Overview of Lecture 1

- Quantum Computers
  - Requirements for quantum computation
  - Current proposals for quantum computation
  - Advantage’s of using optics
- Quantum optics
  - The basics
  - Physical quantum optical states
  - Linear Optics
- Previous Suggestions to use optics for QC
  - Kerr nonlinearity
  - Cavity QED
Quantum Computation

DiVincenzo’s 5 criteria [DiVincenzo 2000]

- A scalable physical system with well characterized qubits.
- The ability to initialize the state of the qubits to a simple fiducial state, such as $|000\ldots\rangle$.
- Long relevant decoherence times, much longer than the gate operation time.
- A “universal” set of quantum gates.
- A qubit-specific measurement capability.

Why optics?

Advantages

- Quantum optics is a well developed field. It is already being widely used for quantum information processing.
- Photons decohere slowly.
- Photons travel well, one reason why they are so widely used for communication.
- The technology is well developed for other purposes.
- Photons can be experimented with at room temperature.

Disadvantages

- Photon do not interact. Two qubit gates are difficult.
- Mode matching in single photon, single photon source/detectors.
Quantum Optics

- **Classical Electromagnetic field**
  - Look for solutions to the source free Maxwell equations $\nabla^2 E(r, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E(r, t) = 0$
  - Solutions have plus and minus frequency part
  - $E(r, t) = i \sum_k \left( \frac{\hbar \omega_k}{2} \right)^{\frac{1}{2}} [a_k u_k(r)e^{-i\omega_k t} + a_k^* u_k^*(r)e^{i\omega_k t}]$
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- **Quantize**
  - Quantize by introducing commutation relations
    - $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$
    - $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$
  - Energy is now given by $\hat{H} = \sum_k \hbar \omega_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2})$
  - If we let
    - $\hat{a} = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{\hbar}{m \omega}} \hat{x} + i \sqrt{\frac{\hbar}{m \omega}} \hat{p} \right)$
  - Energy becomes $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$: Harmonic Oscillator
  - $\hat{a}$ is called an annihilation operator, and $\hat{a}^\dagger$ a creation operator.

![Image of equations and text]

- The eigenstates of $\hat{H}$ are labelled $|n\rangle$ and we call them Fock states. We define $\hat{n}|n\rangle = \hat{a}^\dagger \hat{a}|n\rangle = n|n\rangle$. Also, $\hat{a}|0\rangle = 0$ since $\hat{H}|0\rangle = \frac{1}{2} \hbar \omega |0\rangle$.
- What is the effect of $\hat{a}$ on $|n\rangle$? We can answer this by finding the number of photons in $\hat{a}|n\rangle$:
  - $\hat{n}(\hat{a}|n\rangle) = \hat{a}^\dagger \hat{a}^2|n\rangle = (\hat{a} \hat{a}^\dagger \hat{a} - \hat{a})|n\rangle = (n - 1)$
  - so $\hat{a}|n\rangle$ is a Fock state with $n - 1$ photons. We define this to be $A|n - 1\rangle$.
- In a similar way we can show that $\hat{a}^\dagger|n\rangle$ is a Fock state with $n + 1$ photons, $B|n + 1\rangle$.
- But what is the form of $A$ and $B$?
Use the fact that \( \langle n|\hat{n}|n\rangle = n \):
\[
\langle n|\hat{n}|n\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = A\langle n|\hat{a}^\dagger|n-1\rangle = A^2 = n
\]
\[
\langle n|\hat{n}|n\rangle = A\langle n|\hat{a}^\dagger - 1|n-1\rangle = B^2 - 1 = n
\]

So we see that \( \hat{a} \) acts as an annihilation operator and \( \hat{a}^\dagger \) as a creation operator:
\[
\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle
\]
\[
\hat{a}|n\rangle = \sqrt{n}|n-1\rangle
\]

