A Semi-blind regularization algorithm for inverse problems with application to image deblurring

Computational Methods for Inverse Problems in Imaging

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Outline

We consider inverse problems of the from

$$B(k, f) = g.$$

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We consider inverse problems of the from

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- f: desired solution;

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We consider inverse problems of the from

$$B(k, f) = g.$$

- f: desired solution;
- g: the measured data;
- k: variable on which the operator B depends, e.g., integral kernel.

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Introduction

The problem at hand (continued)

We assume that both g and k are affected by (Gaussian) noise.

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Thus the problem becomes

$$B(k_\epsilon, f) = g_\delta,$$

where

$$\|k - k_\epsilon\| < \epsilon \quad \text{and} \quad \|g - g_\delta\| < \delta.$$

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We would like to construct a method that simultaneously recovers f and k.

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We would like to construct a method that simultaneously recovers f and k.

We refer to this kind of inverse problem as semi-blind.

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Blind and semi-blind problems have been largely investigated, see, e.g., Almeida, Bardsley, Beck, Ben-Tal, Bertero, Bioucas-Dias, Bleyer, Boccacci, Bonettini, Brinicombe, Chan, Cornelio, Dykes, Figueiredo, Fish, He, Jefferies, Kanzow, La Camera, Marquina, Nagy, Ng, Oliveira, Osher, Pesquet, Pike, Plemmons, Porta, Prato, Ramlau, Rebegoldi, Reichel, Soodhalter, Walker, Wong, . . .

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In particular, we would like to propose a model and an algorithm for semi-blind regularization starting from the work in [I.R. Bleyer and R. Ramlau, IP2013-2015].

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Introduction
Inspiring work

We now briefly describe the approach and the results in [I.R. Bleyer and R. Ramlau, IP2013-2015].

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They considered the following minimization problem

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\begin{aligned}
(k^*, f^*) &= \arg\min_{k,f} \left\| B(k,f) - g_\delta \right\|^2 + \gamma \left\| k - k_\epsilon \right\|^2 + \alpha \left\| Lf \right\|^2 + \beta \left\| k \right\|_1 \\
&= \arg\min_{k,f} \tilde{J}^{\delta,\epsilon}_{\alpha,\beta}(k,f),
\end{aligned}
$$

where L is a continuously invertible linear operator.

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&= \arg\min_{k,f} \tilde{J}^{\delta,\epsilon}_{\alpha,\beta}(k,f),
\end{aligned}
$$

where L is a continuously invertible linear operator.

In [I.R. Bleyer and R. Ramlau, IP2013] they proved that

- The minimization above is well posed;
- The minima are stable;
- The minimization above is a regularization method if the parameter are chosen accordingly to the noise.

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Introduction

Inspiring work (continued)

In [I.R. Bleyer and R. Ramlau, IP2015] they developed an algorithm for computing stationary point of $\tilde{J}^{\delta,\epsilon}_{\alpha,\beta}$.

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Introduction
Inspiring work (continued)

In [I.R. Bleyer and R. Ramlau, IP2015] they developed an algorithm for computing stationary point of $\tilde{J}^{\delta,\epsilon}_{\alpha,\beta}$.

They used an alternating minimization algorithm.

$$k^{(j+1)} = \arg\min_k \left\| B\left(k, f^{(j)}\right) - g_\delta \right\|^2 + \gamma \left\| k - k_\epsilon \right\|^2 + \beta \left\| k \right\|_1$$

$$f^{(j+1)} = \arg\min_f \left\| B\left(k^{(j+1)}, f\right) - g_\delta \right\|^2 + \alpha \left\| Lf \right\|^2$$

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$$f^{(j+1)} = \arg\min_f \left\| B\left(k^{(j+1)}, f\right) - g_\delta \right\|^2 + \alpha \left\| Lf \right\|^2$$

They proved that there exist a subsequence $\left\{ \left(k^{(j_i)}, f^{(j_i)} \right) \right\}_{j_i}$ that converges to a stationary point of $\tilde{J}^{\delta,\epsilon}_{\alpha,\beta}$.

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The continuous model
Formulation

We now extend the results of [I.R. Bleyer and R. Ramlau, IP2013-2015] to a more general functional.

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We consider the functional

$$J_{\alpha,\beta}^{\delta,\epsilon}(k,f) = \|B(k,f) - g_\delta\|^2 + \gamma\,\|k - k_\epsilon\|^2 + \alpha^{\mathrm{E}}\,\|f\|^2 + \alpha^{\mathrm{R}}\mathcal{R}_f(f) + \beta\mathcal{R}_k(k),$$

where $\mathcal{R}_f(f)$ and $\mathcal{R}_k(k)$ are convex regularization term.

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where $\mathcal{R}_f(f)$ and $\mathcal{R}_k(k)$ are convex regularization term.

In the following we will assume that $f, k \in H^1$ and we will set

$$\mathcal{R}_f(\cdot) = \mathcal{R}_k(\cdot) = \|\cdot\|_{TV}.$$

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Consequently we use the following notation

$$\alpha^{\mathrm{R}} = \alpha^{\mathrm{TV}}.$$

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We now state some theoretical property of $J_{\alpha,\beta}^{\delta,\epsilon}(k, f)$.

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The continuous model
Theoretical analysis

We now state some theoretical property of $J^{\delta,\epsilon}_{\alpha,\beta}(k, f)$.

Theorem (Existence)

Assume that B is strongly continuous on its domain, then the functional $J^{\delta,\epsilon}_{\alpha,\beta}(f, k)$ has a global minimizer.

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We now state some theoretical property of $J_{\alpha,\beta}^{\delta,\epsilon}(k,f)$.

