Angular Momentum and the Two-Dimensional Free Particle

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The one-dimensional free particle is among the first examples of quantum system that students meet in quantum mechanics and quantum chemistry introductory textbooks (*1-5*). This simple system is widely employed as a *model* system which allows the teacher to introduce very important topics such as the Heisenberg principle, the concept of constant of motion and the solution of the Schrödinger equation in terms of plane waves. The graphical representation of the wavefunction and of its squared modulus for this one-dimensional system is particularly simple, thus allowing the students to better understand the shape and the properties of the wavefunction. The three-dimensional free particle is sometimes studied soon after (*4*) or in conjunction with the theory of scattering (*5*). However, the algebra becomes rather intricate and the ability to plot the wavefunction in a simple manner is lost. In any case the parallels between the properties of the quantum system and those of the classical one are not fully perceived.

In this paper we wish to show that the quantum free particle has many interesting properties usually not mentioned in introductory books (most notably the existence and the conservation of angular momentum) and that it can be used to introduce the students to important topics often encountered much later in the course such as the expansion of the wavefunction on a complete basis and the transformation between different bases. The best compromise between the opposite requests of algebraic simplicity and physical interest is the two-dimensional free particle system. The problem of plotting a complex wavefunction in a

clear and informative way is also addressed. Atomic units ($\hbar = 1$) are used throughout the paper.

The classical system

Before considering the properties of the quantum free particle, it is useful to briefly refresh our memories on its classical counterpart. A free particle is, by definition, not subject to any force or constraint, that is, the force acting upon it is $\mathbf{F} \equiv (F_x, F_y) = 0$ everywhere (we use **boldface** letters to denote vector quantities and the correspondent *italic* letter for their magnitude). The classical equation of motion is $\mathbf{F} = m\,\mathbf{a}$, where m is the constant mass of the particle and $\mathbf{a}$ is its acceleration. We can rewrite this equation defining the linear momentum $\mathbf{p} = m\mathbf{v}$ and using $\mathbf{a} = \frac{d\mathbf{v}}{dt}$; we thus obtain $\frac{d\mathbf{p}}{dt} = 0$. This means that the linear momentum $\mathbf{p}$ is a constant of motion, i. e., a quantity that does not change during the motion of the particle. By integrating the equation of motion it is readily found that the particle moves along a straight line, as intuitively expected. Are there other constants of motion that characterize the particle? The kinetic energy of the particle $E = |\mathbf{p}|^2/2m = p^2/2m$ is clearly a constant of motion. Note that the kinetic energy coincides with the total energy as the particle is not subject to any potential.

At this point many quantum mechanics introductory texts move to the description of the quantum free particle and its description by plane waves. We think however it is worth to further discuss the properties of the classical system to show the similarities with the quantum particle. Both in classical and in quantum mechanics a quantity that is of primary value in discussing the properties of the motion is the angular momentum. In two dimensions the angular momentum of the particle with respect to the origin of the reference frame is a scalar defined as $L = xp_y - yp_x$. It is easily proved that the angular momentum is conserved during the motion of the particle by differentiating its definition with respect to time and using $d\mathbf{p}/dt = 0$. There is then another constant of motion, in addition to the linear momentum and the total energy: the angular momentum. Also note that the conservation of the angular momentum is not related to rotation in the present case.

From the definition of the angular momentum, it is very easy to show that a free particle moving through the origin has a null angular momentum, whereas a particle with non-zero angular momentum does not pass through the origin. The minimum distance b between the particle and the origin is called "impact parameter" and is equal to $|L|/|\mathbf{p}|$. For our later discussion, it is important to note that by fixing the linear momentum **p** the velocity and the energy of the particle are determined, but this is not sufficient to completely determine the graph of the trajectory. To do so we need to specify also the angular momentum. In other words, fixing the energy amounts to fixing the magnitude of **p** but not its direction, but L is left completely unspecified, i. e., the energy and the angular momentum are independent of each other.