In matrix form:
\[
\hat{a} = \begin{pmatrix}
0 & \sqrt{1} & 0 & 0 & \cdots \\
0 & 0 & \sqrt{2} & 0 & \cdots \\
0 & 0 & 0 & \sqrt{3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Remember that we previously showed that
\[
\hat{a} = \frac{1}{\sqrt{2}}(\sqrt{\frac{m\omega}{\hbar}}\hat{x} + i\sqrt{\frac{1}{m\hbar\omega}}\hat{p})
\]

The Fock states form and orthonormal set: \( \langle n|m\rangle = \delta_{nm} \), \( \sum_{n=0}^{\infty} |n\rangle\langle n| = I \). Also, each Fock state may be built up from creation operators: \( |n\rangle = \left(\frac{\hat{a}^\dagger}{\sqrt{n}}\right)^n|0\rangle \).

Exercise show that \( |n\rangle = \left(\frac{\hat{a}^\dagger}{\sqrt{n}}\right)^n|0\rangle \).

The matrix form of the annihilation operator can be seen by using the identity operator twice:
\[
\hat{a} = I\hat{a}I = \sum_{n=0}^{\infty} |n\rangle\langle n|\hat{a} \sum_{m=0}^{\infty} |m\rangle\langle m|
\]
\[
= \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|
\]

It is often more convenient to write the annihilation operator as a linear combination of two Hermitian operators:
\[
\hat{a} = \frac{\hat{Q}_1 + i\hat{Q}_2}{2}
\]

where \( \hat{Q}_1 \) (equivalent to \( \hat{x} \) from before) corresponds to the in-phase component of the electric field amplitude of the spatial-temporal mode and \( \hat{Q}_2 \) (equivalent to \( \hat{p} \) from before) corresponds to the out-of-phase component.

This gives the commutation relation \( [\hat{Q}_1, \hat{Q}_2] = 2i \) and the Heisenberg uncertainty principle
\[
\Delta \hat{Q}_1 \Delta \hat{Q}_2 \geq 1
\]
Exercise: Show the last line for the Coherent state definition is correct using the Campbell-Baker-Hausdorff operator identity:
\[ e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{i}{2}[[\hat{A},\hat{B}]]} \]
for the case when \([\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0\)

- Exercise: show that \(\hat{a}D(\alpha)|0\rangle = \alpha D(\alpha)|0\rangle\)

Working out \(\Delta \hat{Q}_1 \Delta \hat{Q}_2\) for a coherent state, where
\(\langle \Delta \hat{Q}_1 \rangle^2 = \langle \hat{Q}_1^2 \rangle - \langle \hat{Q}_1 \rangle^2\)
We find \(\Delta \hat{Q}_1 = \Delta \hat{Q}_2 = 1\), \(\Delta \hat{Q}_1 \Delta \hat{Q}_2 = 1\)

Coherent states are not orthogonal:
\[ \langle \beta | \alpha \rangle = \exp[-\frac{1}{2}(|\alpha|^2 + |\beta|^2) + \alpha\beta^*] \]

**Minimum Uncertainty States**

- Heisenberg Uncertainty Principle: for two non-commuting operators \(\hat{A}\) and \(\hat{B}\):
  \[ \Delta A \Delta B \geq \frac{1}{2} | [\hat{A}, \hat{B}] | \]
- Example: \(\hat{\Delta} = \hat{x}\) and \(\hat{\Delta} = \hat{p}\), so \(\Delta \hat{x} \Delta \hat{p} \geq \hbar \)

A coherent state \(|\alpha\rangle\) is a minimum uncertainty state
- Defined as: \(\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\)
- Alternatively
  \[ |\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})|0\rangle = \exp(-\frac{|\alpha|^2}{2}) \exp(\alpha \hat{a}^\dagger)|0\rangle \]

And form an over complete set
\[ \int |\alpha\rangle\langle \alpha | d^2\alpha = \pi \]

