Lake Como School of Advanced Studies, May 21-25, 2018

Variational Approach for Missing Data Recovery in Imaging

Raymond H. Chan
Department of Mathematics
Chinese University of Hong Kong

Supported by
HKRGC

Contents

1. Introduction to Imaging
2. Introduction to Tightframe
3. Tightframe Applications
4. High-Resolution Image Reconstruction
5. Parameter Estimation
6. Ground-based Astronomy
7. Segmentation
8. Convergence of Tightframe Algorithm
9. New Tightframe Applications

Chapter 1

Introduction to Imaging

Outline

1. Applications
2. Image Model
3. Denoising
4. Deblurring
5. Boundary Conditions

Inpainting

restored image

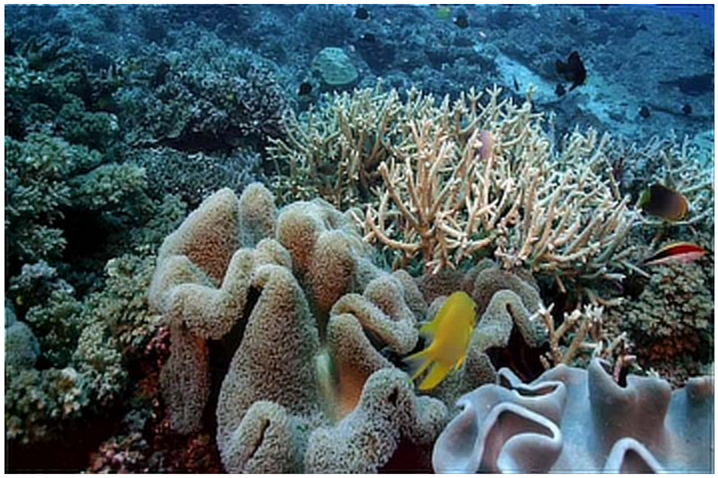
5

Noise Removal

restored image

6

High-Resolution Image Reconstruction



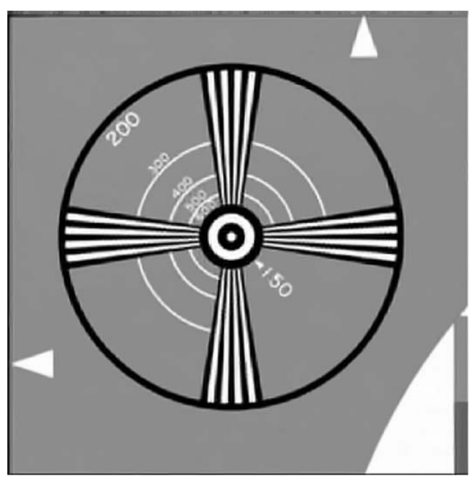
Tightframe interpolation

7

Multi-frame Super-Resolution Reconstruction

19 frames
of size
57×49

Low resolution video



