# Fractional-Tikhonov regularization on graphs

(applied to signal and image restoration)

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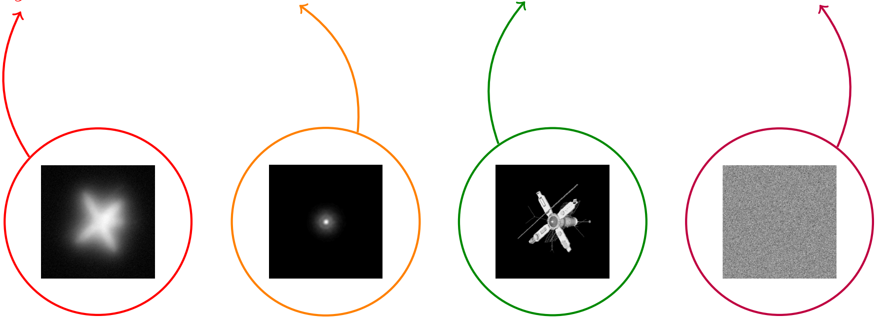
23rd of May, 2018

# Our model problem

$$y^{\delta} = K * x + \text{noise}$$



- $K$ represents the blur and it is severely <u>ill-conditioned</u> (compact integral operator of the first kind);
- $y^{\delta}$ are known measured data (blurred and noisy image);
- $\|\text{noise}\| \leq \delta$.

## Singular value expansion and generalized inverse

Since $K$ is compact, we can write

$$Kx = \sum_{m=1}^{+\infty} \sigma_m \langle x, v_m \rangle u_m,$$

where $(\sigma_m; v_m, u_m)_{m\in\mathbb{N}}$ is the singular value expansion of $K$.

### Generalized inverse

We define $K^\dagger : \mathcal{D}(K^\dagger) \subseteq \mathcal{Y} \to \mathcal{X}$ as

$$K^\dagger y = \sum_{m:\, \sigma_m>0} \sigma_m^{-1} \langle y, u_m \rangle v_m,$$

$$\mathcal{D}(K^\dagger) = \left\{ y \in \mathcal{Y} : \sum_{m:\, \sigma_m>0} \sigma_m^{-2} |\langle y, u_m \rangle|^2 < \infty \right\}.$$

In the free-noise case, we have

$$x^{\dagger} = K^{\dagger} y,$$

but due to the ill-posedness of the problem,

$$x^{\delta} = K^{\dagger} y^{\delta}$$

is not a good approximation of $x^{\dagger}$. Since we are dealing with data affected by noise, i.e., with $y^{\delta}$, then we can not use $K^{\dagger}$ to compute an approximated solution. We have to regularize the operator $K^{\dagger}$.

# Filter based regularization methods

We substitute the $K^\dagger$ operator with a one-parameter family of continuous linear operators $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$,

$$K^\dagger y^\delta = \sum_{m:\,\sigma_m>0} \sigma_m^{-1}\langle y^\delta, u_m\rangle v_m$$

$$\Downarrow$$

$$R_\alpha y^\delta = \sum_{m:\,\sigma_m>0} F_\alpha(\sigma_m)\sigma_m^{-1}\langle y^\delta, u_m\rangle v_m$$

$\alpha = \alpha(\delta, y^\delta)$ is called rule choice.

# Fractional Tikhonov filter functions

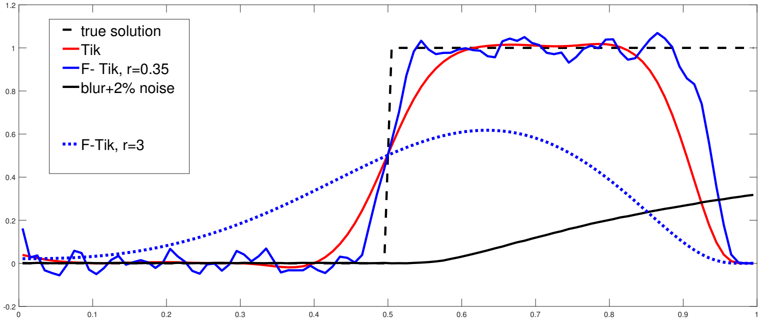
- Standard Tikhonov filter: $F_\alpha(\sigma_m) = \dfrac{\sigma_m^2}{\sigma_m^2 + \alpha}$, with $\alpha > 0$.
- Fractional Tikhonov filter: $F_{\alpha,\gamma}(\sigma_m) = \left(\dfrac{\sigma_m^2}{\sigma_m^2 + \alpha}\right)^\gamma$, with $\alpha > 0$ and $\gamma \in [1/2, \infty)$ (Klann and Ramnlau, 2008).
- Weighted/Fractional Tikhonov filter: $F_{\alpha,r}(\sigma_m) = \dfrac{\sigma_m^{r+1}}{\sigma_m^{r+1} + \alpha}$, with $\alpha > 0$ and $r \in [0, +\infty)$ (Hochstenbach and Reichel, 2011).