Theorem (Existence)

Assume that B is strongly continuous on its domain, then the functional $J_{\alpha,\beta}^{\delta,\epsilon}(f,k)$ *has a global minimizer.*

Theorem (Stability)

With the same notation and assumptions as above, let α^{E}, α^{TV}, β, *and* γ *be fixed. Let* $(g_{\delta_j})_j$ *and* $(k_{\epsilon_j})_j$ *be sequences such that* $g_{\delta_j} \to g_\delta$ *and* $k_{\epsilon_j} \to k_\epsilon$, *let* (k_j, f_j) *be minimizers obtained with data* $g_{\delta_j}, k_{\epsilon_j}$. *Then there exists a convergent subsequence of* (k_j, f_j) *and the limit of every subsequence is a minimizer of* $J_{\alpha,\beta}^{\delta,\epsilon}$.

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We first define the concept of minimum norm solution in our framework

Definition

The minimum norm solution of $B(k_0, f) = g_0$ is

$$f^{\dagger} = \arg\min_{f \in H^1} \{ \|f\|^2 + \|f\|_{TV} : \; B(k_0, f) = g_0 \}.$$

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Theorem (Regularization property)

Let $(g_{\delta_j})_j$ and $(k_{\epsilon_j})_j$ be sequences such that

$$\left\|g_{\delta_j} - g_0\right\| < \delta_j \text{ and } \left\|k_{\epsilon_j} - k_0\right\| < \epsilon_j$$

and such that $\delta_j, \epsilon_j \to 0$ as $j \to \infty$. Let α_j^{E}, α_j^{TV}, and β_j be sequences such that $\alpha_j^{\mathrm{E}}, \alpha_j^{\mathrm{TV}}, \beta_j \to 0$ as $j \to \infty$, moreover, assume that it holds

$$\lim_{j\to\infty} \frac{\delta_j^2 + \gamma\epsilon_j^2}{\alpha_j^{\mathrm{E}}} = 0, \quad \lim_{j\to\infty} \frac{\alpha_j^{\mathrm{TV}}}{\alpha_j^{\mathrm{E}}} = 1, \quad \lim_{j\to\infty} \frac{\beta_j}{\alpha_j^{\mathrm{E}}} = \eta \quad 0 < \eta < \infty.$$

Call $(k_j, f_j) := \left(k_{\alpha_j,\beta_j}^{\delta_j,\epsilon_j}, f_{\alpha_j,\beta_j}^{\delta_j,\epsilon_j}\right)$, then there exists a convergent subsequence of (k_j, f_j) such that $k_j \to k_0$ and the limit of every convergent subsequence of f_j is the minimum norm solution of $B(k_0, f) = g_0$.

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We now formulate an algorithm for computing a stationary point of $J_{\alpha,\beta}^{\delta,\epsilon}(\mathbf{k},\mathbf{f})$, where, for simplicity, we only consider the finite dimensional case, i.e., we assume that $\mathbf{k},\mathbf{f} \in \mathbb{R}^N$.

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We now formulate an algorithm for computing a stationary point of $J^{\delta,\epsilon}_{\alpha,\beta}(\mathbf{k},\mathbf{f})$, where, for simplicity, we only consider the finite dimensional case, i.e., we assume that $\mathbf{k},\mathbf{f} \in \mathbb{R}^N$.

We use the Alternating Directions Multipliers Method (ADMM) and provide a proof of convergence.

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We use the Alternating Directions Multipliers Method (ADMM) and provide a proof of convergence.

We impose some constraints on the solution, i.e., we impose that $(\mathbf{k},\mathbf{f}) \in \Omega_{\mathbf{k}} \times \Omega_{\mathbf{f}}$.

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We impose some constraints on the solution, i.e., we impose that $(\mathbf{k},\mathbf{f}) \in \Omega_{\mathbf{k}} \times \Omega_{\mathbf{f}}$.

Thus we have to solve

$$(\mathbf{k}^*,\mathbf{f}^*) = \arg\min_{\mathbf{k}\in\Omega_{\mathbf{k}},\mathbf{f}\in\Omega_{\mathbf{f}}} \|B(\mathbf{k},\mathbf{f}) - \mathbf{g}_\delta\|^2 + \gamma\|\mathbf{k} - \mathbf{k}_\epsilon\|^2 + \alpha^{\mathrm{E}}\|\mathbf{f}\|^2 + \alpha^{\mathrm{TV}}\|\mathbf{f}\|_{TV} + \beta\|\mathbf{k}\|_{TV}.$$

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Minimization Algorithm

Formulation (continued)

We rewrite the minimization problem in a more useful way

$$(\mathbf{k}^*, \mathbf{f}^*) = \arg \min_{\substack{\tilde{\mathbf{k}}\in\Omega_{\mathbf{k}}, \tilde{\mathbf{f}}\in\Omega_{\mathbf{f}} \\ \hat{\mathbf{k}}, \hat{\mathbf{f}}, \mathbf{k}, \mathbf{f}}} \left\{ \left\| B(\mathbf{k}, \mathbf{f}) - \mathbf{g}_\delta \right\|^2 + \alpha^{\mathrm{E}} \left\| \mathbf{f} \right\|^2 + \alpha^{\mathrm{TV}} \left\| \hat{\mathbf{f}} \right\|_{TV} \right.$$

$$+ \gamma \left\| \mathbf{k} - \mathbf{k}_\epsilon \right\|^2 + \beta \left\| \hat{\mathbf{k}} \right\|_{TV},$$

$$\left. \mathbf{k} = \tilde{\mathbf{k}}, \mathbf{f} = \tilde{\mathbf{f}}, \mathbf{k} = \hat{\mathbf{k}}, \mathbf{f} = \hat{\mathbf{f}} \right\}.$$

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Minimization Algorithm
Formulation (continued)