The quantum system

The total energy is associated to the Hamiltonian operator $\hat{H}$, that is written for the free particle system as

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1}$$

When $\hat{H}$ does not depend explicitly on time, as in the present case, the total energy E is a constant of motion. This conservation law mirrors a *symmetry* of the system, that is, the invariance of the Hamiltonian under a translation of the time axis. The solutions of the stationary Schrödinger equation $\hat{H}\Psi = E\Psi$, i. e., the eigenfunctions of the Hamiltonian operator are the so-called plane waves

$$\Psi_{k_x,k_y}(x,y) = \frac{1}{2\pi} e^{i\left(k_x x + k_y y\right)} = \frac{1}{2\pi} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2}$$

where $\mathbf{r} = (x, y)$, k_x and k_y are real numbers and the factor $1/2\pi$ is a normalization constant which we will drop in the following discussion to avoid unnecessary cluttering of the equations. The eigenvalues are $E = \left(k_x^2 + k_y^2\right)/2m = k^2/2m$. There is no restriction on the values that k_x and k_y can assume, therefore the energy of a free particle is *not* quantized.

Now we turn to the linear momentum, which is associated to the two quantum operators

$$\hat{p}_x = -i\frac{\partial}{\partial x} \quad \text{and} \quad \hat{p}_y = -i\frac{\partial}{\partial y} \tag{3}$$

which form the vector operator $\hat{\mathbf{p}} = \left(\hat{p}_x, \hat{p}_y\right)$. Both components of the linear momentum operator commute with the Hamiltonian $[\hat{H}, \hat{p}_x] = [\hat{H}, \hat{p}_y] = 0$ and then we find again that the linear momentum is a constant of motion. This mathematical result has again a deep physical significance: the linear momentum is constant since the Hamiltonian is left unchanged by a translation of the coordinate axes as it does not depend on the spatial coordinates. It is also noteworthy that the components of the linear momentum commute with each other. This means that all three operators can share the same set of eigenfunctions. These are the plane waves, in fact $\hat{p}_x \Psi_{k_x,k_y} = k_x \Psi_{k_x,k_y}$ and $\hat{p}_y \Psi_{k_x,k_y} = k_y \Psi_{k_x,k_y}$. The components of the vector **k** are then the eigenvalues of the linear momentum and since they can assume any real value, also the linear momentum is *not* quantized.

A particle described by Ψ_{k_x,k_y} is characterized by the eigenvalues E, k_x, and k_y, that is, it has well-defined linear momentum and energy. Note however that, being $E = k^2/2m$, the state is completely defined by specifying only k_x and k_y and then it is sufficient to denote such a state as Ψ_{k_x,k_y} rather than Ψ_{E,k_x,k_y}. The Ψ_{k_x,k_y} can be divided in subsets containing eigenfunctions which differ only in the *direction* of **k**. The eigenfunctions within each subset are said to be degenerate, i. e., they have the same energy but differ in one or more of the eigenvalues needed to completely determine the state of the system. Looking at eq 2, we also see that decreasing (increasing) the linear momentum, and then the energy, corresponds to a "change of scale" which makes the wavefunction to expand (contract).

We are now faced with the problem of plotting complex wavefunctions. The simplest choice is to plot separately the real and imaginary parts. One may argue that such separation is physically unsound: if one multiplicates the wavefunction $e^{i\mathbf{k}\cdot\mathbf{r}}$ by a constant phase shift $e^{i\alpha}$, the new eigenfunction is not physically distinguishable from the original one since it has the same eigenvalues E, k_x, and k_y as before but the real and imaginary parts are obviously changed.

However, this is not yet the whole truth. The new eigenfunction can be written as $e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{a})}$, where **a** is a vector of length α/k parallel to **k**. It then describes the very same system as the original wavefunction but observed from a reference frame *translated* by the vector **a** from the original frame. Thus all properties of the real and imaginary parts of the wavefunction which do not change when the reference frame is translated do physically make sense. With this caveat in mind, a complex wavefunction can be usefully represented by plotting separately its real and imaginary parts. This is another example of the close link between translation and linear momentum eigenfunctions.