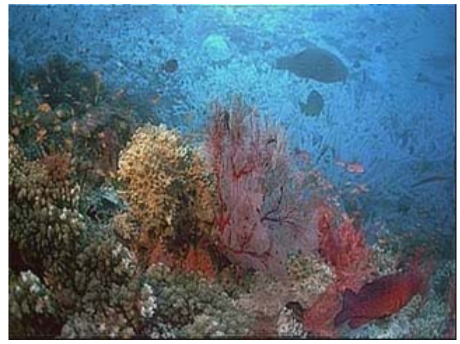
Our method

Source: https://users.soe.ucsc.edu/~milanfar/software/sr-datasets.html

8

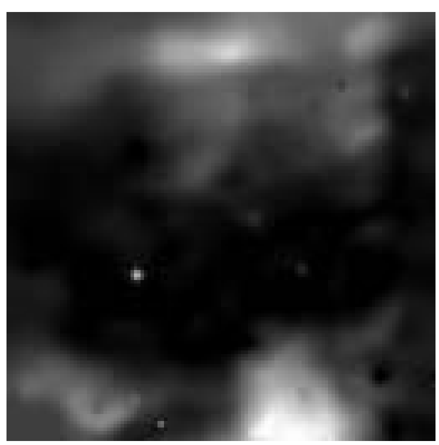
Video Enhancement

input video



high-definition video

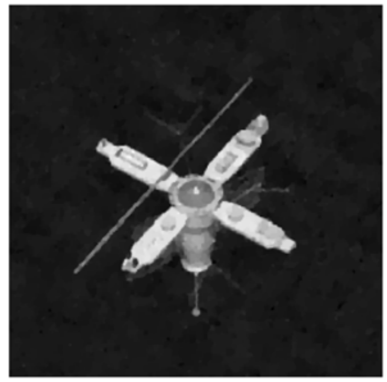
Astronomical Infrared Imaging



Reconstruction image

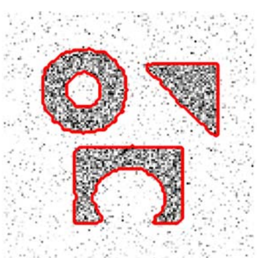
Ground-based Astronomy

observed image
from telescope



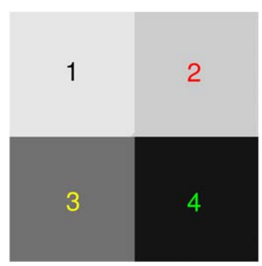
reconstructed image

Segmentation



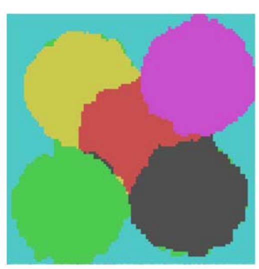
motion + Gaussian

Gamma noise

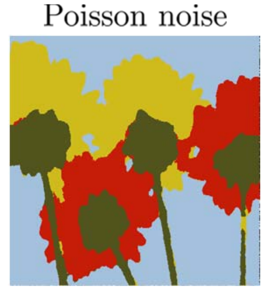


Poisson noise

MRA image

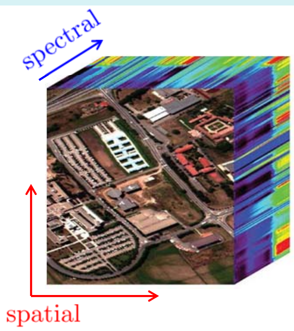


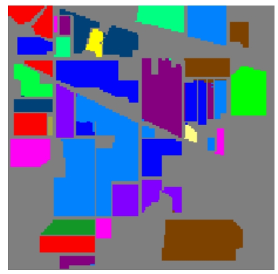
motion + Gaussian



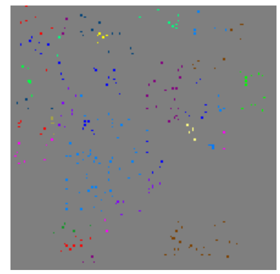
60% pixel loss
+ Gaussian

Classification

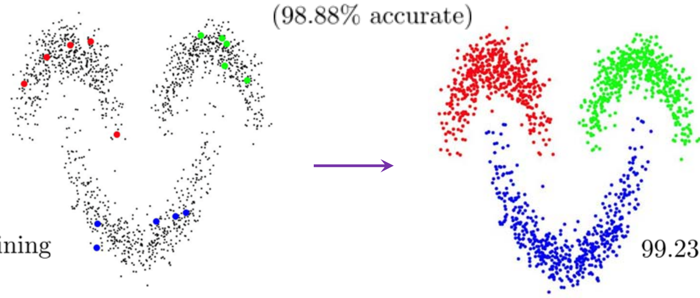




Ground truth
(98.88% accurate)



