Chapter 2

Tight-frames

An Introduction

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Outline

1. Tight-frame
2. Matrix Representation

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Tight Frames

- Definition (Duffin and Schaeffer, Trans. AMS, 1952): Let $\mathcal{X} \subset L^2(\mathbb{R})$ be countable. $\mathcal{X}$ is a tight frame for $L^2(\mathbb{R})$ if

$$\sum_{g \in \mathcal{X}} |\langle f, g \rangle|^2 = \|f\|^2, \quad \forall f \in L^2(\mathbb{R})$$

- This is equivalent to

$$f = \sum_{g \in \mathcal{X}} \langle f, g \rangle g, \quad \forall f \in L^2(\mathbb{R})$$

- An orthonormal basis is a tight frame

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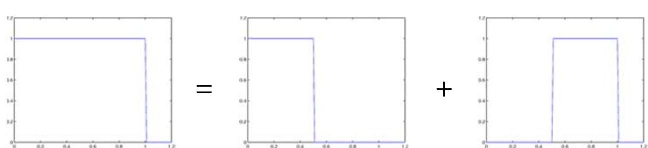
Construction of Haar Wavelet

Define $\phi(x) = 1$ for $x \in [0, 1]$, and 0 otherwise. Then we have the refinement equation:

$$\phi(x) = 1 \cdot \phi(2x) + 1 \cdot \phi(2x - 1).$$



dialation translation

The transfer function is

$$h_0(\omega) = \frac{1}{2} + \frac{1}{2} e^{i\omega} = e^{-i\omega/2} \cos(\omega/2)$$

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Construction of Haar Wavelet

Define

$$h_1(\omega) = \frac{1}{2} + \frac{-1}{2} e^{i\omega} = e^{-i\omega/2} \sin(\omega/2),$$

then

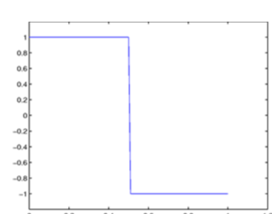
$$\sum_{i=0}^{1} h_i(\omega)\overline{h_i(\omega)} = 1 \quad \text{and} \quad \sum_{i=0}^{1} h_i(\omega)\overline{h_i(\omega+\pi)} = 0.$$

It gives the *perfect reconstruction property* of Haar wavelet.

Construction of Haar Wavelet

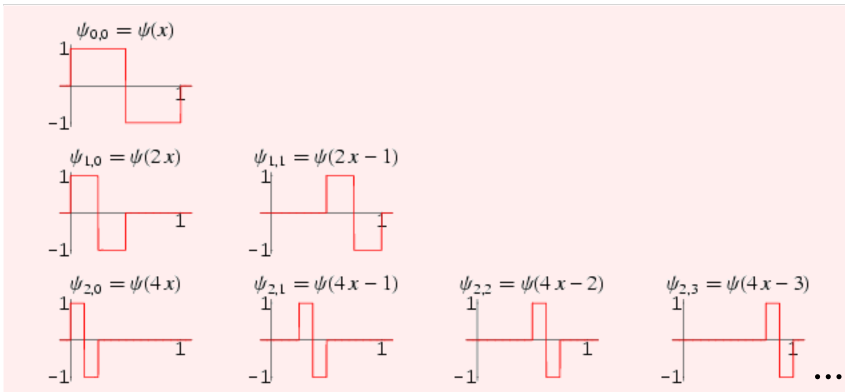
Accordingly, define

$$\begin{aligned} \psi(x) &= 1 \cdot \phi(2x) + (-1) \cdot \phi(2x-1) \\ &= \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \end{cases} \end{aligned}$$



Then $\mathcal{X} = \{\psi_{jk} \mid \psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)\}$ is a wavelet system and hence a tight frame.

Haar Function



The collection of all these functions is the Haar wavelet system and an orthonormal basis for $L_2[0,1]$.