In Figure 1 we report the real and imaginary part of Ψ_{k_x,k_y} for **k** = (1,0), i. e., for a particle traveling parallel to the *x*-axis in the positive direction with energy $E = 1/2m$. The crests and troughs of the real part form ridges perpendicular to the direction of **k** and correspond to the lines where the imaginary part is zero and vice versa. These statements are physically sound since they hold even when the reference frame is translated. On the same grounds a comment such "the imaginary part is zero along the *y*-axis" has no meaning. The probability density $\left|\Psi_{1,0}\right|^2$, also shown in Figure 1, is constant on the whole *x*,*y*-plane. Being the linear momentum exactly defined, the Heisenberg principle tells us that we know nothing about the particle position, so any plane wave gives a constant probability density.

The eigenfunctions of the angular momentum

We have previously seen that the classical free particle has a constant angular momentum, so we should not be surprised that the angular momentum operator $\hat{L} = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ commutes with the Hamiltonian operator. Again, the conservation of the angular momentum is associated to a symmetry of the system: the Hamiltonian is left unchanged by any rotation of the coordinate axes. As before we can choose the solutions of the Schrödinger equation to be also eigenfunction of the angular momentum operator with eigenvalue *l*. However, $\hat{L}$ does not commute with $\hat{p}_x$ and $\hat{p}_y$, as can be easily checked, so these eigenfunctions represent a particle with well-defined energy but undefined linear momentum (we only know its magnitude $k = \sqrt{2mE}$ but not its direction). We then label

the solutions as $\Psi_{E,l}$. On the other hand, Ψ_{k_x,k_y} represents a state with well-defined linear momentum but undefined angular momentum since it is not eigenfunction of $\hat{E}$.

In order to find $\Psi_{E,l}$ we begin by expressing the angular momentum operator in polar coordinates $r = \sqrt{x^2 + y^2}$ and $\varphi = \arctan(y/x)$:

$$\hat{E} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = -i\frac{\partial}{\partial \varphi} \tag{4}$$

The general solution of the angular momentum eigenvalue equation

$$\hat{E}\Phi(\varphi) \equiv -i\frac{\partial}{\partial \varphi}\Phi(\varphi) = l\,\Phi(\varphi) \tag{5}$$

is $\Phi(\varphi) = e^{il\varphi}$ (note the resemblance with eq 2). We now require that $\Phi(\varphi + 2\pi) = \Phi(\varphi)$ since any physical system composed of particles is left unchanged by a rotation through 2π. Since $\Phi(\varphi + 2\pi) = \Phi(\varphi)\, e^{i2\pi l}$ we must impose the condition that $e^{i2\pi l} = 1$, and this leads to the condition that l can only assume the discrete values 0, ±1, ±2, ... In other words, the angular momentum *is* quantized because of a boundary condition that is imposed on physical grounds to the solutions of the differential equation 5. A circular motion of the system is not necessarily involved in the quantization of the angular momentum as students may be driven to think when the quantization of $\hat{E}$ is introduced for the particle-on-a-ring system.

We proceed by expressing also the Hamiltonian operator in polar coordinates

$$\hat{H} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \right) \tag{6}$$

and using the square of the angular momentum operator we can write the Schrödinger equation as

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\hat{L}^2 \right)\Psi = E\Psi \tag{7}$$

Using eqs 5 and 7 we infer that the general solution of eq 7 can be written as $\Psi(r,\varphi) = R(r)\,\Phi(\varphi)$ and that the radial function $R(r)$ must satisfy

$$-\frac{1}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{l^2}{r^2} \right) R(r) = E\ R(r) \tag{8}$$

The solution of this equation can be found by consulting a mathematics handbook. The reader should not be disappointed by the use of this method for solving this equation. In fact, "*This is a very common way of solving differential equation, and the Handbook of Mathematical Functions, M. Abramowitz and I. A. Stegun, Dover, is one of the principal sources for identifying solutions. It is an ideal desert-island book for shipwrecked quantum chemists.*" (ref *2*, p. 65). Another possibility is to use a symbolic algebra package like Maple (*6*) or Mathematica (*7*). The radial function turns out to be a Bessel function of integer order $R(r) = J_l(\sqrt{2mE}\, r)$ where the energy E of the particle is not quantized. We remark here that the appearance of Bessel functions in the solution should not cause any concern, even if they are not familiar to students. Nowadays computers, plotting programs and computer-algebra applications are being used more and more to help teaching quantum mechanics. With these tools, manipulating and plotting the Bessel functions (or any other 'special' function) is not different from working with the well-known elementary functions.