10% training pixels



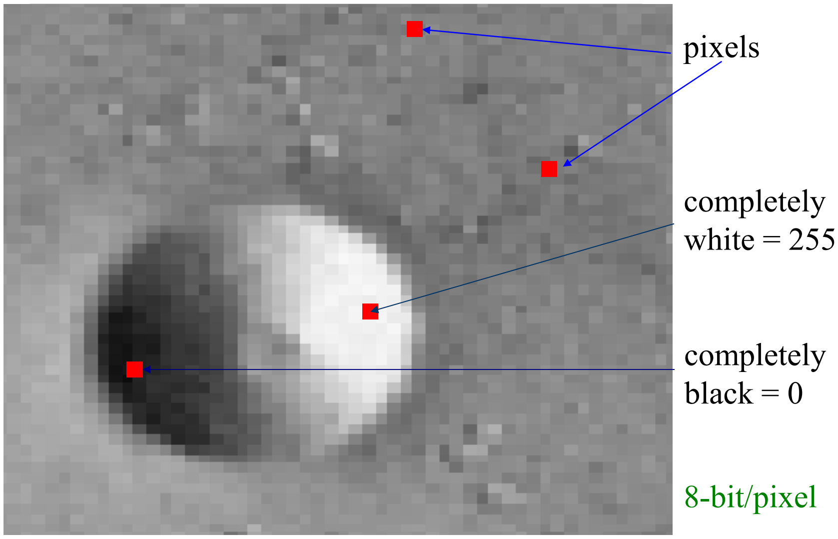
13

Outline

1. **Applications**
2. **Image Model**
3. **Denoising**
4. **Deblurring**
5. **Boundary Conditions**

14

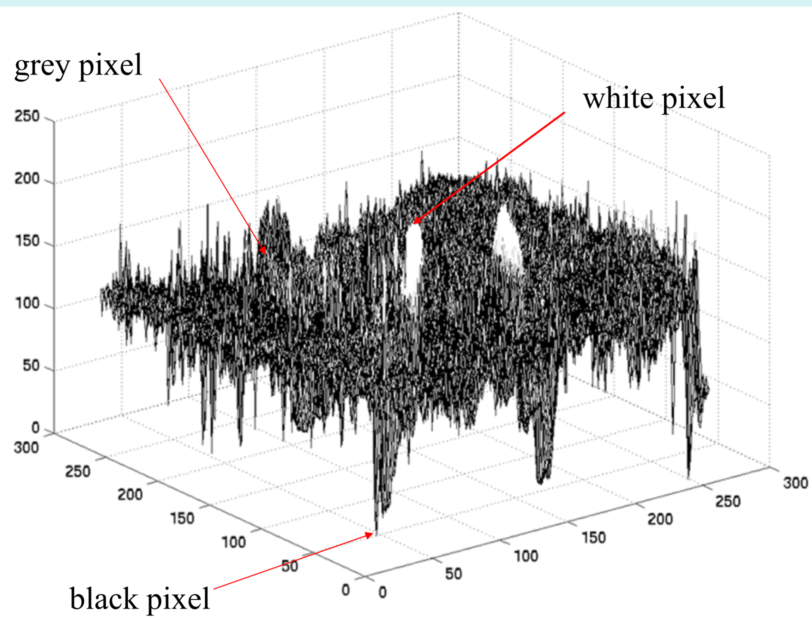
What is (gray-scale) image?



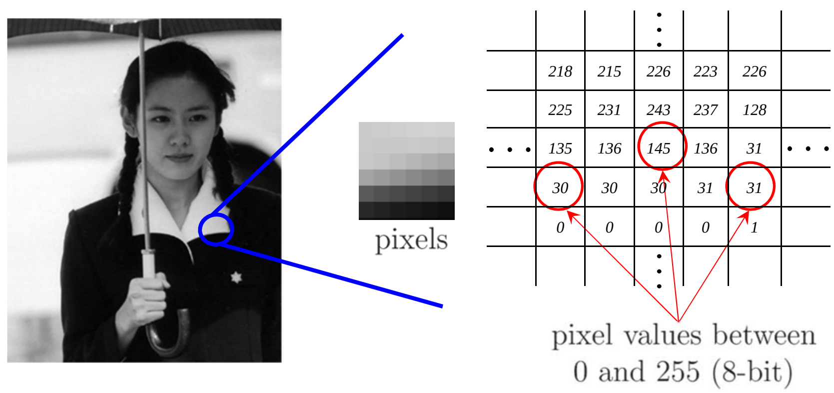
8-bit/pixel

The moon as seen by human

The Moon as Seen by the Computer



What is (gray-scale) image?



