Quantum Communication Cost of Preparing Multipartite Entanglement

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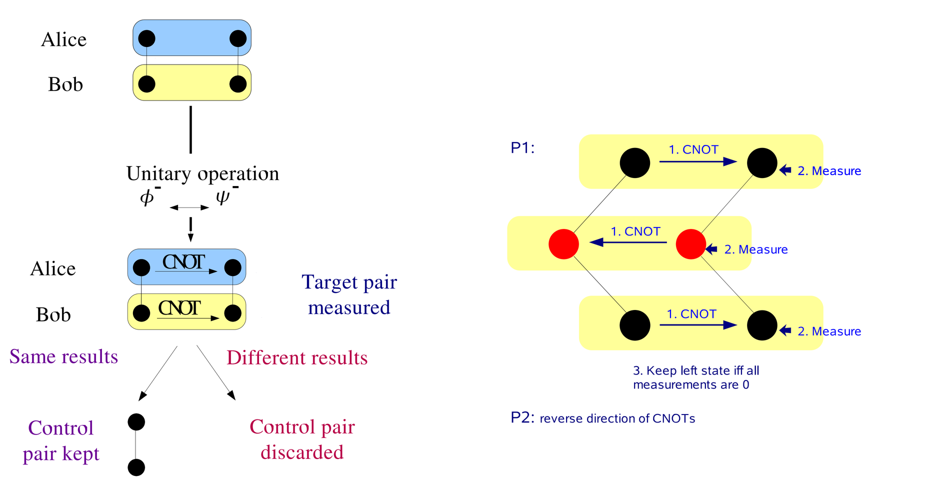
Abstract

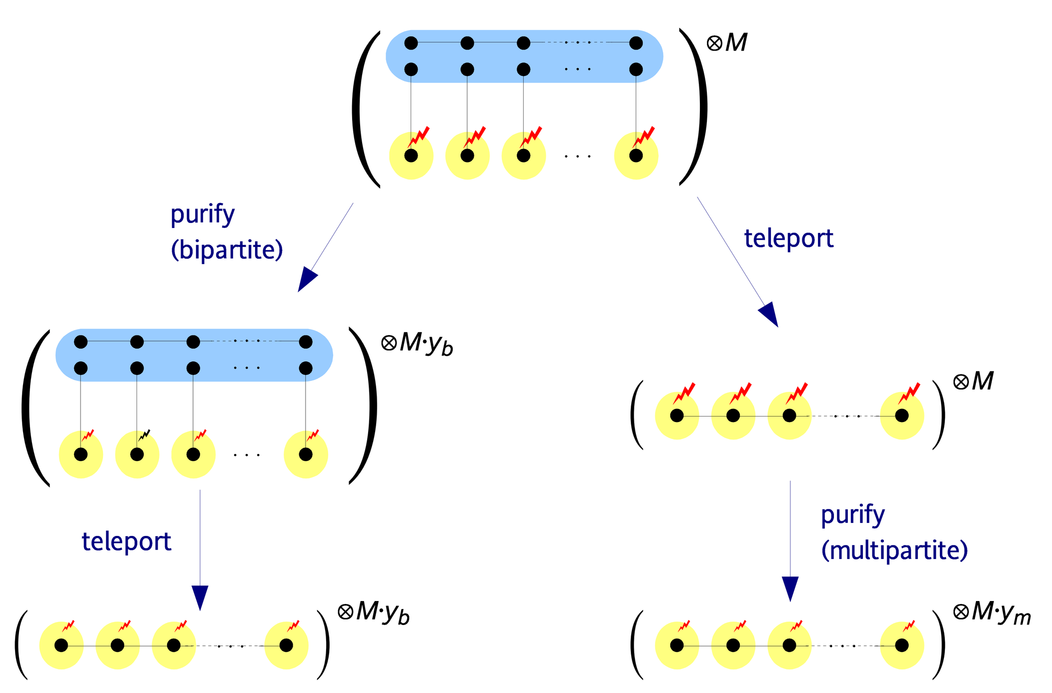
We study the preparation and distribution of high-fidelity multi-party entangled states. There are several possibilities to do that, and which is most efficient depends on the target fidelity one whishes to achieve and on the quality of transmission channel and local operations at one's disposal. We show how to choose the optimal strategy for a variety of settings.