Looking at the wave functions for a coherent state: \(\langle Q_1 | \alpha \rangle = \psi_\alpha(Q_1)\) and \(\langle Q_2 | \alpha \rangle = \tilde{\psi}_\alpha(Q_2)\)
\[
\psi_\alpha(Q_1) = \pi^{-\frac{1}{4}} \exp\left( -\frac{(Q_1 - q_0)^2}{4} + \frac{i}{4} p_0 Q_1 - \frac{i}{8} p_0 q_0 \right)
\]
\[
\tilde{\psi}_\alpha(Q_2) = \pi^{-\frac{1}{4}} \exp\left( -\frac{(Q_2 - p_0)^2}{4} - \frac{i}{4} q_0 Q_2 + \frac{i}{8} q_0 p_0 \right)
\]
where we have decomposed the complex amplitude \(\alpha\) into its real and imaginary parts: \(\alpha = \frac{1}{2}(q_0 + ip_0)\).
We see that it looks like a Gaussian in \(Q_1, Q_2\) phase space.
Linear Optics

What does “Linear Optics” mean?

An optical component is said to be linear if its output modes $\hat{b}_j^\dagger$ are a linear combination of its input modes $\hat{a}_j^\dagger$

$$\hat{b}_j^\dagger = \sum_k M_{jk} \hat{a}_k^\dagger$$

Linear optical components are made up of phase shifters and beam splitters

Phase Shifter

A phase shifter is defined by the transformation $U(P_\theta) : |n\rangle \rightarrow e^{in\phi}|n\rangle$. That is, $(\hat{a}_l^\dagger)^n|0\rangle \rightarrow e^{in\phi}(\hat{a}_l^\dagger)^n|0\rangle$.

Beam Splitter

A beam splitter is defined by the transformation $U(P_{\theta})$.

Squeezed States

Is a state with either $\Delta Q_1 < 1 < \Delta Q_2$ or $\Delta Q_2 < 1 < \Delta Q_1$?

Defined with the Squeezed operator

$$S(\epsilon) = \exp \left( \frac{1}{2} \epsilon^* \hat{a}^2 - \frac{1}{2} \epsilon (\hat{a}^\dagger)^2 \right)$$

where $\epsilon = re^{i\phi}$

Squeezed coherent state $|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)|0\rangle$

If we define a rotated complex plane: $\hat{P}_1 + i\hat{P}_2 = (\hat{Q}_1 + i\hat{Q}_2)e^{-\frac{i}{2}\phi}$

$\Delta \hat{P}_1 = e^{-r}$ and $\Delta \hat{P}_2 = e^r$

Looking at the wave functions for a squeezed coherent state $\langle Q_1|D(\alpha)S(\epsilon)|0\rangle$:

$$\pi^{-\frac{1}{2}}e^{\frac{\epsilon}{2}}\exp\left(-e^{2\epsilon}(Q_1 - q_0)^2 + \frac{i}{4}Q_2Q_1 - \frac{i}{8}p_0q_0\right)$$

where we have decomposed the complex amplitude $\alpha$ into its real and imaginary parts: $\alpha = \frac{1}{2}(q_0 + ip_0)$.

We see that it also looks like a Gaussian in $Q_1, Q_2$ phase space.

The Hamiltonian for a phase shifter is given by the number operator: $U = e^{i\phi H} = e^{i\phi \hat{n}} = e^{i\phi \hat{a}^\dagger \hat{a}}$. So we can see that $e^{i\phi \hat{a}^\dagger \hat{a}}|n\rangle = e^{in\phi}|n\rangle$

Exercise: show that the phase operator takes a coherent state $|\alpha\rangle$ to $|e^{i\phi \alpha}\rangle$.

The Hamiltonian for a beam splitter is given by the matrix

$$U = \begin{pmatrix} \cos \theta & -e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & \cos \theta \end{pmatrix}$$

where the input modes are related to the output modes via $a_i^\dagger|0\rangle \rightarrow \sum_m U_{im}a_m^\dagger|0\rangle$.