1000-by-1000 image = 1000-by-1000 matrix

concatenate into 1M-vector

Color Images

RGB (red, green and blue channels) 24-bit color:

color image　　red channel

1000-by-1000 color image = 1000-by-1000-by-3 tensor

Image Model

$f(x, y) =$

$\approx$ constant

smooth jumps
not preferred

sharp jumps
are preferred

Images are piecewise-smooth functions
with sharp jumps at edges

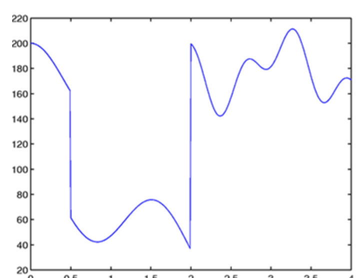
19

What is an image?

$$\text{image} = \text{smooth parts} + \text{jumps}$$

$$= \text{low-frequency parts} + \text{high-frequency part}$$

where high-frequency parts have big magnitudes



20

Outline

1. Applications
2. Image Model
3. Denoising
4. Deblurring
5. Boundary Conditions

Variational Method for Denoising

Restoration = Minimization of a cost functional F

Total Variation of Rudin, Osher and Fatemi:

$$\min_{\mathbf{f}} F_{\mathbf{y}}(\mathbf{f}) = \min_{\mathbf{f}}[\underbrace{\tfrac{1}{2}\|\mathbf{f}-\mathbf{y}\|_{L^2}^2}_{\text{data fitting term}} + \underbrace{\beta \int |\nabla \mathbf{f}|}_{\text{regularization term}}].$$

i.e. $\mathbf{f}$ is close to $\mathbf{y}$ but its TV norm is small.

Minimizing $|\nabla \mathbf{f}|$ means minimizing oscillations.

Variational Method for Denoising

Euler-Lagrange Equation:

$$\frac{\partial F_{\mathbf{y}}(\mathbf{f})}{\partial \mathbf{f}} = \mathbf{0} \Longrightarrow \mathbf{f} - \mathbf{y} - \beta \nabla \cdot \left(\frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) = \mathbf{0}.$$

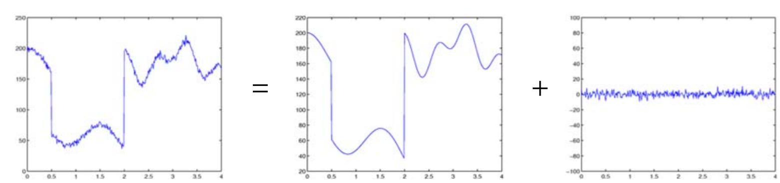
- ☐ Nonlinear PDE—difficult to solve
- ☐ Solve until steady state:

$$\frac{\partial \mathbf{f}}{\partial t} = \beta \nabla \cdot \left(\frac{\nabla \mathbf{f}}{|\nabla \mathbf{f}|} \right) - \mathbf{f} + \mathbf{y}$$

23

Wavelet Denoising

image = low-frequency parts + high-frequency part



- ☐ high frequency components have big magnitudes
- ☐ (Gaussian) noise are high frequency components with small magnitudes

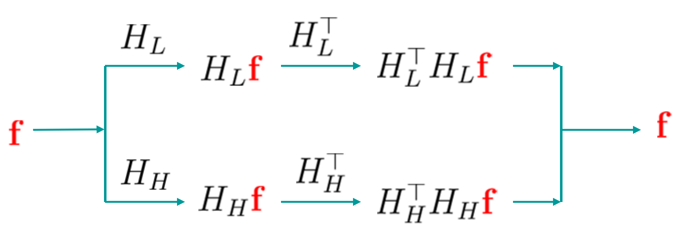
24

Wavelet Denoising

- wavelet transform: orthogonal transform with a low-pass filter H_L and one or more high-pass filters H_H.
- $H_L\mathbf{f}$: low–refrequency parts of $\mathbf{f}$
- $H_H\mathbf{f}$: high–refrequency parts of $\mathbf{f}$
- *perfect-reconstruction formula*:

$$H_L^\top H_L + H_H^\top H_H = I$$

Wavelet Denoising



For image $\mathbf{f}$

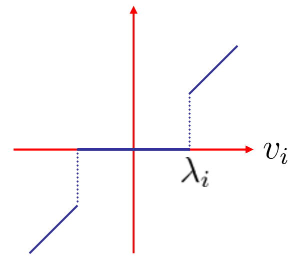
- $H_L\mathbf{f}$: smooth parts of $\mathbf{f}$
- $H_H\mathbf{f}$: jumps of $\mathbf{f}$, and they are big.