Entanglement Purification Protocols

Bipartite purfication (Oxford protocol) | **Multipartite purification** (DAB03; ADB04)

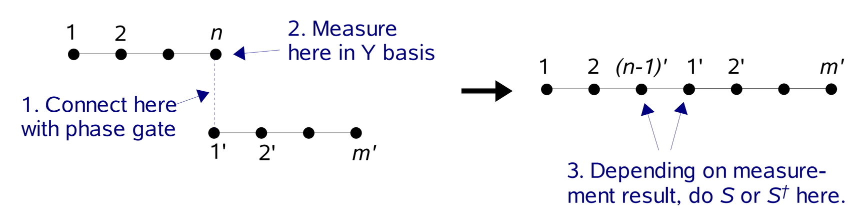


Two Strategies to Distribute High Fidelity Entanglement

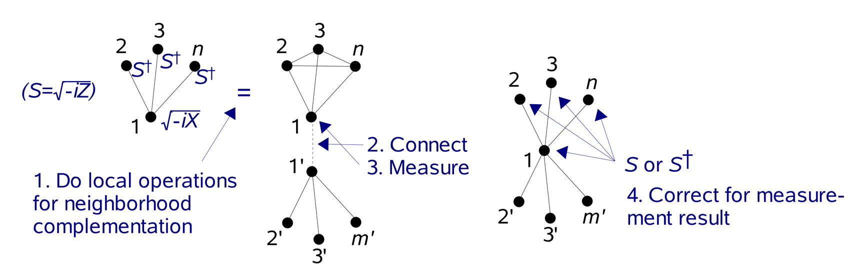


Connecting graph states

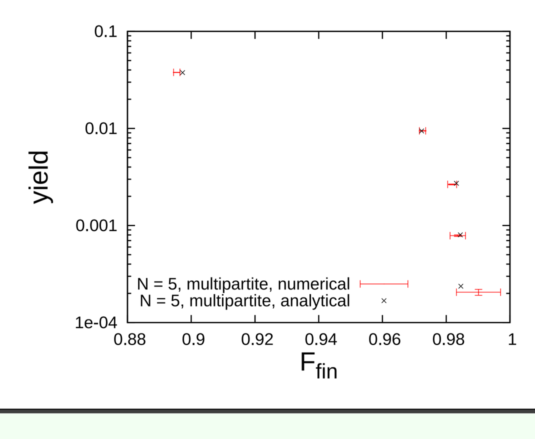
It is easy to connect an n-qubits **1D cluster state** and an m-qubits cluster state to an $(n+m-1)$-qubits cluster state:



To connect **GHZ states**, we need an additional step:



Comparison of analytical and numerical calculations for the simplified noise model:



Noise Model and Monte Carlo Simulation

Every two-qubit operation is followed by the noise map $\rho \mapsto (1-p_l)\rho + p_l \frac{1}{2}$ acting on the affected qubits. A transmission through the channel is modelled as $\rho \mapsto (1-p)\rho + p\frac{1}{2}$.
To simulate the effect of this noise, a Monte Carlo technique is employed:
After each operation, the simulator may add an error event. This happens with probability p_l and is –with equal probabilities– one of the operators in $\mathcal{P} = \{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$.
At the end, the remaining states $|\psi\rangle$ are checked by undoing the graph preparation operations and measuring: If