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$$+ \gamma \left\| \mathbf{k} - \mathbf{k}_\epsilon \right\|^2 + \beta \left\| \hat{\mathbf{k}} \right\|_{TV},$$

$$\mathbf{k} = \tilde{\mathbf{k}}, \mathbf{f} = \tilde{\mathbf{f}}, \mathbf{k} = \hat{\mathbf{k}}, \mathbf{f} = \hat{\mathbf{f}} \Big\}.$$

The associated Augmented Lagrangian is

$$\mathcal{L}\left(\tilde{\mathbf{f}}, \hat{\mathbf{f}}, \mathbf{f}, \tilde{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{k}; \lambda, \xi, \zeta, \mu\right)$$

$$= \left\| B(\mathbf{k}, \mathbf{f}) - \mathbf{g}_\delta \right\|^2 + \alpha^{\mathrm{E}} \left\| \mathbf{f} \right\|^2 + \alpha^{\mathrm{TV}} \left\| \hat{\mathbf{f}} \right\|_{TV} + \gamma \left\| \mathbf{k} - \mathbf{k}_\epsilon \right\|^2 + \beta \left\| \hat{\mathbf{k}} \right\|_{TV}$$

$$+ \frac{\omega}{2} \left\| \tilde{\mathbf{f}} - \mathbf{f} \right\|^2 - \left\langle \lambda, \tilde{\mathbf{f}} - \mathbf{f} \right\rangle + \frac{\omega}{2} \left\| \hat{\mathbf{f}} - \mathbf{f} \right\|^2 - \left\langle \xi, \hat{\mathbf{f}} - \mathbf{f} \right\rangle$$

$$+ \frac{\omega}{2} \left\| \tilde{\mathbf{k}} - \mathbf{k} \right\|^2 - \left\langle \zeta, \tilde{\mathbf{k}} - \mathbf{k} \right\rangle + \frac{\omega}{2} \left\| \hat{\mathbf{k}} - \mathbf{k} \right\|^2 - \left\langle \mu, \hat{\mathbf{k}} - \mathbf{k} \right\rangle.$$

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Minimization Algorithm
Formulation (continued)

We need the following

Assumption

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Minimization Algorithm
Formulation (continued)

We need the following

Assumption

(a) $B(\mathbf{k}, \mathbf{f})$ *is bilinear;*

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Minimization Algorithm
Formulation (continued)

We need the following

Assumption

(a) *$B(\mathbf{k}, \mathbf{f})$ is bilinear;*

(b) *If $\mathbf{k} = \mathbf{0}$ or $\mathbf{f} = \mathbf{0}$ then $B(\mathbf{k}, \mathbf{f}) = \mathbf{0}$;*

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Minimization Algorithm
Formulation (continued)

We need the following

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(a) $B(\mathbf{k}, \mathbf{f})$ *is bilinear;*

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(c) *If for a set* $K = \{\mathbf{k}^{(l)}\}$ *it holds that* $\left\|\mathbf{k}^{(l)}\right\| < C_K$ *then* $A_{\mathbf{k}^{(l)}} = B\left(\mathbf{k}^{(l)}, \cdot\right)$*, have bounded norm;*
If for a set $F = \{\mathbf{f}^{(l)}\}$ *it holds that* $\left\|\mathbf{f}^{(l)}\right\| < C_F$*, then* $A_{\mathbf{f}^{(l)}} = B\left(\cdot, \mathbf{f}^{(l)}\right)$ *have bounded norm;*

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Minimization Algorithm
Formulation (continued)

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If for a set $F=\{\mathbf{f}^{(l)}\}$ *it holds that* $\left\|\mathbf{f}^{(l)}\right\|<C_F$*, then* $A_{\mathbf{f}^{(l)}}=B\left(\cdot,\mathbf{f}^{(l)}\right)$ *have bounded norm;*

(d) *The parameter* ω *is large enough so that*

$$\left\|B(\mathbf{k},\mathbf{f})-\mathbf{g}_\delta\right\|^2+\alpha^{\mathrm{E}}\left\|\mathbf{f}\right\|^2+\frac{\omega}{2}\left\|\hat{\mathbf{f}}-\mathbf{f}\right\|^2-\left\langle\xi,\hat{\mathbf{f}}-\mathbf{f}\right\rangle,$$

$$\left\|B(\mathbf{k},\mathbf{f})-\mathbf{g}_\delta\right\|^2+\gamma\left\|\mathbf{k}-\mathbf{k}_\epsilon\right\|^2+\frac{\omega}{2}\left\|\hat{\mathbf{k}}-\mathbf{k}\right\|^2-\left\langle\mu,\hat{\mathbf{k}}-\mathbf{k}\right\rangle$$

are strongly convex with modulus ρ*.*

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Minimization Algorithm
Formulation (continued)

Applying the ADMM algorithm we have

Algorithm (SeB-A)

for $j = 0, 1, \ldots$ **do**

$$\begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} \\ \mathbf{k}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{f}},\hat{\mathbf{f}},\mathbf{k}} \mathcal{L}\left(\tilde{\mathbf{f}}, \hat{\mathbf{f}}, \mathbf{k} \middle| \tilde{\mathbf{k}}^{(j)}, \hat{\mathbf{k}}^{(j)}, \mathbf{f}^{(j)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \tilde{\mathbf{k}}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} \\ \mathbf{f}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{k}},\hat{\mathbf{k}},\mathbf{f}} \mathcal{L}\left(\tilde{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{f} \middle| \tilde{\mathbf{f}}^{(j+1)}, \hat{\mathbf{f}}^{(j+1)}, \mathbf{k}^{(j+1)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \boldsymbol{\lambda}^{(j+1)} \\ \boldsymbol{\xi}^{(j+1)} \\ \boldsymbol{\zeta}^{(j+1)} \\ \boldsymbol{\mu}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}^{(j)} \\ \boldsymbol{\xi}^{(j)} \\ \boldsymbol{\zeta}^{(j)} \\ \boldsymbol{\mu}^{(j)} \end{pmatrix} - \omega \begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} - \mathbf{f}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} - \mathbf{f}^{(j+1)} \\ \tilde{\mathbf{k}}^{(j+1)} - \mathbf{k}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} - \mathbf{k}^{(j+1)} \end{pmatrix};$$

end

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Minimization Algorithm
Formulation (continued)