For noise $\mathbf{n}$, both $H_L\mathbf{n}$ and $H_H\mathbf{n}$ are small.

Thresholding

Hard-thresholding of $\mathbf{v} = \begin{bmatrix} v_i \\ \vdots \\ v_n \end{bmatrix}$ is

$$\mathcal{T}_\lambda(\mathbf{v}) = \begin{bmatrix} t_{\lambda_1}(v_i) \\ \vdots \\ t_{\lambda_n}(v_n) \end{bmatrix}$$



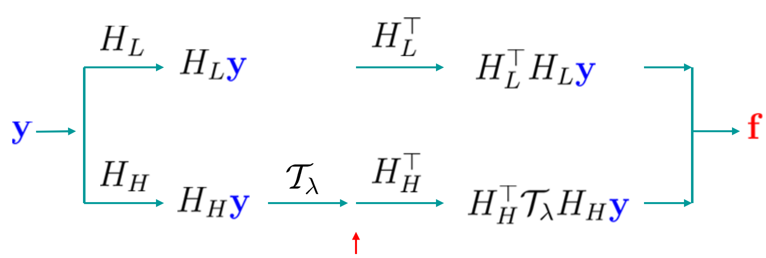
where

$$t_{\lambda_i}(v_i) \equiv \begin{cases} v_i, & \text{if } |v_i| > \lambda_i, \\ 0, & \text{if } |v_i| \le \lambda_i. \end{cases}$$

Wavelet Denoising

Given $\mathbf{y} = \mathbf{f} + \mathbf{n}$

- ❑ decompose $\mathbf{y} = H_L\mathbf{y} + H_H\mathbf{y}$
- ❑ threshold high-frequency part $H_H\mathbf{y}$
- ❑ reconstruct: synthesis high and low parts



noise are thresholded and jumps are kept

Wavelet Denoising

Denoised image using biorthogonal spline wavelet:

noisy image

denoised image

Images taken from Matlab wavelet toolbox.

29

Outline

1. **Applications**
2. **Image Model**
3. **Denoising**
4. **Deblurring**
5. **Boundary Conditions**

30

Blurring

image due to
motion to
the right.

Motion blur

31

Blurring Process

Observed $\mathbf{y}$ obtained from true image $\mathbf{f}$ as follows:

$\mathbf{y} =$ 

$$\mathbf{y}(i) = \mathbf{f}(i) + \mathbf{f}(i+1) + \mathbf{f}(i+2) + \cdots + \mathbf{f}(i+k)$$

32

Blurring Process

Thus for motion blur:

$$\mathbf{y}(i) = \sum_t \mathbf{f}(i-t).$$

More general blur:

$$\mathbf{y}(i) = \sum_t b(t)\mathbf{f}(i-t), \quad \text{with } b(t) \geq 0.$$

In matrix terminology:

blurring matrix → $\mathcal{B}\mathbf{f} = \mathbf{y}$

33

For motion blur, $\mathcal{B}$ is block diagonal Toeplitz matrix:

$$\mathcal{B} = \begin{bmatrix} L & & & & & \\ & L & & \mathbf{0} & & \\ & & L & & & \\ & & & L & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & L \end{bmatrix}$$

with L a lower-Δ Toeplitz matrix.

$$L = \begin{bmatrix} 1 & 0 & & & & & & \\ 1 & 1 & 0 & & & & & \\ 1 & 1 & 1 & 0 & & & 0 & \\ 1 & 1 & 1 & 1 & 0 & & & \\ 0 & 1 & 1 & 1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & & 1 & 1 & 1 & 1 \end{bmatrix}_{n\times n}$$