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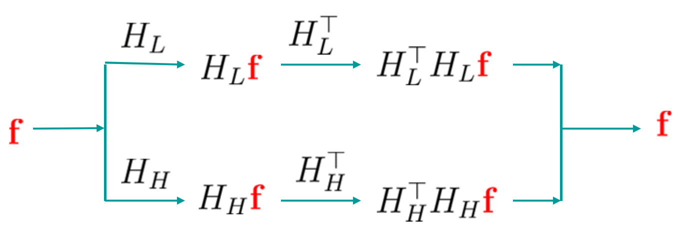
Haar Wavelet Filters

Haar's filters are: $\frac{1}{2}[1,1]$ and $\frac{1}{2}[1,-1]$

Using $\frac{1}{2}[1,1]$ and $\frac{1}{2}[1,-1]$ to construct the low-pass filter H_L and the high-pass filter H_H respectively, we have the perfect reconstruction formula.

$$H_L^\top H_L + H_H^\top H_H = I$$



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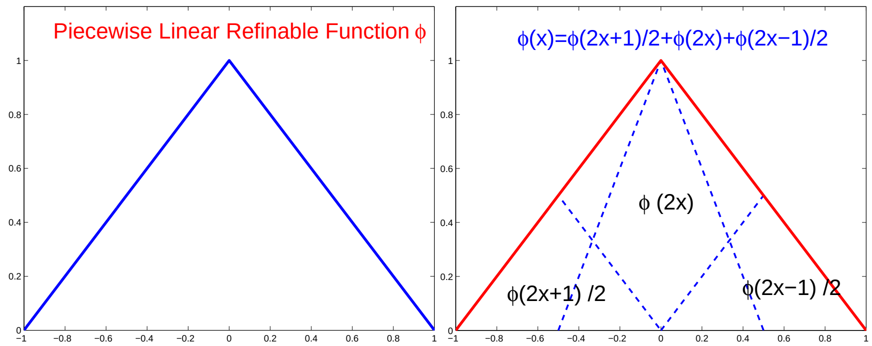
Piecewise Linear Tight Frame



$$\phi(x) = \frac{1}{2}\phi(2x+1) + 1 \cdot \phi(2x) + \frac{1}{2}\phi(2x-1)$$

$$h_0(\omega) = \frac{1}{4}e^{i\omega} + \frac{1}{2} + \frac{1}{4}e^{-i\omega}$$

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Piecewise Linear Tight Frame

Let

$$h_1(\omega) = \frac{\sqrt{2}}{4}e^{i\omega} - \frac{\sqrt{2}}{4}e^{-i\omega}$$

$$h_2(\omega) = -\frac{1}{4}e^{i\omega} + \frac{1}{2} - \frac{1}{4}e^{-i\omega}$$

Then

$$\sum_{i=0}^{2} h_i(\omega)\overline{h_i(\omega)} = 1 \quad \text{and} \quad \sum_{i=0}^{2} h_i(\omega)\overline{h_i(\omega+\pi)} = 0.$$

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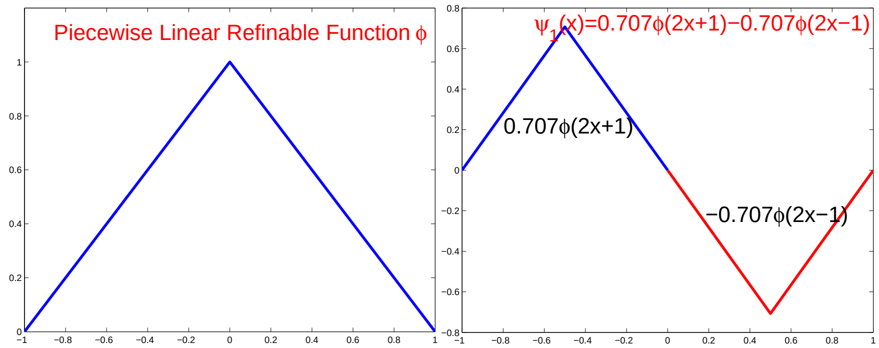
Piecewise Linear Tight Frame



$$
\begin{aligned}
h_1(\omega) &= \frac{\sqrt{2}}{4}e^{i\omega} - \frac{\sqrt{2}}{4}e^{-i\omega} \\
\psi_1(x) &= \frac{1}{\sqrt{2}}\phi(2x+1) - \frac{1}{\sqrt{2}}\phi(2x-1)
\end{aligned}
$$