Most of the minimizations above have closed form

$$\tilde{\mathbf{f}}^{(j+1)} = P_{\Omega_\mathbf{f}}\left(\mathbf{f}^{(j)} + \frac{\boldsymbol{\lambda}^{(j)}}{\omega}\right)$$

$$\mathbf{k}^{(j+1)} = \left(2A^*_{\mathbf{f}^{(j)}}A_{\mathbf{f}^{(j)}} + 2(\gamma+\omega)I\right)^{-1}\left(2A^*_{\mathbf{f}^{(j)}}\mathbf{g}_\delta + 2\gamma\mathbf{k}_\epsilon - \zeta^{(j)} + \omega\tilde{\mathbf{k}}^{(j)} - \boldsymbol{\mu}^{(j)} + \omega\hat{\mathbf{k}}^{(j)}\right)$$

$$\tilde{\mathbf{k}}^{(j+1)} = P_{\Omega_\mathbf{k}}\left(\mathbf{k}^{(j+1)} + \frac{\zeta^{(j)}}{\omega}\right)$$

$$\mathbf{f}^{(j+1)} = \left(2A^*_{\mathbf{k}^{(j+1)}}A_{\mathbf{k}^{(j+1)}} + 2(\alpha^{\mathrm{E}}+2\omega)I\right)^{-1}\left(2A^*_{\mathbf{k}^{(j+1)}}\mathbf{g}_\delta - \boldsymbol{\lambda}^{(j)} + \omega\tilde{\mathbf{f}}^{(j+1)} - \boldsymbol{\xi}^{(j)} + \omega\hat{\mathbf{f}}^{(j+1)}\right)$$

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Minimization Algorithm

Formulation (continued)

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$$\tilde{\mathbf{k}}^{(j+1)} = P_{\Omega_{\mathbf{k}}}\left(\mathbf{k}^{(j+1)} + \frac{\zeta^{(j)}}{\omega}\right)$$

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Whereas the minimizations w.r.t. $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ does not

$$\hat{\mathbf{f}}^{(j+1)} = \arg\min_{\hat{\mathbf{f}}} \left\|\hat{\mathbf{f}}\right\|_{TV} + \frac{\omega}{2\alpha^{\mathrm{TV}}}\left\|\hat{\mathbf{f}} - \left(\mathbf{f}^{(j)} + \frac{\boldsymbol{\xi}^{(j)}}{\omega}\right)\right\|^2$$

$$\hat{\mathbf{k}}^{(j+1)} = \arg\min_{\hat{\mathbf{k}}} \left\|\hat{\mathbf{k}}\right\|_{TV} + \frac{\omega}{2\beta}\left\|\hat{\mathbf{k}} - \left(\mathbf{k}^{(j+1)} + \frac{\boldsymbol{\mu}^{(j)}}{\omega}\right)\right\|^2$$

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Minimization Algorithm

Formulation (continued)

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$$\tilde{\mathbf{k}}^{(j+1)} = P_{\Omega_{\mathbf{k}}}\left(\mathbf{k}^{(j+1)} + \frac{\zeta^{(j)}}{\omega}\right)$$

$$\mathbf{f}^{(j+1)} = \left(2A^*_{\mathbf{k}^{(j+1)}}A_{\mathbf{k}^{(j+1)}} + 2(\alpha^{\mathrm{E}}+2\omega)I\right)^{-1}\left(2A^*_{\mathbf{k}^{(j+1)}}\mathbf{g}_\delta - \boldsymbol{\lambda}^{(j)} + \omega\tilde{\mathbf{f}}^{(j+1)} - \boldsymbol{\xi}^{(j)} + \omega\hat{\mathbf{f}}^{(j+1)}\right)$$

Whereas the minimizations w.r.t. $\hat{\mathbf{f}}$ and $\hat{\mathbf{k}}$ does not

$$\hat{\mathbf{f}}^{(j+1)} = \arg\min_{\hat{\mathbf{f}}} \left\|\hat{\mathbf{f}}\right\|_{TV} + \frac{\omega}{2\alpha^{\mathrm{TV}}}\left\|\hat{\mathbf{f}} - \left(\mathbf{f}^{(j)} + \frac{\boldsymbol{\xi}^{(j)}}{\omega}\right)\right\|^2$$

$$\hat{\mathbf{k}}^{(j+1)} = \arg\min_{\hat{\mathbf{k}}} \left\|\hat{\mathbf{k}}\right\|_{TV} + \frac{\omega}{2\beta}\left\|\hat{\mathbf{k}} - \left(\mathbf{k}^{(j+1)} + \frac{\boldsymbol{\mu}^{(j)}}{\omega}\right)\right\|^2$$

For the resolution of these problems we will have to resort to iterative methods.

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Minimization Algorithm
Theoretical analysis

We perform the theoretical analysis on the unconstrained model, i.e., assuming that $\Omega_{\mathbf{f}} = \Omega_{\mathbf{k}} = \mathbb{R}^N$. In this case we can ignore the Lagrangian multipliers $\boldsymbol{\lambda}$ and $\boldsymbol{\zeta}$ and the auxiliary variables $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{f}}$.

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The proof of convergence of SeB-A is inspired by [M. Hong, Z.-Q. Luo, and M. Razaviyayn, SIOPT2016].