For $1/2 \leq \gamma < 1$ and $0 \leq r < 1$, fractional and weighted filters smooth the reconstructed solution less than standard Tikhonov.

# An easy $1d$ example of oversmoothing - part 1

Blur taken from *Heat*$(n, \kappa)$ in $\mathrm{Regtools}$, $n = 100, \kappa = 1$ and $2\%$ $\mathrm{noise.}$ True solution:

$$\mathbf{x}^{\dagger} : [0, 1] \to \mathbb{R} \qquad \text{s.t.} \qquad \mathbf{x}^{\dagger}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 0.5, \\ 1 & \text{if } 0.5 < t \leq 1. \end{cases}$$



# Let's reformulate the problem

- Tikhonov: $\underset{\mathbf{x}\in\mathbb{R}^n}{\operatorname{argmin}} \|K\mathbf{x}-\mathbf{y}\|_2^2 + \alpha\|\mathbf{x}\|_2^2$

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- F. Tikhonov: $\underset{\mathbf{x}\in\mathbb{R}^n}{\operatorname{argmin}} \|K\mathbf{x}-\mathbf{y}\|_W^2 + \alpha\|\mathbf{x}\|_2^2$, with $W = (KK^*)^{\frac{r-1}{2}}$.

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- Tikhonov: $\operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^n} \|K\mathbf{x}-\mathbf{y}\|_2^2 + \alpha\|\mathbf{x}\|_2^2$

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- Generalized Tikhonov: $\operatorname*{argmin}_{\mathbf{x}\in\mathbb{R}^n} \|K\mathbf{x}-\mathbf{y}\|_2^2 + \alpha\|L\mathbf{x}\|_2^2$, with $L$ semi-positive definite and $\ker(L)\cap\ker(K) = \vec{0}$. $\ker(L)$ should '*approximate the features* 'of $\mathbf{x}^\dagger$.

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# Laplacian - Finite Difference approximation

Poisson (Sturm-Liouville) problem on $[0, 1]$:

$$\begin{cases} -\Delta \mathbf{x}(t) = \mathbf{f}(t) & t \in (0,1), \\ \alpha_1 \mathbf{x}(0) + \beta_1 \mathbf{x}'(0) = \gamma_1, \\ \alpha_2 \mathbf{x}(1) + \beta_2 \mathbf{x}'(1) = \gamma_2. \end{cases}$$

If we consider Dirichlet homogeneous boundary conditions $(\mathbf{x}(0) = \mathbf{x}(1) = 0)$ and 3-point stencil FD approximation:

$$-\Delta \mathbf{x}(t) \approx \frac{-\mathbf{x}(t-h) + 2\mathbf{x}(t) - \mathbf{x}(t+h)}{h^2}, \quad h^2 = n^{-2},$$

$$L = \begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 2 \end{bmatrix} \qquad \ker(L) = \vec{0}.$$

# Laplacian - Finite Difference approximation

$$\cdots$$

If we consider Neumann homogeneous boundary conditions $(\mathbf{x}'(0) = \mathbf{x}'(1) = 0)$ and 3-point stencil FD approximation:

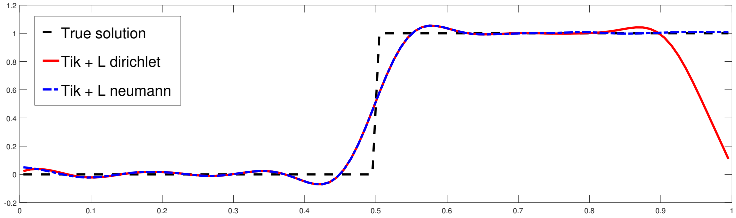
$$-\Delta\mathbf{x}(t) \approx \frac{-\mathbf{x}(t-h) + 2\mathbf{x}(t) - \mathbf{x}(t+h)}{h^2}, \quad h^2 = n^{-2},$$