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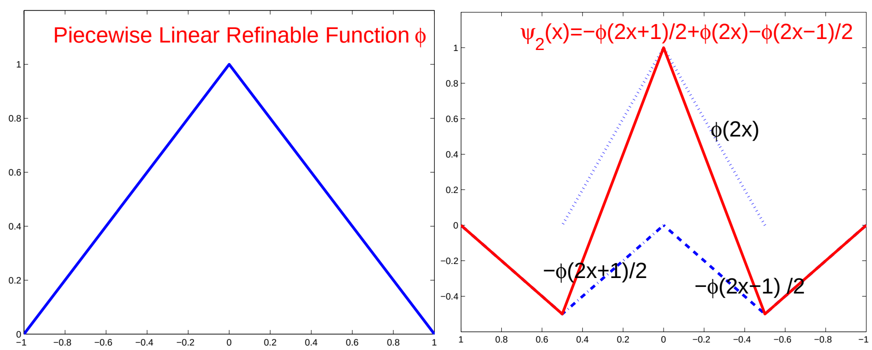
Piecewise Linear Tight Frame



$$
\begin{aligned}
h_2(\omega) &= -\frac{1}{4}e^{i\omega} + \frac{1}{2} - \frac{1}{4}e^{-i\omega} \\
\psi_2(x) &= -\frac{1}{2}\phi(2x+1) + 1 \cdot \phi(2x) - \frac{1}{2}\phi(2x-1)
\end{aligned}
$$

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Piecewise Linear Tight Frame

The system obtained by dilation and translation:

$$\mathcal{X} = \{2^{k/2}\psi_i(2^k \cdot -j) : k,\, j \in \mathbb{Z}; i = 1, 2\}$$

is the *piecewise linear tight framelet system*.

Given h_0, is it easy to find h_1 and h_2 such that

$$\sum_{i=0}^{2} h_i(\omega)\overline{h_i(\omega)} = 1 \quad \text{and} \quad \sum_{i=0}^{2} h_i(\omega)\overline{h_i(\omega+\pi)} = 0?$$

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Unitary Extension Principle

Theorem (Ron-Shen, 97): Let $\phi \in L^2(\mathbb{R})$ be a refinable function whose refinement equation is

$$\widehat{\phi}(2\cdot) = h_0(\cdot)\widehat{\phi}(\cdot),$$

(h_0 is called *refinement mask* or *low-pass filter*.)

Let h_i, $i = 1, \ldots, m$ be highpass filters satisfying

$$\sum_{i=0}^{m} h_i(\omega)\overline{h_i(\omega)} = 1 \quad \text{and} \quad \sum_{i=0}^{m} h_i(\omega)\overline{h_i(\omega+\pi)} = 0.$$

($\{h_i\}_{i=1}^m$ are called *framelet masks*.)

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Unitary Extension Principle

Define $\Psi := \{\psi_1, \ldots, \psi_m\}$ with

$$\widehat{\psi}_i(2\cdot) = h_i(\cdot)\widehat{\phi}(\cdot).$$

Then $\mathcal{X}(\Psi)$ is a *tight frame* of $L^2(\mathbb{R})$ and $\{\psi_i\}_{i=1}^m$ are called *framelets*.