We perform the theoretical analysis on the unconstrained model, i.e., assuming that $\Omega_{\mathbf{f}} = \Omega_{\mathbf{k}} = \mathbb{R}^N$. In this case we can ignore the Lagrangian multipliers λ and ζ and the auxiliary variables $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{f}}$.

The proof of convergence of SeB-A is inspired by [M. Hong, Z.-Q. Luo, and M. Razaviyayn, SIOPT2016].

For the proof of convergence we need the following

Assumption

The norm of the iterates $\mathbf{f}^{(j)}$ and $\mathbf{k}^{(j)}$ generated by SeB-A are uniformly bounded.

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We can now state some preliminary results

Lemma

Let $\xi^{(j)}, \mu^{(j)}, \mathbf{f}^{(j)}, \mathbf{k}^{(j)}$ be the iterations generated by SeB-A. Then we have

$$\left\| \xi^{(j+1)} - \xi^{(j)} \right\| \leq C \left\| \mathbf{f}^{(j+1)} - \mathbf{f}^{(j)} \right\|,$$

$$\left\| \mu^{(j+1)} - \mu^{(j)} \right\| \leq C \left\| \hat{\mathbf{k}}^{(j+1)} - \hat{\mathbf{k}}^{(j)} \right\|$$

where $C > 0$ is a constant.

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Minimization Algorithm

Theoretical analysis (continued)

Proposition

It holds that

$$\begin{aligned}
&\mathcal{L}\left(\mathbf{k}^{(j+1)}, \mathbf{f}^{(j+1)}, \hat{\mathbf{k}}^{(j+1)}, \hat{\mathbf{f}}^{(j+1)}; \boldsymbol{\xi}^{(j+1)}, \boldsymbol{\mu}^{(j+1)}\right) \\
&- \mathcal{L}\left(\mathbf{k}^{(j)}, \mathbf{f}^{(j)}, \hat{\mathbf{k}}^{(j)}, \hat{\mathbf{f}}^{(j)}; \boldsymbol{\xi}^{(j)}, \boldsymbol{\mu}^{(j)}\right) \\
&\leq \left(\frac{C^2}{\omega} - \frac{\rho}{2}\right)\left(\left\|\mathbf{f}^{(j+1)} - \mathbf{f}^{(j)}\right\|^2 + \left\|\hat{\mathbf{k}}^{(j+1)} - \hat{\mathbf{k}}^{(j)}\right\|^2\right) \\
&- \frac{\rho}{2}\left(\left\|\hat{\mathbf{f}}^{(j+1)} - \hat{\mathbf{f}}^{(j)}\right\|^2 + \left\|\mathbf{k}^{(j+1)} - \mathbf{k}^{(j)}\right\|^2\right).
\end{aligned}$$

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Minimization Algorithm

Theoretical analysis (continued)

Lemma

Let $\mathcal{L}$ be the Augmented Lagrangian defined above and $\mathbf{k}^{(j)}, \mathbf{f}^{(j)}, \hat{\mathbf{k}}^{(j)}, \hat{\mathbf{f}}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\mu}^{(j)}$ the iterates generated by SeB-A. Assume that $\frac{C^2}{\omega} - \frac{\rho}{2} < 0$,then we have that

$$\lim_{j\to\infty} \mathcal{L}\left(\mathbf{k}^{(j)}, \mathbf{f}^{(j)}, \hat{\mathbf{k}}^{(j)}, \hat{\mathbf{f}}^{(j)}; \boldsymbol{\xi}^{(j)}, \boldsymbol{\mu}^{(j)}\right) \geq \nu,$$

where ν is the global minimum of $J^{\delta,\epsilon}_{\alpha,\beta}(\mathbf{k}, \mathbf{f})$.

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Minimization Algorithm
Theoretical analysis (continued)

We are now in position to state our main result

Theorem

The iterates generated by SeB-A converge to a limit point $\mathbf{p}_* = \left(\mathbf{k}_*, \mathbf{f}_*, \hat{\mathbf{k}}_*, \hat{\mathbf{f}}_*, \xi_*, \boldsymbol{\mu}_*\right)$. *Moreover, the followings hold*

(a) $\mathbf{p}_*$ *is a stationary point*

(b) *Assume now that* $\Omega_{\mathbf{f}} \times \Omega_{\mathbf{k}}$ *is convex and compact then*

$$\lim_{j\to\infty} \operatorname{dist}\left(\left(\mathbf{f}^{(j)}, \mathbf{k}^{(j)}, \hat{\mathbf{f}}^{(j)}, \hat{\mathbf{k}}^{(j)}; \xi^{(j)}, \boldsymbol{\mu}^{(j)}\right), Z^*\right) = 0,$$

where Z^* *denotes the set of stationary points and* dist *the Euclidean distance between sets and points.*

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Numerical Example
Implementation of SeB-A

Before giving a numerical example we discuss the implementation of the SeB-A algorithm and the construction of $\Omega_\mathbf{f}$ and $\Omega_\mathbf{k}$.

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Numerical Example

Implementation of SeB-A

Before giving a numerical example we discuss the implementation of the SeB-A algorithm and the construction of $\Omega_{\mathbf{f}}$ and $\Omega_{\mathbf{k}}$.