Exercise: The Hamiltonian for a beam splitter can be given by $\hat{H} = e^{i\phi \hat{a}^\dagger \hat{b} + e^{-i\phi \hat{a}\hat{b}^\dagger}}$. Exercise: check that this is Hermitian.
The unitary transformation for a beam splitter then looks like $\exp\left(-i\theta(e^{i\phi}\hat{a}_k^\dagger\hat{a}_l + e^{-i\phi}\hat{a}_k\hat{a}_l^\dagger}\right)$ where $k$ and $l$ are the two modes being acted upon by the beam splitter.

Some examples

$|10\rangle \rightarrow \cos \theta |10\rangle + e^{i\phi} \sin \theta |01\rangle$

$|01\rangle \rightarrow -e^{i\phi} \sin \theta |10\rangle + \cos \theta |01\rangle$

$|11\rangle \rightarrow -\sqrt{2}e^{i\phi} \cos \theta \sin \theta |20\rangle + (\cos^2 \theta - \sin^2 \theta)|11\rangle$ $+ \sqrt{2}e^{-i\phi} \cos \theta \sin \theta |02\rangle$

$|20\rangle \rightarrow \cos^2 \theta |20\rangle + e^{-i\phi} \sqrt{2} \cos \theta \sin \theta |11\rangle + e^{-2i\phi} \sin^2 \theta |02\rangle$

$|02\rangle \rightarrow e^{2i\phi} \sin^2 \theta |20\rangle - e^{i\phi} \sqrt{2} \cos \theta \sin \theta |11\rangle + \cos^2 \theta |02\rangle$

Exercise: confirm these examples.

This uses a nonlinear crystal, one whose index of refraction $n$ is proportional to the total intensity: $n = n_0 + n_2 E^2 = n_0 + n_2 I$, where $n_0$ is the normal refractive index and $n_2$ is the correction term necessary for Kerr materials.

The Kerr Hamiltonian is of the form

$$H_I = \hbar \chi \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2$$

The original proposal for a Fredkin gate is a piece of Kerr media in either arm of a Mach-Zehnder interferometer, as shown above.

In one arm the Kerr media has the control beam incident on it.
The beam splitter on the left side has $\theta = \frac{\pi}{4}, \phi = 0$ and the right hand side $\theta = \frac{\pi}{4}, \phi = \pi$. The effect of the Kerr media is to act the unitary operator $U = \exp \left( i\epsilon H_f \right)$. Remember that $e^{i\phi_n} |m\rangle = e^{im\phi} |m\rangle$

For example, look at $|010\rangle$

$$|010\rangle \rightarrow \frac{1}{\sqrt{2}}(|010\rangle + |001\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}} e^{i\epsilon c a^\dagger a} e^{i\epsilon b^\dagger b d^\dagger d} (|010\rangle + |001\rangle)$$

$$= \frac{1}{\sqrt{2}} (e^{i\epsilon c a^\dagger a} |010\rangle + e^{i\epsilon b^\dagger b d^\dagger d} |001\rangle)$$

$$= \frac{1}{\sqrt{2}} (|010\rangle + |001\rangle) \rightarrow |010\rangle$$

where mode $d$ is always in the vacuum mode $|0\rangle$.

Next look at $|101\rangle$

$$|101\rangle \rightarrow \frac{1}{\sqrt{2}}(|101\rangle - |110\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}} (-e^{i\epsilon c a^\dagger a} |110\rangle + e^{i\epsilon b^\dagger b d^\dagger d} |101\rangle)$$

$$= \frac{1}{\sqrt{2}} (-e^{i\epsilon} |110\rangle + e^{i\epsilon} |101\rangle + |101\rangle + |110\rangle)$$

Problems with this scheme: (1) difficult to achieve the high non-linearities, especially those required for a $\pi$ phase change; (2) at high non-linearities the crystal exhibits other detrimental affects, such as absorption.