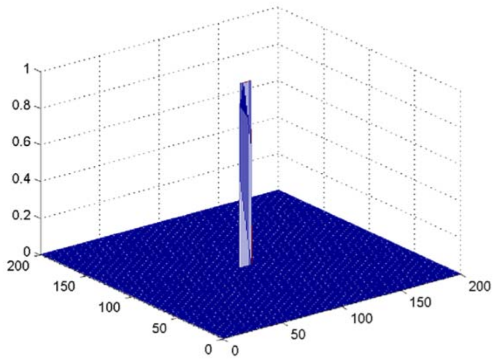
for n-by-n image

34

Point-spread Function

For general blur, $\mathcal{B}$ is block-Toeplitz-Toeplitz-block, hence determined by the middle row.

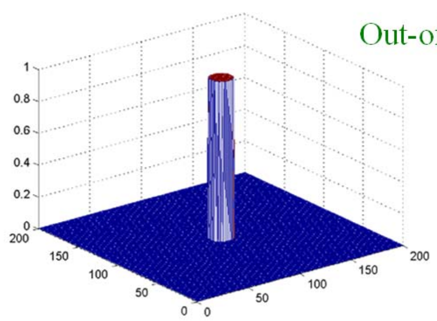
Reshape the middle row into an n-by-n matrix and display it as a 2D function or an image:



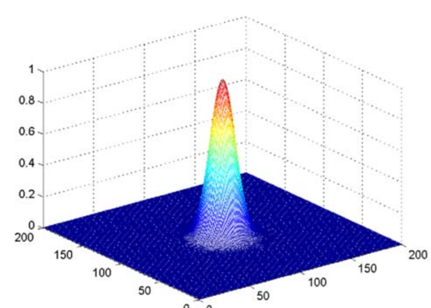
Point-spread function for motion blur

35

The point-spread function tells how every pixel in the image is blurred.



Out-of-focus blur



Gaussian blur

Given the point-spread function, we can form the BTTB $\mathcal{B}$ blurring matrix accordingly.

36

Blur Model

observed image= blurred image + noise

$$\mathbf{y} = \mathcal{B}\mathbf{f} + \mathbf{n}$$

Naive inversion: $\mathcal{B}^{-1}\mathbf{y} = \mathbf{f} + \mathcal{B}^{-1}\mathbf{n}$ ← noise got amplified

37

Variational Method for Deblurring

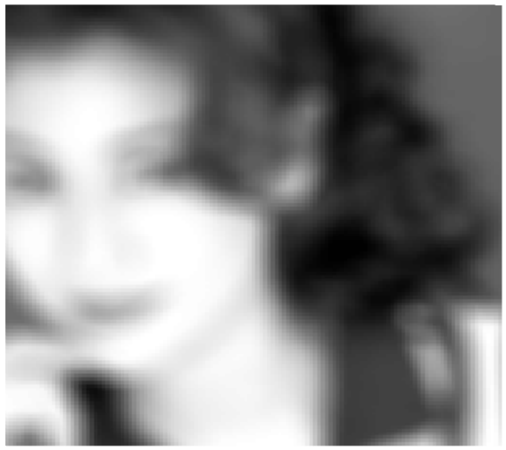
Restoration = Minimization of a cost functional F

$$\min_{\mathbf{f}} F_{\mathbf{y}}(\mathbf{f}) = \min_{\mathbf{f}} \{ \underbrace{\|\mathcal{B}\mathbf{f} - \mathbf{y}\|}_{\text{data fitting term}} + \underbrace{\beta \cdot \|\mathcal{D}\mathbf{f}\|}_{\text{regularization term}} \},$$

i.e. $\mathcal{B}\mathbf{f}$ is close to $\mathbf{y}$ and yet $\mathbf{f}$ is smooth.