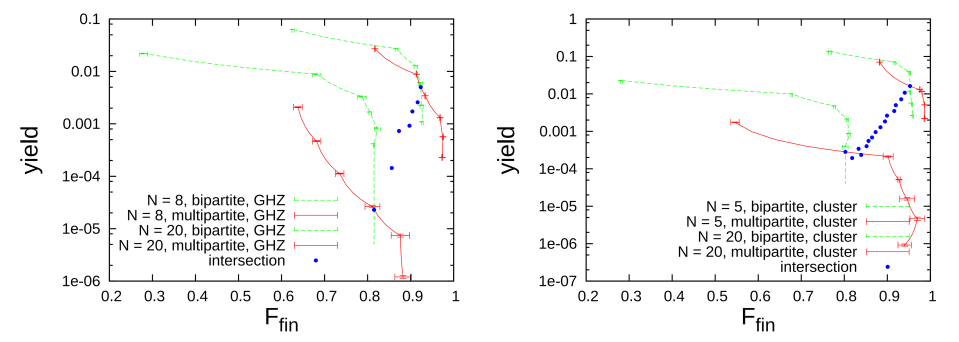
$$\prod_{i \in V} H_i \prod_{\{i,j\} \in E} \Lambda Z_{ij} |\psi\rangle = |0\rangle^{\otimes N},$$

the state is counted as "good", otherwise "bad" (i. e. erroneous). After summing up counts over many runs, we estimate

$$\text{yield} = \frac{\text{good} + \text{bad}}{\text{initial}}, \qquad \text{fidelity} = \frac{\text{good}}{\text{good} + \text{bad}}$$

Applicability Range of Both Strategies

Examples for $q = 0.9$ (i. e. 10% channel noise) and $q_l = 0.99$ (i. e. 1% local noise) — *preliminary data, unverified*



For high target fidelities, multipartite purification gives better yield than bipartite one. The blue disks mark the fielity/yield values where this break-even occurs for different numbers N of qubits.

Analytical recurrence formulae

We could obtain analytical recurrence formulae to describe the multipartite purification protocol for GHZ states for a simplified noise model. Alice's qubit only suffers bit-flipping noise, the other qubits only phase-flipping noise. Then, the state keeps the diagonal, partly permutation-invariant form

$$r_0 |\mathbf{0}\rangle\langle\mathbf{0}| + r_1 \sum_{i_1=2}^{N} |0\ldots0\overset{\overset{i_1}{\downarrow}}{1}0\ldots\rangle\langle 0\ldots0\overset{\overset{i_1}{\downarrow}}{1}0\ldots| + \\ + r_2 \sum_{\substack{i_1,i_2=2\\ i_1<i_2}}^{N} |0\ldots\overset{\overset{i_1}{\downarrow}}{1}\ldots\overset{\overset{i_2}{\downarrow}}{1}\ldots\rangle\langle\ldots| \cdots + r_{N-1}|011\ldots1\rangle\langle\ldots|,$$

and the effect of noise can be described by a recurrence formula for the coefficients r_i:

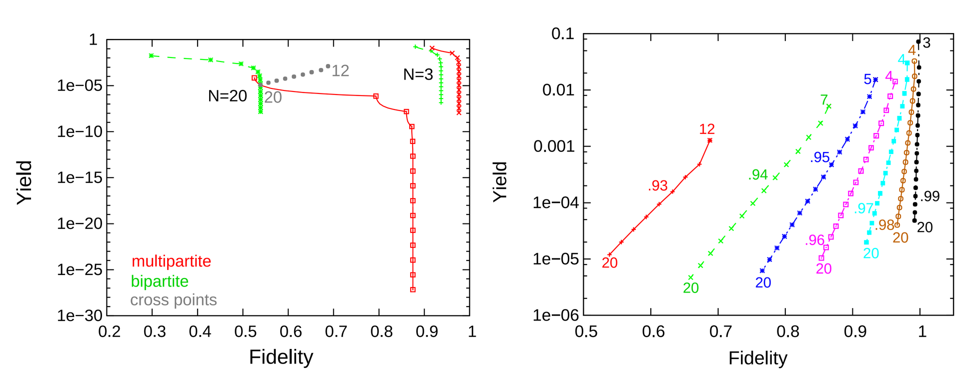
$$r'_m = \sum_{l=0}^{N-1} \lambda_{ml} r_l,$$

$$\lambda_m l = \sum_{s=0}^{l} \frac{\binom{N-1}{l}}{\binom{N-1}{m}} \binom{l}{s} \binom{N-1-l}{m-l+s} (1-q)^{m-l+2s} q^{N-1-m+l-2s}$$

The subsequent purification step can be written this way:

$$r''_m = q r'_m + (1-q) r'_{N+1-m}$$

We can use this formulae to study the protocols, and to check our numerics (switched to the simplified noise model) against it.



