermediate statistics**
  $\rightarrow$ If $<$ Tr$U^p >_p \sim \kappa\sqrt{N}$ then $K(0) = |\kappa|^2$ and statistics are integrable ($|\kappa|^2 = 1$) or intermediate ($|\kappa|^2 < 1$)
  $\rightarrow$ If $<$ Tr$U^p >_p \sim 1$ then $K(0) = 0$ (Wigner statistics)
- **Quadratic gain compared to classical computation**
- **Can be used with one qubit of quantum information**

Spectrum: phase estimation

A. Kitaev, quant-ph/9511026, D. Abrams and S. Lloyd, Phys. Rev. Lett. 83, 5162 (1999): Given an unitary operator U and an eigenvector $|u\rangle$, find efficiently the associated eigenvalue $e^{2i\pi\omega_u}$.

- **Start from** $1/\sqrt{N}\sum_{t=0}^{N-1}|t\rangle|u\rangle$
- **Transform it into** $1/\sqrt{N}\sum_{t=0}^{N-1}|t\rangle|U^t u\rangle = 1/\sqrt{N}\sum_{t=0}^{N-1}e^{2i\pi\omega_u t}|t\rangle|u\rangle$
- **Quantum Fourier transform of first register** $\Rightarrow |\omega_u\rangle|u\rangle$

Exponential gain but needs: i) an operator U whose exponentially large iterates are efficiently implementable and ii) a good approximation of one eigenvector

Note: Shor's algorithm can be reinterpreted as phase estimation on the operator $U|y\rangle = |ay \text{ mod}(N)\rangle$ (indeed Shor's algorithm computes the period r of the integer function $f(x) = a^x \text{ mod}(N)$. Since $|y_s\rangle = 1/\sqrt{r}\sum_{k=0}^{r-1}e^{\frac{-2i\pi sk}{r}}|a^k \text{ mod}(N)\rangle$ is eigenvector of U for $0 \leq s \leq r-1$, phase estimation gives the eigenvalue $e^{\frac{2i\pi s}{r}}$, which gives r)

Amplitude amplification

(Brassard, Hoyer, Mosca, Tapp, quant-ph/0005055)

$\Rightarrow$ **Generalization of Grover's algorithm. Amplitude amplification increases the amplitude of a** *whole subspace* H**. Let** P **be a projector on this subspace** H **and** $\hat{V}$ **an operator taking** $|0\rangle$ **to a state having some projection on** H**.**

$\Rightarrow$ **Repeated iterations of** $\hat{V}(I-2|0\rangle\langle 0|)\hat{V}^{-1}(I-2P)$ **on** $\hat{V}|0\rangle$ **will increase the projection. Indeed, if one write** $\hat{V}|0\rangle = P\hat{V}|0\rangle + (I-P)\hat{V}|0\rangle$**, the result of one iteration is to rotate the state toward** $P\hat{V}|0\rangle$ **staying in the subspace spanned by** $P\hat{V}|0\rangle$ **and** $(I-P)\hat{V}|0\rangle$**. One can check that after one iteration the state has a component along** $(I-P)\hat{V}|0\rangle$ **decreased by an amount which depends on** $|P\hat{V}|0\rangle|^2$

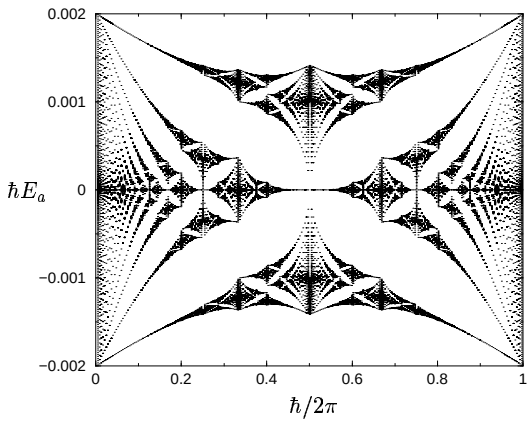
$\Rightarrow$ **If** N **is the dimension of the total Hilbert space and** M **the dimension of** H**,** $\sqrt{N/M}$ **iterations are needed to bring the probability to be in** H **close to** 1**.**

Spectrum: phase estimation + amplification

Works if i) and ii) are not fulfilled; evolution operator should be efficiently implementable.