A state with definite energy and angular momentum is then described by the wavefunction

$$\Psi_{E,l}(r,\varphi) = e^{il\varphi} J_l(\sqrt{2mE}\, r) = e^{il\varphi} J_l(k\, r) \qquad l = 0, \pm 1, \pm 2, \mathrm{K} \tag{9}$$

where E and l are the eigenvalues of $\hat{H}$ and $\hat{L}$ and the last equality holds since $E = k^2/2m$. Of course, two wavefunctions with the same eigenvalue E but with different l are degenerate. Equation 9 shows that the energy of the particle is independent of the angular momentum, just like we found in the classical case, but in contrast with the case of the particle moving on a ring. Changing the sign of the angular momentum l does not change the radial part of the wavefunction apart from an insignificant multiplicative factor, since for Bessel functions of integer order $J_{-l}(x) = (-1)^l J_l(x)$. This means that $e^{il\varphi} J_l(\sqrt{2mE}\, r)$ and $e^{-il\varphi} J_l(\sqrt{2mE}\, r)$ represent two states which differ only in the sign of the angular momentum (here it is improper to talk about the *direction* of the angular momentum since the two-dimensional angular momentum is a scalar, not a vector quantity). Consider now the effect of changing the energy on the wavefunction. Suppose to increase the energy while l is fixed. The argument of the Bessel function becomes larger and the wavefunction contracts without changing its overall

shape. It is again a sort of change of scale. We can then plot the eigenfunctions for a fixed energy $E = 1/2m$ without losing generality.

Now we are faced again with the problem of how to plot a complex function. As in the preceding Section we just plot the real and imaginary part of the wavefunction. To find out the limitations of such a picture we ask how the multiplication of the wavefunction $\Psi_{E,l}$ by a constant phase shift $e^{i\alpha}$ can be interpreted. From Equation 9 it is readily found that the new wavefunction describes the very same system as the original one but observed from a reference frame *rotated* through the angle α/l with respect to the original frame. Note that rotation (l) plays now the role played by translation (**k**) in the case of the linear-momentum eigenfunctions.

In Figure 2 are plotted the real and imaginary parts of the wavefunction $\Psi_{E,l}$ for $E = 1/2m$ and $l = 0, 1, 2$. The angular dependence of $\Psi_{E,l}$ is entirely given by $e^{il\varphi}$. It is thus fixed by the eigenvalue of $\hat{E}$ and not by the energy. The case $l = 0$ is special in that the imaginary part of $\Psi_{E,0}$ is zero everywhere and the real part has cylindrical symmetry. When $l \neq 0$, the real and imaginary parts have the same shape but are rotated through the angle $\pi/2l$ with respect to each other. In Figure 3a are reported the probability densities $\left|\Psi_{E,l}\right|^2 = \left[J_l\left(\sqrt{2mE}\, r\right)\right]^2$ for $E = 1/2m$ and $l = 0, 2, 4, 6$. Note that the probability density is isotropic, i. e., it does not depend on φ. The main difference with respect to $\left|\Psi_{k_x,k_y}\right|^2$ is that now we do know something about the particle position since the squared modulus of $\Psi_{E,l}$ is not constant. There are points where the probability of finding the particle is large, points where it is small, and even points where the particle cannot be found. For instance, the quantum particle can be found at the origin only when $l = 0$ just like the classical particle passes through the origin only when $L = 0$. We can also draw an interesting parallel with the classical case by considering the radial probability $RPD_{E,l}(r)\, dr$, i. e., the probability of finding the particle at a distance from the origin comprised between r and $r+dr$:

$$RPD_{E,l}(r) = \int_0^{2\pi} \left|\Psi_{E,l}(r,\varphi)\right|^2 r\, d\varphi = 2\pi r\, J_l^2(k\, r) \tag{10}$$

Near the origin the Bessel function $J_l(kr)$ can be approximated by $(kr/2)^l/l!$ and then the radial probability density behaves like r^{2l+1}. As can be seen in Figure 3b, $RPD(r)$ is very small inside a circle of radius $b_l(k) = |l|/|\mathbf{k}|$ and this implies that a particle described by $\Psi_{E,l}$ is practically unaffected by what happens inside the circle. In classical mechanics the minimum distance from the origin attained by the particle is the impact parameter $b = |L|/|\mathbf{p}|$, which exactly parallels the expression for $b_l(k)$ making the substitution $L \rightarrow l$, $\mathbf{p} \rightarrow \mathbf{k}$. The angular momentum wavefunctions appear in the theory of scattering in two dimensions (*8*) and are the counterpart of the spherical waves usually employed in the analysis of three-dimensional scattering. Both sets of functions are used to describe the behavior of the particle after interaction with the target.