Related proposal using Kerr nonlinearity

In Chuang and Yamamoto showed how to implement one-bit Deutsch's problem from multiple Fredkin gates

Only need one Kerr crystal for a Fredkin gate

The simplified gate is given below. The left hand and right hand beam splitter have $\theta = \frac{\pi}{4}, \phi = 0$ and $\theta = \frac{\pi}{4}, \phi = \pi$.

Logic here is the same as before, vacuum is logical 0 and the 1 photon Fock state is logical 1.

The input modes are $|c, a, b\rangle$, where mode $c$ is the control.

Let's work through an example

$$|110\rangle \rightarrow \frac{1}{\sqrt{2}}(|110\rangle + |101\rangle)$$

$$\rightarrow \frac{1}{\sqrt{2}} (e^{i\chi c a^\dagger a} |110\rangle + |101\rangle)$$

$$= \frac{1}{2} (e^{i\chi} |110\rangle - e^{i\chi} |101\rangle + |101\rangle + |110\rangle)$$

$$= \frac{1}{2} (1 + e^{i\chi}) |110\rangle + \frac{1}{2} (1 - e^{i\chi}) |101\rangle$$

If we choose $\chi = \pi$ we end up with the state $|101\rangle$, as required for a Fredkin gate.
Cavity Quantum Electrodynamics [Turchett 1995]

As with the Fredkin gate, we need a nonlinearity to make the two qubit gate.

Use the interaction of photons with a single three state atom in a Fabry-Perot cavity to induce an effective Kerr non-linearity between two photons.

The Jaynes-Cummings Hamiltonian for the interaction of photons with a two level atom:

\[ \hat{H} = \frac{1}{2} \hbar \omega_0 (-\hat{Z}) + \hbar \omega \hat{a}^\dagger \hat{a} + g (\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+) \]

where \( \omega_0 \) = the frequency of the atom, \( \omega = \) the frequency of the field and \( g \) is the coupling constant between the field and atom.

The interaction term \( H_I \) shows that an atom in the state \( |1\rangle \) can decay to give a photon, or a photon can excite the atom from a state \( |0\rangle \).

\( \sigma_\pm \) are the atomic raising and lowering operators, \( \sigma_+ = |1\rangle \langle 0| = \frac{1}{2}(X + iY), \sigma_- = |0\rangle \langle 1| = \frac{1}{2}(X - iY) \).

We can rewrite the Jaynes-Cummings Hamiltonian in the following form:

\[ \hat{H} = \hbar \omega N + \delta Z + g (\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+) \]

where \( N = \hat{a}^\dagger \hat{a} + \frac{1}{2} Z \) and \( \delta = \frac{1}{2} \hbar (\omega_0 - \omega) \).

We may neglect \( N \) from any further analysis, this just leading to phase factors.

The atom energy term \( \hat{H}_{\text{atom}} = \frac{1}{2} \hbar \omega_0 Z \). We have a two level atom and we say the energy of the top level \( |1\rangle \) is \( E_0 + \frac{1}{2} \hbar \omega_0 \) and that of the bottom level \( |0\rangle \) is \( E_0 - \frac{1}{2} \hbar \omega_0 \).

If we set \( E_0 \) to 0 we have that the lower energy is negative and the upper is positive. Now say we have the operator \( \hat{H}_{\text{atom}} = \frac{1}{2} \hbar \omega_0 (|1\rangle \langle 1| - |0\rangle \langle 0|) \), then \( \langle 0| \hat{H}_{\text{atom}} |0\rangle = -\frac{1}{2} \hbar \omega_0 \) and \( \langle 1| \hat{H}_{\text{atom}} |1\rangle = \frac{1}{2} \hbar \omega_0 \).

Hence the \( Z = -(|1\rangle \langle 1| - |0\rangle \langle 0|) \) term.