For the implementation of the SeB-A algorithm we reformulate following [R.H. Chan, M. Tao, and X. Yuan, SIMS2013] the minimization of $J_{\alpha,\beta}^{\delta,\epsilon}$ in another way

$$(\mathbf{k}^*, \mathbf{f}^*) = \arg \min_{\substack{\tilde{\mathbf{k}}\in\Omega_{\mathbf{k}}, \tilde{\mathbf{f}}\in\Omega_{\mathbf{f}} \\ \hat{\mathbf{k}}, \hat{\mathbf{f}}, \mathbf{k}, \mathbf{f}}} \Bigg\{ \left\| B(\mathbf{k}, \mathbf{f}) - \mathbf{g}_\delta \right\|^2 + \alpha^{\mathrm{E}} \left\| \mathbf{f} \right\|^2 + \alpha^{\mathrm{TV}} \sum_{i=1}^{N} \left\| \hat{\mathbf{f}}_i \right\|$$

$$+\gamma \left\| \mathbf{k} - \mathbf{k}_\epsilon \right\|^2 + \beta \sum_{i=1}^{N} \left\| \hat{\mathbf{k}}_i \right\|,$$

$$\mathbf{k} = \tilde{\mathbf{k}}, \mathbf{f} = \tilde{\mathbf{f}}, D_i\mathbf{k} = \hat{\mathbf{k}}_i, D_i\mathbf{f} = \hat{\mathbf{f}}_i \Bigg\},$$

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Numerical Example
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$$(\mathbf{k}^*,\mathbf{f}^*) = \arg \min_{\substack{\tilde{\mathbf{k}}\in\Omega_{\mathbf{k}},\tilde{\mathbf{f}}\in\Omega_{\mathbf{f}} \\ \hat{\mathbf{k}},\hat{\mathbf{f}},\mathbf{k},\mathbf{f}}} \left\{ \left\|B(\mathbf{k},\mathbf{f}) - \mathbf{g}_\delta\right\|^2 + \alpha^{\mathrm{E}} \left\|\mathbf{f}\right\|^2 + \alpha^{\mathrm{TV}} \sum_{i=1}^{N} \left\|\hat{\mathbf{f}}_i\right\| \right.$$

$$\left. +\gamma \left\|\mathbf{k} - \mathbf{k}_\epsilon\right\|^2 + \beta \sum_{i=1}^{N} \left\|\hat{\mathbf{k}}_i\right\|, \right.$$

$$\left. \mathbf{k} = \tilde{\mathbf{k}}, \mathbf{f} = \tilde{\mathbf{f}}, D_i\mathbf{k} = \hat{\mathbf{k}}_i, D_i\mathbf{f} = \hat{\mathbf{f}}_i \right\},$$

Applying the ADMM algorithm to this reformulation we obtain the CSeB-A algorithm.

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Algorithm (CSeB-A)

for $j = 0, 1, \ldots$ **do**

$$\begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} \\ \mathbf{k}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{f}},\hat{\mathbf{f}},\mathbf{k}} \mathcal{L}\left(\tilde{\mathbf{f}}, \hat{\mathbf{f}}, \mathbf{k} \middle| \tilde{\mathbf{k}}^{(j)}, \hat{\mathbf{k}}^{(j)}, \mathbf{f}^{(j)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \tilde{\mathbf{k}}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} \\ \mathbf{f}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{k}},\hat{\mathbf{k}},\mathbf{f}} \mathcal{L}\left(\tilde{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{f} \middle| \tilde{\mathbf{f}}^{(j+1)}, \hat{\mathbf{f}}^{(j+1)}, \mathbf{k}^{(j+1)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \boldsymbol{\lambda}^{(j+1)} \\ \boldsymbol{\xi}^{(j+1)} \\ \boldsymbol{\zeta}^{(j+1)} \\ \boldsymbol{\mu}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}^{(j)} \\ \boldsymbol{\xi}^{(j)} \\ \boldsymbol{\zeta}^{(j)} \\ \boldsymbol{\mu}^{(j)} \end{pmatrix} - \omega \begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} - \mathbf{f}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} - D\mathbf{f}^{(j+1)} \\ \tilde{\mathbf{k}}^{(j+1)} - \mathbf{k}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} - D\mathbf{k}^{(j+1)} \end{pmatrix};$$

end

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Algorithm (CSeB-A)

for $j = 0, 1, \ldots$ **do**

$$\begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} \\ \mathbf{k}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{f}}, \hat{\mathbf{f}}, \mathbf{k}} \mathcal{L}\left(\tilde{\mathbf{f}}, \hat{\mathbf{f}}, \mathbf{k} \middle| \tilde{\mathbf{k}}^{(j)}, \hat{\mathbf{k}}^{(j)}, \mathbf{f}^{(j)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \tilde{\mathbf{k}}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} \\ \mathbf{f}^{(j+1)} \end{pmatrix} = \arg\min_{\tilde{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{f}} \mathcal{L}\left(\tilde{\mathbf{k}}, \hat{\mathbf{k}}, \mathbf{f} \middle| \tilde{\mathbf{f}}^{(j+1)}, \hat{\mathbf{f}}^{(j+1)}, \mathbf{k}^{(j+1)}; \boldsymbol{\lambda}^{(j)}, \boldsymbol{\xi}^{(j)}, \boldsymbol{\zeta}^{(j)}, \boldsymbol{\mu}^{(j)}\right);$$

$$\begin{pmatrix} \boldsymbol{\lambda}^{(j+1)} \\ \boldsymbol{\xi}^{(j+1)} \\ \boldsymbol{\zeta}^{(j+1)} \\ \boldsymbol{\mu}^{(j+1)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}^{(j)} \\ \boldsymbol{\xi}^{(j)} \\ \boldsymbol{\zeta}^{(j)} \\ \boldsymbol{\mu}^{(j)} \end{pmatrix} - \omega \begin{pmatrix} \tilde{\mathbf{f}}^{(j+1)} - \mathbf{f}^{(j+1)} \\ \hat{\mathbf{f}}^{(j+1)} - D\mathbf{f}^{(j+1)} \\ \tilde{\mathbf{k}}^{(j+1)} - \mathbf{k}^{(j+1)} \\ \hat{\mathbf{k}}^{(j+1)} - D\mathbf{k}^{(j+1)} \end{pmatrix};$$

end

The minimizations in CSeB-A are easily computed and all have a closed form. However, we are not able to provide a rigorous convergence analysis.