Back and forth between representations

The eigenfunctions of the Hamiltonian operator always form a complete basis, so we can expand a generic state of the particle using either the plane waves of eq 2 or the functions with definite angular momentum of eq 9. This means that we can expand an eigenfunction with definite linear momentum using eigenfunctions with definite angular momentum, and vice versa. This is not a mere mathematical exercise but it has a physical meaning: suppose we prepare the particle in a state with definite energy and linear momentum, we can then ask what is the probability of finding a certain angular momentum as the outcome of an experiment. Likewise we would like to know what is the probability of measuring a certain linear momentum for a particle having definite energy and angular momentum. To answer these questions we must look at the squared moduli of the expansion coefficients.

We begin by expanding a plane wave in eigenfunctions of the angular momentum. Without losing generality we first focus on a plane wave with the **k** vector directed along the x axis: $\mathbf{k}_0 = (k, 0)$. Then, to obtain the general plane wave expansion, we will rotate the system by an arbitrary angle. Since eq 9 is expressed in polar coordinates, we express our plane wave as

$$\Psi_{k,0} = e^{ikx} = e^{ikr\cos(\varphi)} = \cos\left[kr\cos(\varphi)\right] + i\sin\left[kr\cos(\varphi)\right] \tag{11}$$

where we put in evidence the real and imaginary parts. It is possible to transform the previous expression using

$$\begin{aligned}\cos[z\cos(\varphi)] &= J_0(z) + 2\sum_{l=1}^{\infty}(-1)^l J_{2l}(z)\cos(2l\varphi)\\ \sin[z\cos(\varphi)] &= 2\sum_{l=0}^{\infty}(-1)^l J_{2l+1}(z)\cos[(2l+1)\varphi]\end{aligned} \tag{12}$$

(see ref. *9*, eqs 9.1.44 and 9.1.45) and, after some tedious but trivial algebra, we arrive at

$$e^{ikr\cos(\varphi)} = J_0(kr) + \sum_{l=1}^{\infty} i^l\,(e^{+il\varphi} + e^{-il\varphi})\,J_l(kr) \tag{13}$$

Recalling that $J_{-l}(x) = (-1)^l J_l(x)$, the last expression can be written more concisely as

$$\Psi_{k,0} \equiv e^{ikr\cos(\varphi)} = \sum_{l=-\infty}^{+\infty} i^l\,\Psi_{E,l} \tag{14}$$

where $E = k^2/2m$. What is left is to find the expansion of an arbitrary plane wave. If we look at the particle from a new reference frame (x', y') rotated clockwise by the *arbitrary* angle θ, the linear momentum is $\mathbf{k'} = (k'_x,\ k'_y) = (k\ \cos\theta,\ k\ \sin\theta)$, and the particle is described the wavefunction

$$\begin{aligned}\Psi_{k'_x,k'_y} &= e^{i\mathbf{k'}\cdot\mathbf{r'}} = e^{i\left(k'_x x' + k'_y y'\right)}\\ &= e^{ikr[\cos(\theta)\cos(\varphi') + \sin(\theta)\sin(\varphi')]}\end{aligned} \tag{15}$$

which is the generic free particle wavefunction since θ is arbitrary. As we already know, to get the angular momentum eigenfunctions in the new reference frame, we just add the phase factor $e^{-il\theta}$ to the old ones. Dropping the primes, we are now able to rewrite eq 14 as

$$\Psi_{k_x,k_y} = \sum_{l=-\infty}^{+\infty} i^l\,e^{-il\theta}\,\Psi_{E,l} \tag{16}$$