- **Start with** $\sum_{t=0}^{N-1} |t\rangle|\psi_0\rangle$, **for example** $|\psi_0\rangle = 2^{-nq}\sum_n |n\rangle$
- **Transform it into** $2^{-nq}\sum_{t=0}^{N-1} |t\rangle|U^t\psi_0\rangle$ **in** $O(N)$ **operations,**
- **QFT of the first register** $\Rightarrow$ **peaks centered at eigenvalues of** U
- **Measurement of the first register** $\Rightarrow$ **one (possible) eigenvalue of** U **in** $O(N)$ **operations**
- **Amplitude amplification: all eigenvalues in a given interval in** $O(N\sqrt{N})$ **operations.**

Compare with $O(N^2)$ **operations classically (quantum maps),** $O(N^3)$ **(general system)**



fractal spectrum kicked Harper: eigenphases as a function of $\hbar$ $K = L = 10^{-3}$

Quantum phase space distribution function: Wigner function

$$W(p,q) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-\frac{i}{\hbar}p.q'} \psi(q+\frac{q'}{2})^* \psi(q-\frac{q'}{2}) dq'$$

Real but can take negative values

$\int_q W(p,q)dq = |\psi(p)|^2$

$\int_p W(p,q)dp = |\psi(q)|^2$

On a N-dimensional Hilbert space (e. g. kicked rotator): use $2N \times 2N$ points.

$$W(\Theta,n) = \frac{1}{2N} \sum_{m=0}^{N-1} e^{-\frac{2i\pi}{N}n(m-\Theta/2)} \psi(\Theta-m)^* \psi(m)$$

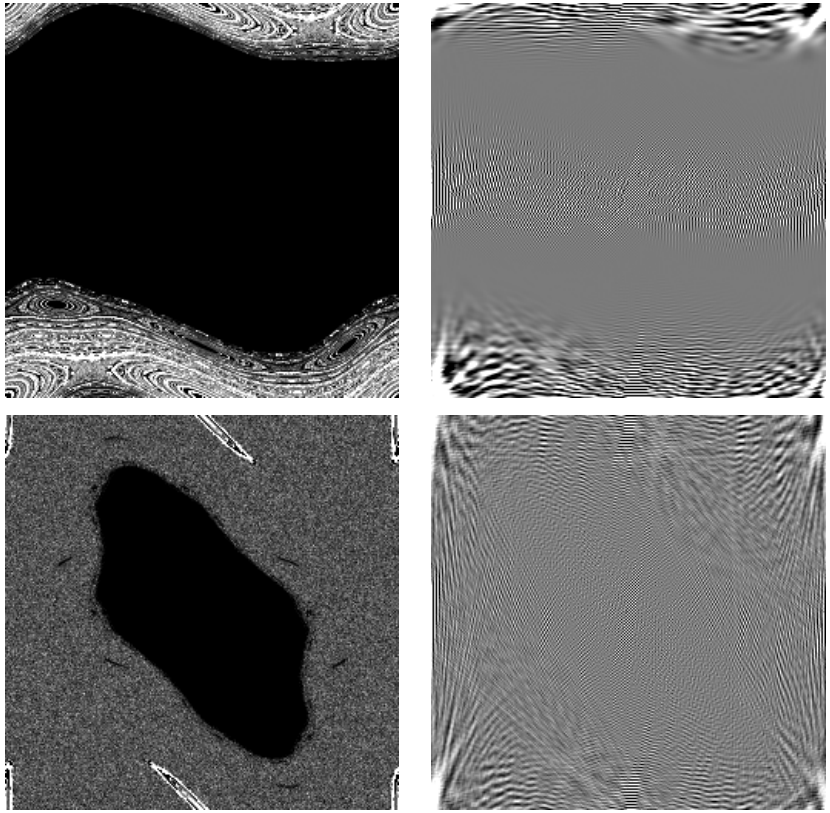


Figure: Evolution of classical phase space density (left) and Wigner function of a wavepacket (right) for kicked rotator, $K = 0.9$ (top), $K = 2$ (bottom).

Wigner function on a quantum computer: algorithm I

C. Miquel, J. Paz, M. Saraceno, E. Knill, R. Laflamme, C. Negrevergne, Nature 418, 59 (2002): Measures the Wigner function at one given location using an ancilla qubit.