The \( \hat{H}_{\text{field}} \) term comes from the harmonic oscillator for light (there should also be a \( \frac{1}{2} \hbar \omega_0 \) term here which is dropped due to it not being important).

We have:

\[ \hat{H} = \delta Z + g (\hat{a}^\dagger \sigma_- + \hat{a} \sigma_+) \]

Now we can work out the matrix elements of the Hamiltonian in the basis \{00\}, \{01\}, \{10\}, \}
where we have \{|field, atom\}\}

That is \( \hat{H} \) is given by

\[
\begin{pmatrix}
\langle 00| \hat{H} |00\rangle & \langle 01| \hat{H} |00\rangle & \langle 10| \hat{H} |00\rangle \\
\langle 00| \hat{H} |01\rangle & \langle 01| \hat{H} |01\rangle & \langle 10| \hat{H} |01\rangle \\
\langle 00| \hat{H} |10\rangle & \langle 01| \hat{H} |10\rangle & \langle 10| \hat{H} |10\rangle 
\end{pmatrix}
= \begin{pmatrix}
\delta & 0 & 0 \\
0 & \delta & g \\
0 & g & -\delta
\end{pmatrix}
\]

and the evolution operator \( U = e^{-i\hat{H}t} \) looks like.

\[
\begin{pmatrix}
e^{-i\delta t} & 0 & 0 \\
0 & \cos \Omega t + i\frac{g}{\Omega} \sin \Omega t & -i\frac{g}{\Omega} \sin \Omega t \\
0 & -i\frac{g}{\Omega} \sin \Omega t & \cos \Omega t - i\frac{g}{\Omega} \sin \Omega t
\end{pmatrix}
\]

where \( \Omega = \sqrt{g^2 + \delta^2} \).
Now, we want to utilize a single atom to obtain a non-linear interaction between photons.

Need two optical modes, instead of just the one used above. Need to generalize the Jaynes-Cummings Hamiltonian for these two optical modes, $a$ and $b$. For a $V$ type atom we write the three possible levels as:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The Hamiltonian becomes

$$\hat{H} = \delta \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) + g_a \left( \hat{a} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \hat{a}^\dagger \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$+ g_b \left( \hat{b} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \hat{b}^\dagger \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

From this we can work out all the matrix elements

$$H = \begin{pmatrix} \langle 000|H|000\rangle & \cdots & \langle 002|H|000\rangle \\ \langle 000|H|100\rangle & \cdots & \langle 002|H|100\rangle \\ \vdots & \cdots & \vdots \\ \langle 000|H|011\rangle & \cdots & \langle 002|H|011\rangle \\ \langle 000|H|102\rangle & \cdots & \langle 002|H|102\rangle \end{pmatrix}$$

and we can exponentiate.

Then we have to generalized $\sigma_{\pm}$. We now have $|1\rangle \langle 0|$, $|0\rangle \langle 1|$ and $|2\rangle \langle 0|$, $|0\rangle \langle 2|$. That is,

$$|1\rangle \langle 0| = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, |0\rangle \langle 1| = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$|2\rangle \langle 0| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, |0\rangle \langle 2| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we look at states of the form $|a, b, \text{atom}\rangle$.

We have used the following basis set:

$$\{|000\rangle, |100\rangle, |001\rangle, |010\rangle, |002\rangle, |110\rangle, |011\rangle, |102\rangle\}.$$
We find $|000\rangle \rightarrow |000\rangle$, $|100\rangle \rightarrow e^{i\varphi_a}|100\rangle$, $|010\rangle \rightarrow e^{i\varphi_b}|010\rangle$ and $|110\rangle \rightarrow e^{i(\varphi_a+\varphi_b+\Delta)}|110\rangle$ where $\varphi_{ab} = \varphi_a + \varphi_b + \Delta$. This is the photon non-linearity. If $\Delta$ were 0 the system would be linear and the photons would not be interacting. This is the key for implementing quantum logic with these methods.