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 1 \end{bmatrix} \qquad \ker(L) = \mathsf{Span}\{\vec{1}\}.$$

# An easy $1d$ example of oversmoothing - part 2

Blur taken from *Heat*$(n, \kappa)$ in $\mathrm{Regtools}$, $n = 100, \kappa = 1$ and $2\%$ $\mathrm{noise.}$ True solution:

$$\mathbf{x}^\dagger : [0,1] \to \mathbb{R} \qquad \text{s.t.} \qquad \mathbf{x}^\dagger(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 0.5, \\ 1 & \text{if } 0.5 < t \leq 1. \end{cases}$$



## Graph Laplacian

- An image/signal $\mathbf{x}$ can be represented by a weighted undirected graph $\mathcal{G} = (V, E, w)$:
  - the nodes $v_i \in V$ are the pixels of the image/signal and $\mathbf{x}_i \geq 0$ is the color intensity of $\mathbf{x}$ at $v_i$.
  - an edge $e_{i,j} \in E \subseteq V \times V$ exists if the pixels $v_i$ and $v_j$ are connected, i.e., $v_i \sim v_j$.
  - $w : E \to \mathbb{R}$ is a similarity (positive) weight function, $w(e_{i,j}) = w_{i,j}$.
- The graph Laplacian is defined as $\Delta_w^{(n)} \mathbf{x}_i = \sum_{v_j \sim v_i} w_{i,j} \left(\mathbf{x}_i - \mathbf{x}_j\right)$.

**Remark**

$$\int_{([0,1],\mu)} \mathbf{x}''(t)\overline{\phi(t)}\, d\mu(t) = \int_{([0,1],\mu)} \mathbf{x}(t)\overline{\phi''(t)}\, d\mu(t)$$

# Graph Laplacian - Example

**Example**. In the $1d$ case, if we define

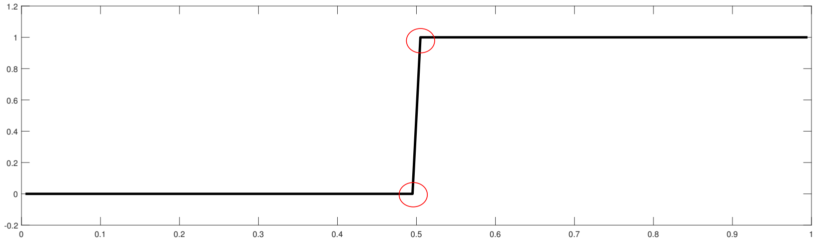
$$v_i \sim v_j \text{ iff } i = j+1 \text{ or } i = j-1, \qquad w_{i,j} = \begin{cases} 1 & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

then it holds

$$\Delta_w^{(n)} = L_w^{(n)} = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 1 \end{bmatrix}$$