- **Apply one Hadamard gate to the ancilla qubit**
- **Apply** $A(\Theta, n) = S^{\Theta} R V^{-n} \exp(2i\pi\Theta n/2N)$ **to the system conditioned by the value of the ancilla qubit (**S**=shift in** Θ **basis,** $S(|\Theta\rangle) = |\Theta + 1\rangle$**;** V**=shift in the** n **basis;** R**=reflection operator,** $R(|n\rangle) = |N - n\rangle$**)**
- **Apply one Hadamard gate to the ancilla qubit**

Expectation value of the ancilla: $\langle\sigma^z\rangle = Re[Tr(U(\Theta, n)\rho)] = 2NW(\Theta, n)$ **(**ρ **is the density matrix and** $N = 2^{nq}$ **is the dimension of the Hilbert space).**

Problem: exponential gain in principle, but practical efficiency depends on the value of W**: if very small, may require exponentially many iterations.**

Wigner function on a quantum computer: algorithm II

M. Terraneo, B. G. and D. Shepelyansky, Phys. Rev. E 71, 066215 (2005): builds a state whose amplitudes in a chosen basis gives the Wigner function.

- **Start from** $|\psi_0\rangle|\psi_0\rangle$**; transform it into** $|U^t\psi_0\rangle|U^t\psi_0\rangle = \sum_{\theta,\theta'}\psi(\theta)\psi^*(\theta')|\theta\rangle|\theta'\rangle$**.**
- **Add an extra qubit, transform into** $\sum_{\theta,\theta'}\psi(\theta)\psi^*(\theta')|\theta+\theta'\rangle|\theta'\rangle$ **(addition)**
- **Fourier transform of second register** $\Rightarrow \sum_\Theta\sum_n(\sum_{\theta'}e^{-\frac{2i\pi}{N}n\theta'}\psi(\Theta-\theta')\psi^*(\theta'))|\Theta\rangle|n\rangle$ $=2\sqrt{N}\sum_\Theta\sum_n W(\Theta,n)e^{-\frac{2i\pi}{N}n\Theta/2}|\Theta\rangle|n\rangle$ **where** $\Theta=\theta+\theta'$ **and** Θ **varies from** 0 **to** $2N-1$ **and** n **from** 0 **to** $N-1$**.**
- **Add an extra qubit in the state** $|0\rangle$**, apply Hadamard gate** $\Rightarrow$ $\sqrt{2N}(\sum_\Theta\sum_{n=0}^{N-1}W(\Theta,n)e^{-\frac{2i\pi}{N}n\Theta/2}|\Theta\rangle|n\rangle+\sum_\Theta\sum_{n=N}^{2N-1}W(\Theta,n)e^{-\frac{2i\pi}{N}(n-N)\Theta/2}|\Theta\rangle|n\rangle)$
- **Multiply by the phases** $e^{-\frac{2i\pi}{N}n\Theta/2}$ **and** $e^{-\frac{2i\pi}{N}(n-N)\Theta/2}$ $\Rightarrow$ $|\psi_f\rangle = \sqrt{2N}\sum_{\Theta=0}^{2N-1}\sum_{n=0}^{2N-1}W(\Theta,n)|\Theta\rangle|n\rangle$

Allows further data treatment (amplitude amplification, wavelet transform).

Wavelet transforms

⇒ **Wavelet transforms are recent generalizations of Fourier transform. Based on wavelet bases, which differ from the usual Fourier basis by the fact that each basis vector is localized in position as well as momentum, with different scales.**

⇒ **The basis vectors are obtained by translations and dilations of an original function and their properties enable to probe the different scales of the data as well as localized features, both in space and frequency.**

⇒ **Wavelet transforms are used in a large number of applications involving classical data treatment, in particular they allow to reach large compression rates for classical images in standards like MPEG.**

⇒ **Algorithms for implementing such transforms on quantum computers were developed and were shown to be efficient, requiring polynomial resources to treat an exponentially large vector.**

Wigner function on a quantum computer: measurements

Result of numerical simulations for kicked rotator on a N dimensional space after t iterations:

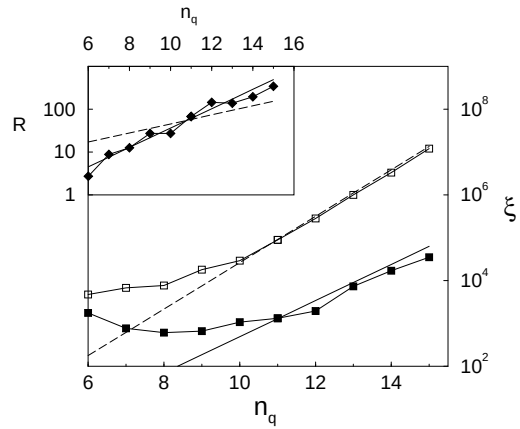
1) Wigner function IPR $\sim N^\alpha$ with $\alpha \approx 1.8 - 2$ $\Rightarrow \sim N^\alpha$ direct measurements needed $\Rightarrow$ quantum algorithm $O(tN^\alpha)$; classical $O(tN^2 \log N)$ $\Rightarrow$ Small polynomial gain