In [Turchette 95] Cesium atoms and two photons ($a$ and $b$) were used to implement quantum logic. Qubits were encoded in orthogonal polarization modes on each photon, $|1^\pm\rangle_a|1^\pm\rangle_b$. The photons were frequency non-degenerate. One photon (say $a$) was coupled to the atomic $|1\rangle$ state and the other (say $b$) to the atomic $|2\rangle$ state.

That is, this unitary matrix was setup between two photon polarization modes.

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i\varphi_a} & 0 & 0 \\
0 & 0 & e^{i\varphi_b} & 0 \\
0 & 0 & 0 & e^{i(\varphi_a+\varphi_b+\Delta)}
\end{pmatrix}
$$

The phases induced by interaction with the atom were then measured using heterodyne measurement.

Both photons were generated with attenuated weak laser pulses.

The following logical transformations were generated

$$
\begin{align*}
|1^-\rangle_a|1^-\rangle_b & \rightarrow |1^-\rangle_a|1^-\rangle_b \\
|1^+\rangle_a|1^-\rangle_b & \rightarrow e^{i\varphi_b}|1^+\rangle_a|1^-\rangle_b \\
|1^-\rangle_a|1^+\rangle_b & \rightarrow e^{i\varphi_a}|1^-\rangle_a|1^+\rangle_b \\
|1^+\rangle_a|1^+\rangle_b & \rightarrow e^{i(\varphi_a+\varphi_b+\Delta)}|1^+\rangle_a|1^+\rangle_b
\end{align*}
$$

where $\phi_a \approx (17.5 \pm 1)^o$, $\phi_b \approx (12.5 \pm 1)^o$ and $\Delta \approx (16 \pm 3)^o$.

Progress with Linear Optics

Decomposition of unitaries

We can break any unitary into linear optical components [Reck 94]

The unitary transformation acts on the creation operators

$$
U = \left( T_{N,N-1} \cdot T_{N,N-2} \cdot T_{N,N-3} \cdots T_{N,1} \cdot T_{N-1,N-2} \cdot T_{N-1,N-3} \cdots T_{2,1}D \right)^{-1}
$$

Here $T_{p,q}$ is the $N$ dimensional identity with the $\{p,q\}$ elements replaced with the beam splitter matrix and $D$ is $N \times N$ matrix with phases on the diagonal.
The general linear optical network for a unitary matrix $U$ is a triangular array of beam splitter and phase shifters.

An example, what does the linear optical circuit for the following unitary look like?

$$U = \begin{pmatrix}
1 - \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{\sqrt{2}} - 2} \\
\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} - \frac{1}{\sqrt{2}} \\
\sqrt{\frac{3}{\sqrt{2}} - 2} & \frac{1}{2} - \frac{1}{\sqrt{2}} & \sqrt{2 - \frac{1}{2}}
\end{pmatrix}$$

Exercise: check that this is unitary.

Here we have a unitary matrix with $N=3$, so we can say that $U = (T_{3,2} \cdot T_{3,1} \cdot T_{2,1} \cdot D)^{-1} = D^\dagger \cdot T_{2,1}^\dagger \cdot T_{3,1}^\dagger \cdot T_{3,2}^\dagger$.

We find

$$T_{3,2} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & e^{-i\phi_1} \sin \theta_1 \\
0 & -e^{i\phi_1} \sin \theta_1 & \cos \theta_1
\end{pmatrix}$$

$$T_{3,1} = \begin{pmatrix}
\cos \theta_2 & 0 & e^{-i\phi_2} \sin \theta_2 \\
0 & 1 & 0 \\
-e^{i\phi_2} \sin \theta_2 & 0 & \cos \theta_2
\end{pmatrix}$$

$$T_{21} = \begin{pmatrix}
\cos \theta_3 & e^{-i\phi_3} \sin \theta_3 & 0 \\
-e^{i\phi_3} \sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{pmatrix}$$

Optical Simulation of Quantum Logic

Cerf et. al. proposed quantum logic with linear optical devices only [Cerf 1998].