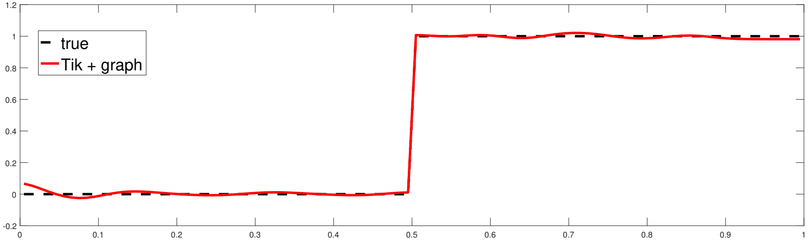
# Question

**Why should the red points be connected?**



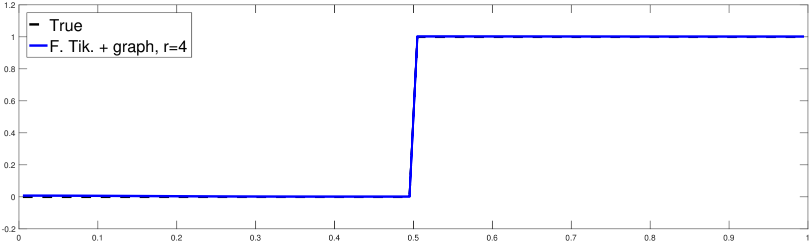
# Answer

**They should not, indeed**

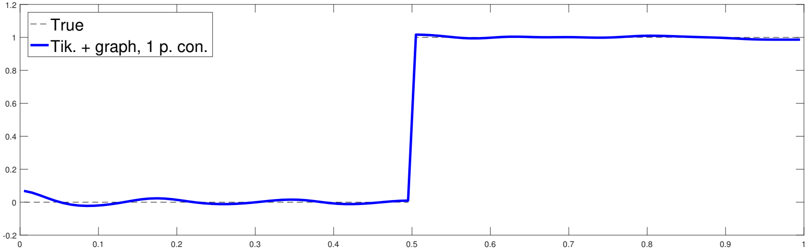



$$L_w^{(n)} = \begin{bmatrix} L_w^{(n/2)} & 0 \\ 0 & L_w^{(n/2)} \end{bmatrix}$$

# Fractional Tikhonov + Graph Laplacian

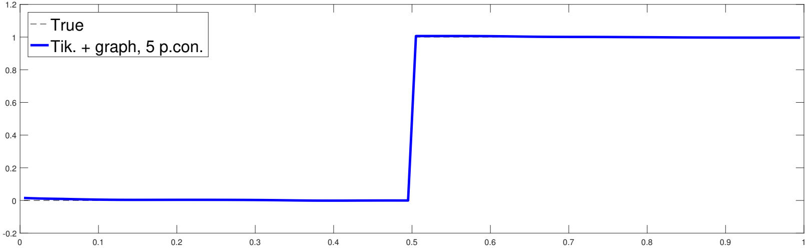


# Remark 1/2



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
True
Tik. + graph, 1 p. con.

# Remark 2/2


1.2
1
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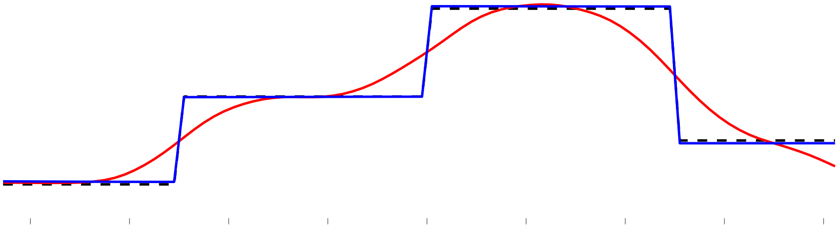
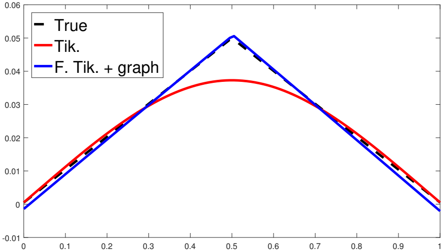
0
-0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
True
Tik. + graph, 5 p.con.

# another example - heat$(n, 1)$ 5% noise



$$L_w^{(n)} = \text{diag}\left(L_w^{(n/4)}, L_w^{(n/4)}, L_w^{(n/4)}, L_w^{(n/4)}\right)$$

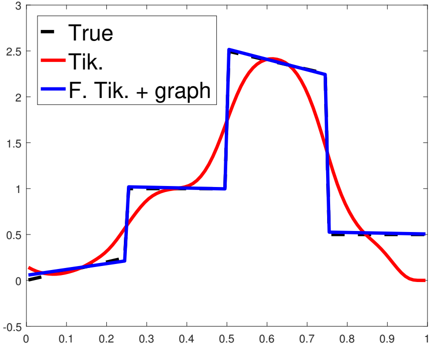
# another another example - deriv2$(n, 3)$, 2% noise




$$L_w^{(n)} = \begin{bmatrix} L_w^{(n/2)} & 0 \\ 0 & L_w^{(n/2)} \end{bmatrix} \quad L_w^{(n/2)} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$\ker(L_w^{(n/2)}) = \text{Span}\{\vec{1}, \vec{t}\}$$

# (another)$^3$ example - heat$(n, 1)$ 2% noise



## Some references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