This is a remarkable result, since we have just discovered how to expand a plane wave using the eigenfunctions of the angular momentum. Note that the summation extends over all possible values of l, whereas the energy is restricted to the value $E = k^2/2m$, compatible with the magnitude k of the linear momentum. A state with a well-defined linear momentum **k** can

be decomposed as a series of infinite states with fixed energy and well-defined angular momentum l. The squared modulus of any expansion coefficient is $\left|i^l e^{il\theta}\right|^2 = 1$, independent of l. This means that if we measure the angular momentum of a particle prepared as a plane wave we get as outcome any possible angular momentum with equal probability. This is analogous to the classical case where a particle with fixed energy and linear momentum can have any impact parameter and consequently any angular momentum.

In Figure 4 we can see a graphical representation of the expansion we have just derived. There are reported the probability densities of the approximate $\left|\Psi_{k_x,k_y}\right|^2$ for $\mathbf{k} = (1,0)$ obtained by truncating the series in eq 21 either on the positive and on the negative side at $|l| = 1, 2, 4, 6, 8, 10$. The constant probability density typical of plane waves quickly develops around the origin and gets larger on adding more and more angular momentum eigenfunctions. This behavior can be understood when we recall that high-l functions have zero amplitude near the origin.

Consider now a state with angular momentum l_0 and energy E. To find the formula that expands the eigenfunction of such a state over the functions Ψ_{k_x,k_y} it is sufficient to multiply eq 16 by $e^{il_0\theta}$ and to integrate with respect to θ from 0 to 2π:

$$\int_0^{2\pi} e^{il_0\theta}\Psi_{k_x,k_y}\,d\theta = \sum_{l=-\infty}^{+\infty} i^l \Psi_{E,l_0} \int_0^{2\pi} e^{i(l_0-l)\theta}\,d\theta \tag{17}$$

The integral in the right-hand side is a standard one and it is zero if $l \neq l_0$ or 2π if $l = l_0$. The final result is then

$$\Psi_{E,l_0} = \frac{(-i)^{l_0}}{2\pi}\int_0^{2\pi} e^{il_0\theta}\Psi_{k_x,k_y}\,d\theta \tag{18}$$

Not all possible values of **k** are present under the integral: the integration is over all the *directions* that the vector **k** can assume while satisfying the condition $k^2 = E/2m$. This is analogous to the classical case, where for a given energy E there is an infinite number of **p** vectors which differ in orientation while satisfying $E = p^2/2m$. The squared modulus of any expansion coefficient is independent of **k** and therefore we reach a similar conclusion as before: measuring the linear momentum of a particle prepared with a definite energy and angular

momentum can give as outcome any vector **k** with magnitude $k = \sqrt{2mE}$ with equal probability.

Similar to Figure 4, we report in Figure 5 the successive approximations to $|\Psi_{E,l}|^2$ for E = 1/2m and l = 4, obtained by approximating the integral over θ as a finite sum. We choose n (n = 1, 4, 8, 10, 14, 18) **k** vectors with $k = \sqrt{2mE}$ which point towards the vertices of the regular polygon with n sides and add the correspondent plane waves. On increasing n the probability density loses its constant value because of the destructive interference between the plane waves and the typical undulatory character with cylindrical symmetry of $|\Psi_{E,l}|^2$ develops.

Conclusions

In this paper we have shown that the two-dimensional free particle is a system with many interesting properties that are not present in the one-dimensional case. This system, which can be treated at an elementary level, can be used to introduce the students to advanced concepts like the quantization of the angular momentum and the expansion of a wavefunction on a basis set. The wavefunctions can be easily plotted allowing students to easily grasp the meaning of the mathematical formulae and associated concepts.