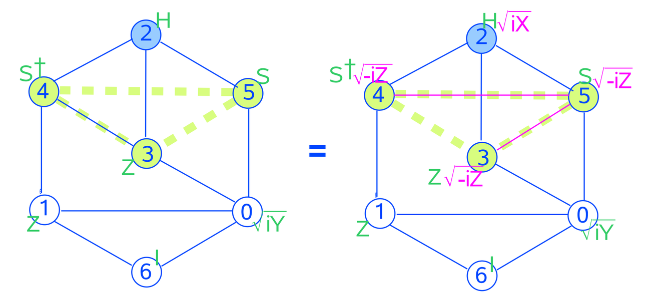
Results of the analytical model for GHZ states, 10% channel noise, and 7% local noise (left), or 7% to 1% local noise (right).

The Graph State Simulator

Entanglement purification protocols use only Clifford gates. According to the Gottesman-Knill theorem, they can hence be simulated efficiently on a classical computer.
We have developped such a simulator (AnBr05). Different from the program of Aaronson and Gottesman (AaGo04), it does not directly rely on the stabilizer formalism, but represents a state as

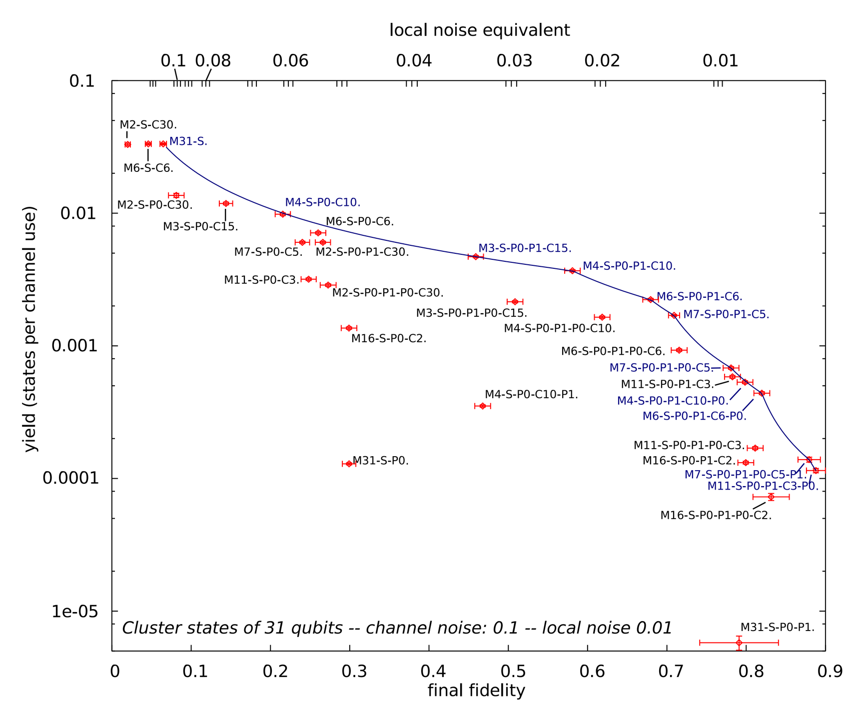
- a graph (given as adjacency list), representing a graph state
- a list of local Clifford operators, which transform this state into the actual one.

This leads to signifanctly improved performance.
Measurements are simulated using the rules of (HEB03). Single qubit gates just change the local Clifford operators, and the two-qubit phase gate changes the graph. For the latter, a clever use of the "LU rule" (HEB03; NDM03) is essential.



Intermediate Strategies

The optimal strategy for purification depends on a trade-off between yield and fidelity. Often, it is of advantage to first produce medium-sized states, purify them, connect, and repurify them.



1D cluster states of 31 qubits; channel noise 10%, local noise 1% (preliminary data, unverified)

Legend for the "instruction string": e. g. "M4-S-P0-C4-P1" means **M**ake **4**-qubit states, **S**end them through the channel, **P**urify once (direction **0**), **C**onnect **4** states to one, **P**urify again (direction **1**).

References

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