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Numerical Example

Implementation of SeB-A (continued)

$$\tilde{\mathbf{f}}^{(j+1)} = P_{\Omega_{\mathbf{f}}}\left(\mathbf{f}^{(j)} + \frac{\boldsymbol{\lambda}^{(j)}}{\omega}\right), \quad \tilde{\mathbf{k}}^{(j+1)} = P_{\Omega_{\mathbf{k}}}\left(\mathbf{k}^{(j+1)} + \frac{\boldsymbol{\zeta}^{(j)}}{\omega}\right)$$

$$\hat{\mathbf{f}}_i^{(j+1)} = \frac{\left(D_i\mathbf{f}^{(j)} + \frac{1}{\omega}\xi_i^{(j)}\right)}{\left\|D_i\mathbf{f}^{(j)} + \frac{1}{\omega}\xi_i^{(j)}\right\|} \circ \left(\left\|D_i\mathbf{f}^{(j)} + \frac{1}{\omega}\xi_i^{(j)}\right\| - \frac{\alpha^{\mathrm{TV}}}{\omega}\right)_+$$

$$\mathbf{k}^{(j+1)} = \left(2A^*_{\mathbf{f}^{(j)}}A_{\mathbf{f}^{(j)}} + (2\gamma + \omega)I + \omega D^* D\right)^{-1} \cdot \left(2A^*_{\mathbf{f}^{(j)}}\mathbf{g}_\delta + 2\gamma\mathbf{k}_\epsilon - \zeta^{(j)} + \omega\tilde{\mathbf{k}}^{(j)} - D^*\boldsymbol{\mu}^{(j)} + \omega D^*\hat{\mathbf{k}}^{(j)}\right)$$

$$\hat{\mathbf{k}}_i^{(j+1)} = \frac{\left(D_i\mathbf{k}^{(j+1)} + \frac{1}{\omega}\boldsymbol{\mu}_i^{(j)}\right)}{\left\|D_i\mathbf{k}^{(j+1)} + \frac{1}{\omega}\boldsymbol{\mu}_i^{(j)}\right\|} \circ \left(\left\|D_i\mathbf{k}^{(j+1)} + \frac{1}{\omega}\boldsymbol{\mu}_i^{(j)}\right\| - \frac{\beta}{\omega}\right)_+$$

$$\mathbf{f}^{(j+1)} = \left(2A^*_{\mathbf{k}^{(j+1)}}A_{\mathbf{k}^{(j+1)}} + 2(\alpha^{\mathrm{E}} + \omega)I + \omega D^* D\right)^{-1} \cdot \left(2A^*_{\mathbf{k}^{(j+1)}}\mathbf{g}_\delta - \boldsymbol{\lambda} + \omega\tilde{\mathbf{f}}^{(j+1)} - D^*\xi^{(j)} + \omega D^*\hat{\mathbf{f}}^{(j+1)}\right)$$

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We are going to consider the framework of space invariant image deblurring.

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Numerical Example
Constraints

We are going to consider the framework of space invariant image deblurring.

- $\mathbf{k}$ $\rightarrow$PSF;

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- **k** $\rightarrow$PSF;
- **g** $\rightarrow$Blurred image;

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We are going to consider the framework of space invariant image deblurring.

- **k** $\rightarrow$PSF;
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- **f** $\rightarrow$True image;

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Numerical Example
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[1] For simplicity we impose periodic boundary conditions

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Thus we are going to impose nonnegativity and flux constraints.

[1] For simplicity we impose periodic boundary conditions

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Numerical Example
Constraints (continued)

We briefly discuss the flux constraint.

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Numerical Example
Constraints (continued)

We briefly discuss the flux constraint.

For the blurring phenomenon it holds

(i) $\mathbf{k}_i \geq 0$;

(ii) $\text{flux}\,(\mathbf{k}) := \mathbf{1}^t\mathbf{k} = 1$, where $\mathbf{1} = (1, 1, \ldots, 1)^t$.

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Numerical Example

Constraints (continued)

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Then we have

- $A_{\mathbf{k}}$ has no negative entries;
- the row-sum and column-sum of $A_{\mathbf{k}}$ is 1;
- If $\mathbf{y} = A_{\mathbf{k}}\mathbf{z}$, then $\text{flux}\,(\mathbf{y}) = \text{flux}\,(\mathbf{z})$.

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Numerical Example
Constraints (continued)

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Then it holds

$$\text{flux}\,(\mathbf{f}) = \text{flux}\,(\mathbf{g})$$

In the noisy case: $\mathbf{g}_\delta = \mathbf{g} + \eta$. Then

$$\text{flux}\,(\mathbf{g}_\delta) = \text{flux}\,(\mathbf{g}) + \text{flux}\,(\eta) \approx \text{flux}\,(\mathbf{g}) + 0 = \text{flux}\,(\mathbf{g})\,.$$

Numerical Example

Constraints (continued)

We set

$$\Omega_{\mathbf{f}} = \{\mathbf{x} : \mathbf{x}_i \geq 0\} \cap \{\mathbf{x} : \text{flux}(\mathbf{x}) = \text{flux}(\mathbf{g}_\delta)\} = \Omega_0 \cap \Omega_{\text{flux}}^{\mathbf{g}_\delta}$$

$$\Omega_{\mathbf{k}} = \{\mathbf{x} : \mathbf{x}_i \geq 0\} \cap \{\mathbf{x} : \text{flux}(\mathbf{x}) = 1\} = \Omega_0 \cap \Omega_{\text{flux}}^{1}$$

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Numerical Example
Constraints (continued)

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Since the projection on either $\Omega_{\mathbf{f}}$ or $\Omega_{\mathbf{k}}$ is not trivial we will split the constraints and use two auxiliary variables in the ADMM algorithm.