Appendix

In this appendix is reported a short Matlab (*10*) script which generates the figures reported in the paper. It can be used to explore how the shape of the discussed wavefunctions and of their squared moduli changes as the parameters (energy, linear and angular momentum) are varied.

```
% Figure 1 : Linear-momentum eigenfunctions psi_k.

x=(-10:0.5:10); y=x; r=sqrt(x.^2+y.^2);
[xx,yy]=meshgrid(x,y); rr=sqrt(xx.^2+yy.^2); phi=atan2(yy,xx);
kx = 1.0; ky = 0.0; [kxx,kyy]=meshgrid(kx,ky);
psi_k = exp(i*(kxx.*xx+kyy.*yy));
```

```
psi2_k = abs(psi_k).^2;
figure(1); colormap cool;
subplot(2,2,1); mesh(xx,yy,real(psi_k));
grid on; axis([-10 10 -10 10 -1 1]); view(-30,55);
xlabel('x'); ylabel('y'); title(['Re(Psi_k)']);
subplot(2,2,2); mesh(xx,yy,imag(psi_k));
grid on; axis([-10 10 -10 10 -1 1]); view(-30,55);
xlabel('x'); ylabel('y'); title(['Im(Psi_k)']);
subplot(2,2,3); mesh(xx,yy,psi2_k);
grid on; axis([-10 10 -10 10 -1 1]); view(-30,55);
xlabel('x'); ylabel('y');
title(['|Psi_k|^2']);
```

```
% Figure 2 : Angular-momentum eigenfunctions psi_El with energy E = 1/2m.

x=(-08:0.5:08); y=x; r=sqrt(x.^2+y.^2);
[xx,yy]=meshgrid(x,y); rr=sqrt(xx.^2+yy.^2); phi=atan2(yy,xx);
E = 0.5; l = 6;
figure(2); colormap cool;
psi_El = exp(i*l*phi).* besselj(l, sqrt(2*E)*rr);
subplot(2,1,1); mesh(xx,yy,real(psi_El));
grid on; axis([-8 8 -8 8 -0.5 0.5]); view(-30,65)
xlabel('x'); ylabel('y'); title(['Re(Psi_E,l)']);
set(gca,'XTick',[-8 0 8],'YTick',[-8 0 8]);
subplot(2,1,2); mesh(xx,yy,imag(psi_El));
psi_El_m = sin(l*phi).* besselj(l, sqrt(2*E)*rr);
grid on; axis([-8 8 -8 8 -0.5 0.5]); view(-30,65)
xlabel('x'); ylabel('y'); title(['Im(Psi_E,l)']);
set(gca,'XTick',[-8 0 8],'YTick',[-8 0 8]);
```

```
% Figure 3a : Squared-modulus angular-momentum eigenfunctions
%             |psi_El|^2 with energy E = 1/2m.

x=(-08:0.5:08); y=x; r=sqrt(x.^2+y.^2);
[xx,yy]=meshgrid(x,y); rr=sqrt(xx.^2+yy.^2); phi=atan2(yy,xx);
E=0.5; l=6;
figure(3); colormap cool;
psi2_El = besselj(l, sqrt(2*E)*rr) .^ 2;
mesh(xx,yy,psi2_El);
grid on; axis([-8 8 -8 8 0 0.5]); view(-30,55);
xlabel('x'); ylabel('y'); title(['|Psi_E,l|^2']);
set(gca,'XTick',[-8 0 8],'YTick',[-8 0 8]);
```

```
% Figure 3b : Radial distibution function of angular
%             momentum eigenfunctions psi_El with energy E = 1/2m.
r = (0:.1:20);
E = 0.5; M = 2;
R_EM = 2*pi* r .* besselj(M, sqrt(2*E)*r).^2;
plot(r, R_EM);
xlabel('r'); ylabel('RPD(r)');
```

```
%Figure 4 : psi_k as linear combination of psi_El with energy E = k^2/2m.

x=(-08:0.5:08); y=x; r=sqrt(x.^2+y.^2);
[xx,yy]=meshgrid(x,y); rr=sqrt(xx.^2+yy.^2); phi=atan2(yy,xx);
kx=1.0; ky=0.0; theta=atan2(ky,kx); E=(kx.^2+ky.^2)/2;
figure(4); colormap cool;
lmax=8;
psi = zeros(size(xx));
for m = 0 : lmax
   b = besselj(m,sqrt(2*E)*rr);
      psi = psi + (i)^m*exp(-i*m*theta)*exp(i*m*phi).*b;
end
for m = -lmax : -1
   b = (-1)^m * besselj(abs(m),sqrt(2*E)*rr);
   psi = psi + (i)^m*exp(-i*m*theta)*exp(i*m*phi).*b;
end
mesh(xx,yy,abs(psi).^2);
grid on; axis([-8 8 -8 8 0 1.5]); view(-30,55)
xlabel('x'); ylabel('y'); title(['lmax = ' num2str(lmax)]);
set(gca,'XTick',[-8 0 8],'YTick',[-8 0 8], 'ZTick',[0 .75 1.5]);
```

```
%Figure 5 : psi_El as linear combination of psi_k with |k| = 1.

figure(5); colormap cool;
x=(-08:0.5:08); y=x; r=sqrt(x.^2+y.^2);
[xx,yy]=meshgrid(x,y); rr=sqrt(xx.^2+yy.^2); phi=atan2(yy,xx);
l = 4; E = 1/2;
k = sqrt(2*E); nk = 8;
psi = zeros(size(xx));
for j = 1 : nk
  theta =j *2*pi/nk;
  psi = psi + exp(i*k*(cos(theta)*xx+sin(theta)*yy)).*exp(i*l*theta)/(i)^l;
end
psi = psi/nk;
mesh(xx,yy,abs(psi).^2);
grid on; axis([-8 8 -8 8 0 1.0]); view(-30,55);
xlabel('x'); ylabel('y'); title(['nkappa = ' num2str(nk)]);
set(gca,'XTick',[-8 0 8],'YTick',[-8 0 8]);
```