38

Variational Method for Deblurring



$$\min_{\mathbf{f}} F_{\mathbf{y}}(\mathbf{f}) = \min_{\mathbf{f}}\{\|\mathcal{B}\mathbf{f} - \mathbf{y}\|_2^2 + \beta\|\nabla\mathbf{f}\|_2^2\},$$

39

Variational vs Tight-frame

From observed $\mathbf{y}$ to reconstructed $\mathbf{f}$

Variational Approach	*Tight-Frame Approach*
$\min_{\mathbf{f}} F_{\mathbf{y}}(\mathbf{f})$	$\mathcal{A}\mathbf{z}$ and $\mathcal{A}^\top\mathbf{z}$,
or	and
$\frac{\partial F_{\mathbf{y}}(\mathbf{f})}{\partial \mathbf{f}} = \mathbf{0}$	thresholding
	where $\mathcal{A}^\top\mathcal{A} = \mathcal{I}$

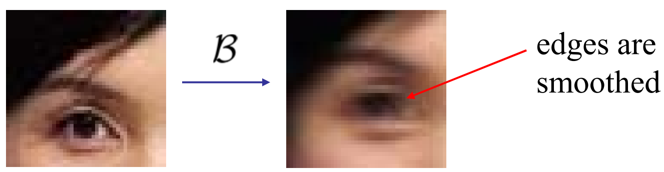
40

Blurs are Low-pass Filters

Blurring has the form:

$$\mathbf{y}(i) = \sum_t b(t)\mathbf{f}(i-t), \quad \text{with } b(t) \geq 0$$

- blurring is a weighted averaging
- high frequencies in $\mathbf{f}$ (e.g. edges) averaged out or smoothed
- $\mathcal{B}\mathbf{f}$ consists mainly of low frequencies



41

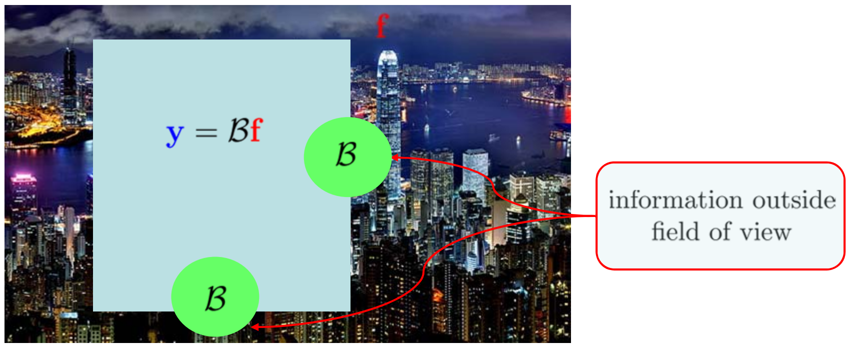
Outline

1. **Applications**
2. **Image Model**
3. **Denoising**
4. **Deblurring**
5. **Boundary Conditions**

42

Boundary Conditions

Blurred image $\mathbf{y}$ involves information of true image $\mathbf{f}$ outside the field of view.



43

Periodic Boundary Conditions

Gonzalez and Woods, 93

Assume data are periodic near the boundary

44

Dirichlet (Zero) Boundary Conditions

Boo and Bose, IJIST 97

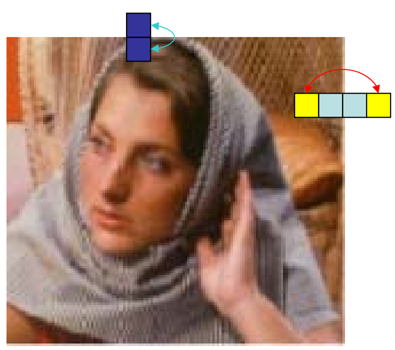
Assume data are zero outside the boundary

45

Neumann Boundary Conditions

Ng, C. & Tang, SISC (2000)

Assume data are mirror reflective near the boundary

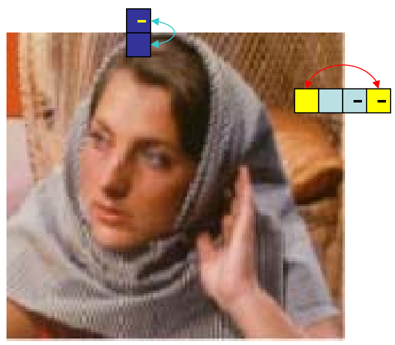


46

Anti-reflexive Boundary Conditions

Serra, SISC (2003)

Assume data are negated and reflected near the boundary



Effect of Boundary Conditions

periodic boundary condition is used here