2) Coarse-grained measurement $\Rightarrow$ possible exponential gain

3) Amplitude amplification on a $N_D \times N_D$ square: $O(tN + N_D N)$ classically, $O(tN_D^\alpha N)$ quantum $\Rightarrow$ Small polynomial gain

4) Wavelet transformed Wigner function: $O(tN^\beta)$ with $1.4 \leq \beta \leq 2$ quantum, $O(tN^2 \log N)$ classical $\Rightarrow$ Larger polynomial gain

Gain is larger in chaotic regime



Inverse Participation Ratio (IPR) of Wigner function (open squares) and wavelet transform of it (full squares) for $K = 2$. Straight lines are resp. $\alpha = 1.8$ and $\beta = 1.4$.

Quantum phase space distribution function: Husimi function

Unlike classical phase space distribution function, Wigner function can be negative
Smoothing of Wigner function over cells of size $\hbar$ $\Rightarrow$ real nonnegative function

Smoothing by a Gaussian:Husimi function
Smoothing by a box (in p): modified Husimi function

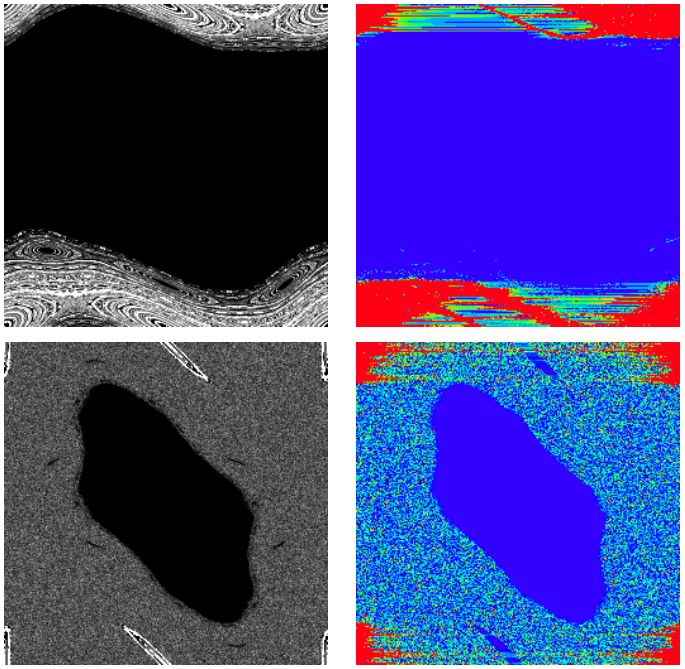


Figure: Evolution of classical phase space density (left) and modified Husimi function of a wavepacket (right) for the kicked rotator, $K = 0.9$ (top), $K = 2$ (bottom).

Husimi function on a quantum computer

Husimi function: **J. P. Paz, A. J. Roncaglia and M. Saraceno, Phys. Rev. A 69, 032312 (2004).**

$\rho_H(\theta_0, n_0) = |\langle\phi_{(\theta_0,n_0)}|\psi\rangle|^2$ **where** $\phi_{(\theta_0,n_0)}(\theta, n) = A\sum_n e^{-(n-n_0)^2/4a^2 - i\theta_0 n}|n\rangle$ **is a Gaussian coherent state centered on** (θ_0, n_0) **with width** a **(**A **is a normalization constant).**

Construct a Gaussian using ground state of Harper Hamiltonian.

Much simpler: modified Husimi function: **K. M. Frahm, R. Fleckinger and D. L. Shepelyansky, Eur. Phys. J. D 29, 139 (2004).**

$\rho_H^{(p)}(\theta_0, n_0) = |\langle\phi^{(p)}_{(\theta_0,n_0)}|\psi\rangle|^2$ **where** $\phi^{(p)}_{(\theta_0,n_0)}(\theta, n) = (1/N^{1/4})\sum_{n=n_0}^{n_0+\sqrt{N}-1} e^{-i\theta_0 n}|n\rangle$

Quantum Fourier transform to first half of the qubits: $\psi \Rightarrow |\psi_H\rangle = \sum_{\theta,n} H(\theta, n)|\theta\rangle|n\rangle$ **where** θ **and** n **take only** $\sqrt{N}$ **values each and** $|H(\theta, n)|^2$ **is the modified Husimi function.**