Literature Cited

1. Merzbacher, E. *Quantum Mechanics*, 2nd edition; Wiley, New York, 1970.
2. Atkins, P. W. *Molecular Quantum Mechanics*, 2nd edition; Oxford University Press, Oxford, 1983.

3. Pilar, F. L. *Elementary Quantum Chemistry*, 2nd edition; McGraw Hill, New York, 1990.
4. Levine, I. N. *Quantum Chemistry*, 4th edition; Prentice Hall, Englewood Cliff, 1991.
5. Cohen-Tannoudji, C.; Diu, B.; Laloë, F. *Quantum Mechanics*, 2nd edition; Wiley, New York, 1977.
6. *Maple V, Release 3*, Waterloo Maple Software, 1995.
7. *Mathematica, Version 2.2*. Wolfram Research, Inc., 1994 .
8. Lapidus, I. R. *Am. J. Phys.,* **1982**, *50*, 45-47.
9. Abramowitz, M.; Stegun, I. A. *Handbok of Mathematical Functions*, 9th edition; Dover, New York, 1972
10. *Matlab, Version 4.2c*. The Mathworks, Inc., 1994 .

Figure Captions

Figure 1. Real and imaginary parts and squared modulus of the linear momentum eigenfunctions for a free particle with mass m, linear momentum $\mathbf{k} = (1, 0)$, and energy $E = 1/2m$.

Figure 2. Real and imaginary parts of the angular momentum eigenfunctions for three free particles with equal mass m and energy $E = 1/2m$, but different angular momentum. Top row: $l = 0$; middle row: $l = 1$; bottom row: $l = 2$. Note the greater number of nodes at larger l values.

Figure 3. Probability densities for four free particles with equal mass m and energy $E = 1/2m$, but different angular momentum $l = 0, 2, 4, 6$. a) Normal probability density equal to the squared modulus of the angular momentum eigenfunctions. b) Radial probability density $\mathrm{RPD}_{E,l}(r)$. Solid line: $l = 0$; dotted line: $l = 2$; dashed line: $l = 4$; dash-dot line: $l = 6$.

Figure 4. Picture of the expansion of a linear momentum eigenfunction with $\mathbf{k} = (1, 0)$ over the angular momentum eigenfunctions. For the sake of clarity we plotted the probability density instead of the complex wavefunction. The infinite series is approximated by a finite sum which ranges from $l = -l_{\max}$ to $l = +l_{\max}$ in unit steps. On increasing $l_{\max}$, the probability density flattens and quickly converges to the constant value 1.

Figure 5. Picture of the expansion of an angular momentum eigenfunction with energy $E = 1/2m$ and angular momentum $l = 6$ over the linear momentum eigenfunctions. For the sake of clarity we plotted the probability density instead of the complex wavefunction. The expansion involves only the plane waves with $k = 1$. The expansion integral is approximated by a finite sum with n terms. The plots should be compared with the corresponding one appearing in Figure 3.