- Easy to find $\{h_i\}_{i=1}^m$ if

$$|h_0(\cdot)|^2 + |h_0(\cdot + \pi)|^2 \leq 1$$

- Explicit formula for h_i for B-spline tightframes

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1D Piecewise Linear Tight Frame

1. Start with linear B-spline ϕ (the hat function).
2. Its Fourier transform is $\widehat{\phi}(\omega) = \frac{\sin^2(\omega/2)}{(\omega/2)^2}$.
3. Define framelets

$$\widehat{\psi}_i(\omega) = \widehat{h}_i(\omega/2)\widehat{\phi}(\omega/2)$$

with framelet masks

$$\widehat{h}_i(\omega) = \sqrt{\binom{2}{i}} \sin^i(\omega/4)\cos^{2-i}(\omega/4),$$

for $1 \leq i \leq 2$.

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1D Piecewise Linear Tight Frame

4. The system

$$\mathcal{X} = \{2^{k/2}\psi_i(2^k \cdot -j) : k,\ j \in \mathbb{Z}; i = 1, 2\}$$

is the *piecewise linear tight framelet system*.

5. The filters are:

$$\begin{aligned} h_0 &= \frac{1}{4}[1, 2, 1], \\ h_1 &= \frac{\sqrt{2}}{4}[1, 0, -1], \\ h_2 &= \frac{1}{4}[-1, 2, -1]. \end{aligned}$$

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Outline

1. **Tight-frame**
2. **Matrix Representation**

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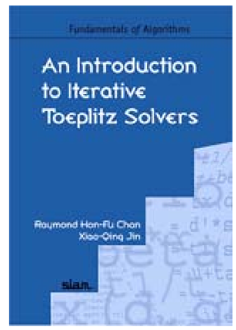
Matrix Representation

To apply a filter onto a signal is equivalent to pre-multiply the signal vector by a Toeplitz matrix.

E.g. $h_0 = \frac{1}{4}[1, 2, 1]$ corresponds to

$$h_0 \longleftrightarrow H_0 = \frac{1}{4}\begin{bmatrix} 2 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ 0 & & & 1 & 2 \end{bmatrix}.$$



Toeplitz-type matrix (C. & Jin, SIAM book, 2007).

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Matrix Representation

Usually, one uses reflexive boundary condition to minimize boundary artifacts—resulting a Toeplitz-like matrix.

E.g. $h_0 = \frac{1}{4}[1, 2, 1]$ corresponds to

$$h_0 \longleftrightarrow H_0 = \frac{1}{4}\begin{bmatrix} 3 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ 0 & & & 1 & 3 \end{bmatrix}.$$

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Analysis and Synthesis Operators

The tight-frame transform is obtained by:

$$\mathcal{A} = \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} \quad \leftrightarrow \quad \text{analysis operator}$$

$$\mathcal{A}^\top = \begin{bmatrix} H_0{}^\top & H_1{}^\top & H_2{}^\top \end{bmatrix} \quad \leftrightarrow \quad \text{synthesis operator}$$

$$\mathcal{A}\mathbf{f} = \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} \mathbf{f} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} \quad \leftrightarrow \quad \text{framelet coefficients}$$

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Important Observation

Tight Frames = Redundant Bases

If

$$\mathcal{A} = \text{analysis operator}$$

and

$$\mathcal{A}^\top = \text{synthesis operator},$$

then

$$\mathcal{A}^\top \mathcal{A} = \mathcal{I},$$

but

$$\mathcal{A}\mathcal{A}^\top \neq \mathcal{I}.$$

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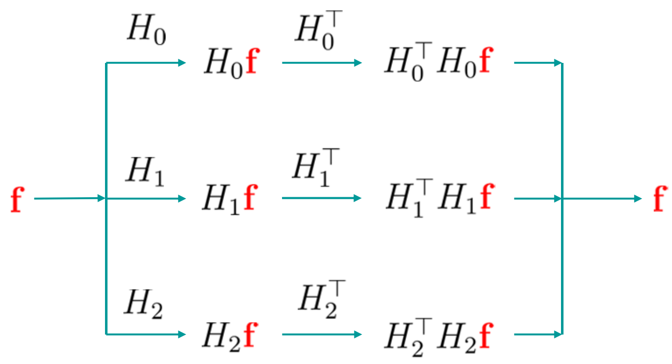
Perfect Reconstruction Formula

$$\mathcal{A}^\top \mathcal{A} = \mathcal{I} \Leftrightarrow H_0^\top H_0 + H_1^\top H_1 + H_2^\top H_2 = I$$