To simulate $n$ qubits put a single photon in $2^n$ different paths. Three simple circuits are given below.

Exercise: show that a solution to this is given by the angles: $\theta_1 = 12.8^o, \phi_1 = \pi, \theta_2 = 20.4^o, \phi_2 = 0, \theta_3 = 63.8, \phi_3 = 0, \phi_4 = \pi, \phi_5 = 0, \phi_6 = 0$.

Exercise: What transformation does the implement on the input state $(\alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle)|10\rangle$? What is the transformation if we measure $|10\rangle$ in modes 2 and 3? This will be important later!

Exercise: What transformation does the implement on the input state $(\alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle)|10\rangle$? What is the transformation if we measure $|10\rangle$ in modes 2 and 3? This will be important later!
A $\sqrt{\text{NOT}}$ gate is given by a beam splitter with $\theta = \frac{\pi}{4}$ and $\phi = -\frac{\pi}{2}$, where
\[
\sqrt{\text{NOT}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\]
and remember that
\[
U = \begin{pmatrix} \cos \theta & -e^{i\phi} \sin \theta \\ e^{-i\phi} \sin \theta & \cos \theta \end{pmatrix}
\]
for a beam splitter.

$$|0\rangle_L = |01\rangle \rightarrow \frac{1}{\sqrt{2}}(i|10\rangle + |01\rangle)$$
$$|1\rangle_L = |10\rangle \rightarrow \frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle)$$

To implement a CNOT we encode a qubit in position and polarization. The location is the control and the polarization the target. For the control: $|0\rangle_L = |01\rangle$, $|1\rangle_L = |10\rangle$. For the target: $|0\rangle_L = |H\rangle$, $|1\rangle_L = |V\rangle$.

The circuit for this is a polarization rotator on the upper arm, as seen in figure (b). That is, if a photon is present in the top arm, its polarization will be flipped.

$$(\alpha|01\rangle + \beta|10\rangle) \otimes (\gamma|H\rangle + \delta|V\rangle)$$
$$\rightarrow \alpha \gamma |01\rangle |H\rangle + \alpha \delta |01\rangle |V\rangle + \beta \gamma |10\rangle |V\rangle + \beta \delta |10\rangle |H\rangle$$

To implement a Hadamard gate we use a $\theta = \frac{\pi}{4}$, $\phi = -\frac{\pi}{2}$ beam splitter and two $-\frac{\pi}{2}$ phase shifters, as seen in figure (a).

$$|0\rangle_L = |01\rangle \rightarrow -i|01\rangle \rightarrow -\frac{i}{\sqrt{2}}(|01\rangle + i|10\rangle)$$
$$\rightarrow -\frac{i}{\sqrt{2}}(|01\rangle + i(-i)|10\rangle) = -\frac{i}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|1\rangle_L = |10\rangle \rightarrow |10\rangle \rightarrow \frac{1}{\sqrt{2}}(i|01\rangle + |10\rangle)$$
$$\rightarrow \frac{1}{\sqrt{2}}(i|01\rangle + -i|10\rangle) = \frac{i}{\sqrt{2}}(|01\rangle - |10\rangle)$$

To implement a reverse CNOT we simply need a polarizing beam splitter (PBS), where horizontal is reflected. As before, the location is the control and the polarization the target. This is seen in figure (c):

$$(\alpha|01\rangle + \beta|10\rangle) \otimes (\gamma|H\rangle + \delta|V\rangle)$$
$$\rightarrow \alpha \gamma |01\rangle |H\rangle + \alpha \delta |01\rangle |V\rangle + \beta \gamma |10\rangle |V\rangle + \beta \delta |10\rangle |H\rangle$$

$n$ qubits requires $2^n$ paths which requires $2^n - 1$ beam splitters. With one qubits encoded in polarization: we need $2^{n-1}$ optical paths.
References