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The projections into $\Omega_{\text{flux}}^{\mathbf{g}_\delta}$ and Ω_{flux}^{1} can be computed in $O(N)$ operations.

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Numerical Example
Constraints (continued)

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Since the projection on either $\Omega_{\mathbf{f}}$ or $\Omega_{\mathbf{k}}$ is not trivial we will split the constraints and use two auxiliary variables in the ADMM algorithm.

The projections into $\Omega_{\text{flux}}^{\mathbf{g}_\delta}$ and Ω_{flux}^{1} can be computed in $O(N)$ operations.

In particular

$$\mathcal{P}_{\Omega_{\text{flux}}^{\mathbf{g}_\delta}}(\mathbf{x}) = \frac{\text{flux}(\mathbf{g}_\delta) - \text{flux}(\mathbf{x})}{N}\mathbf{1} + \mathbf{x}$$

$$\mathcal{P}_{\Omega_{\text{flux}}^{1}}(\mathbf{x}) = \frac{1 - \text{flux}(\mathbf{x})}{N}\mathbf{1} + \mathbf{x}$$

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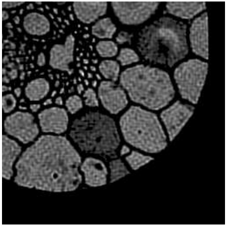
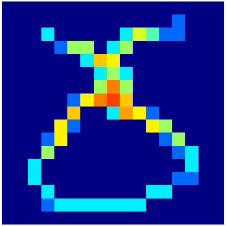
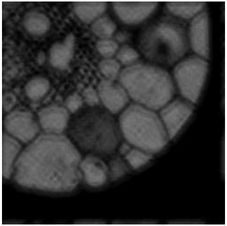
Numerical Example
Experiment



$\mathbf{f}$

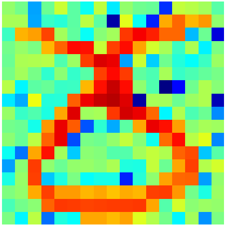


$\log(|\mathbf{k}|)$



$\mathbf{g}_\delta$

$\delta = 0.01\,\|\mathbf{g}\|$



$\log(|\mathbf{k}_\epsilon|)$

$\epsilon = 0.8\,\|\mathbf{k}\|$

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Numerical Example
Experiment (continued)

	BID-ADMM	TV	CSeB-A
SNR **f**	11.924	12.844	23.268
SNR **k**	1.597	$--$	22.925

- BID-ADMM: [M. S. Almeida and M. A. Figueiredo, IEEE2013];
- TV: $\mathbf{f}^* = \arg\min_{\mathbf{f}\in\Omega_\mathbf{f}} \left\|B(\mathbf{k}_\epsilon, \mathbf{f}) - \mathbf{g}_\delta\right\|^2 + \alpha^{\mathrm{E}} \left\|\mathbf{f}\right\|^2 + \alpha^{\mathrm{TV}} \left\|\mathbf{f}\right\|_{TV}$.

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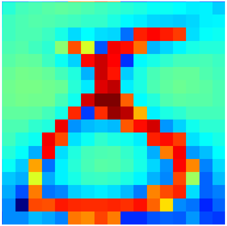
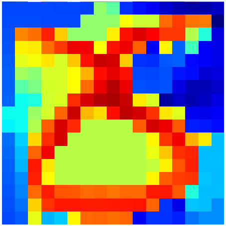
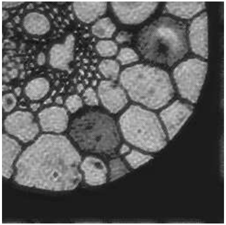
Numerical Example
Experiment (continued)



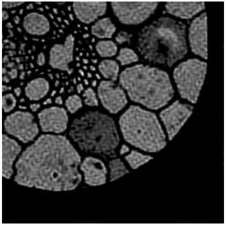
BID-ADMM



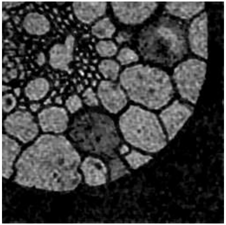
CSeB-A



BID-ADMM



CSeB-A

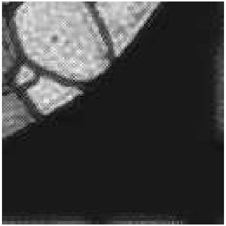


TV

Numerical Example

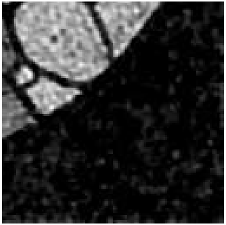
Experiment (continued)

True



BID-ADMM

CSeB-A



TV

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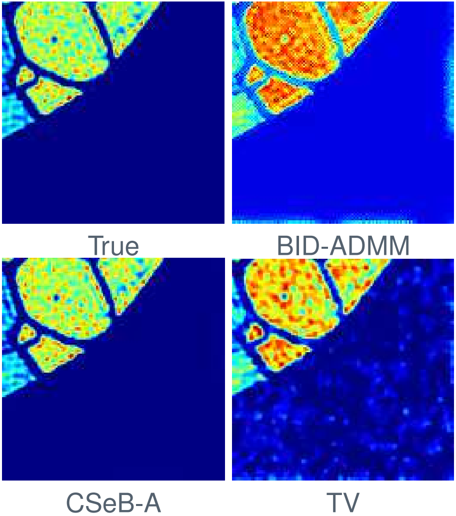
Numerical Example
Experiment (continued)



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Conclusions & Future work

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Conclusions & Future work

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- Extend to non-convex priors.

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Thank you for your attention!