Modified Husimi function: measurements

Result of numerical simulations for kicked rotator on a N dimensional space after t iterations:
1) Husimi function $O(tN^{\gamma})$ with $0.5 \leq \gamma \leq 0.7$; classical $O(tN)$ $\Rightarrow$ Polynomial gain for direct measurements
2) Coarse-grained measurement $\Rightarrow$ possible exponential gain
3) Amplitude amplification on a $N_D \times N_D$ square: $O(t\sqrt{N}N_D^{\gamma-1/2})$ quantum, $O(tN)$ classical $\Rightarrow$ Polynomial gain, independent of IPR
4) Wavelet transformed Husimi function $\Rightarrow$ Polynomial gain

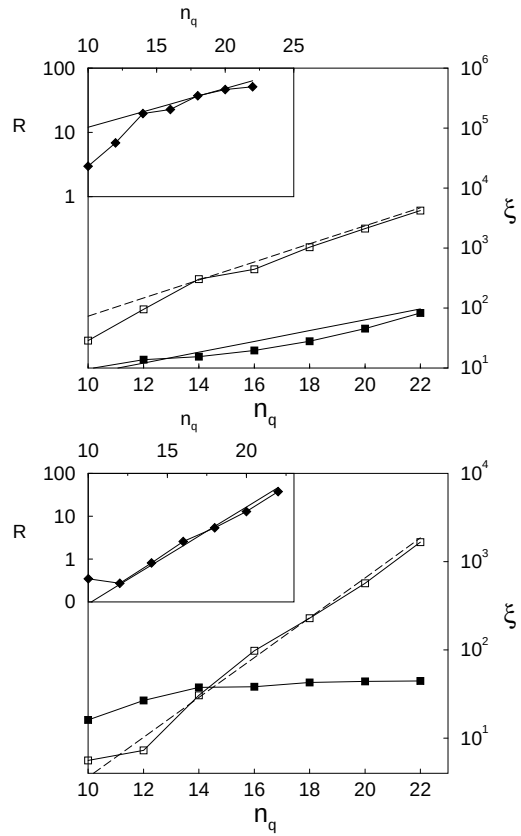


Figure: IPR of Husimi function and wavelet transform of it for $K = 0.9$, $K = 2$; dashed lines: $N^{0.5}, N^{0.7}$

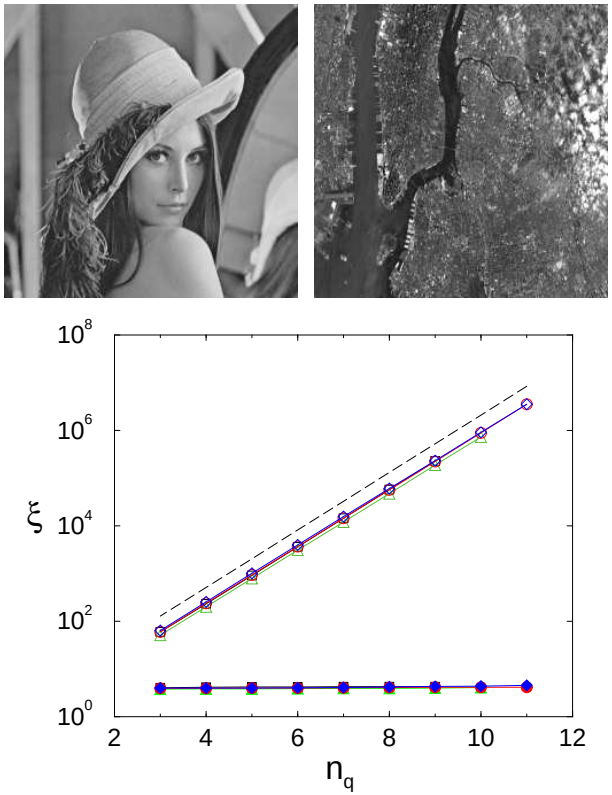
Standard images

Numerical simulations $\longrightarrow$ wavelet transform is very efficient at compressing information for standard images (Figure: number of significant components for image and wavelet-transformed image)

Largest wavelet coefficients can be obtained in polynomial time for exponentially large images. Total gain depends on efficiency of encoding the image

Reconstruction of image from such largest wavelet coefficients leads to large loss of information

Figure: Example of images; IPR of coefficients of the images, and of their wavelet transform.



Concluding remarks

- **Quantum maps: especially simple dynamics, yet complex behaviour**
- **Adapted to small-size quantum computers; some of them already implemented.**
- **At least polynomial gain can be obtained; possibly exponential**
- **Have inspired pseudorandom operators, which can produce efficiently quasi-random vectors on quantum computers (J. Emerson, Y. Weinstein, M. Saraceno, S. Lloyd and D. Cory, Science 302, 2098 (2003)).**