No need for $\mathcal{A}\mathcal{A}^\top = \mathcal{I}$

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Multi-level Decomposition without Down-sampling

For piecewise linear tight frame:

$h_0 = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$, $h_1 = [-\frac{\sqrt{2}}{4}, 0, \frac{\sqrt{2}}{4}]$, $h_2 = [-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}]$

define h_0 at level ℓ to be

$$h_0^{(\ell)} = \left[\frac{1}{4}, \underbrace{0, \cdots, 0}_{2^{(\ell-1)}-1}, \frac{1}{2}, \underbrace{0, \cdots, 0}_{2^{(\ell-1)}-1}, \frac{1}{4}\right]$$

and the masks $h_1^{(\ell)}$ and $h_2^{(\ell)}$ similarly.

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Multi-level Decomposition without Down-sampling

Let $H_i^{(\ell)}$ be the matrix corresponding to $h_i^{(\ell)}$. Then

$$\mathcal{A} = \begin{bmatrix} \prod_{\ell=0}^{L-1} H_0^{(L-\ell)} \\ \hline H_1^{(L)} \prod_{\ell=1}^{L-1} H_0^{(L-\ell)} \\ H_2^{(L)} \prod_{\ell=1}^{L-1} H_0^{(L-\ell)} \\ \vdots \\ H_1^{(1)} \\ H_2^{(1)} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{A}_L \\ \hline \\ \mathcal{A}_H \\ \\ \end{bmatrix}.$$

We still have

$$\mathcal{A}^\top \mathcal{A} = \mathcal{A}_L^\top \mathcal{A}_L + \mathcal{A}_H^\top \mathcal{A}_H = I.$$

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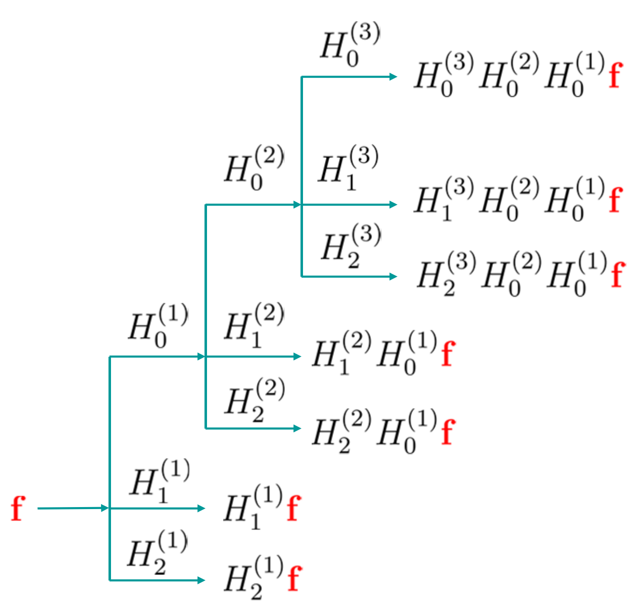
Multi-level Decomposition



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2D Tight Frame

We use tensor product to produce a tight framelet system in $\mathcal{L}^2(\mathbb{R}^2)$:

- framelet masks:

$$\widehat{h}_{i,j}(\omega_1, \omega_2) = \widehat{h}_i(\omega_1)\widehat{h}_j(\omega_2)$$

- filters:

$$h_{ij} = h_i^t h_j$$

for $i, j = 0, 1, 2.$

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2D Piecewise Linear Framelets

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \frac{\sqrt{2}}{16}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \frac{1}{16}\begin{bmatrix} -1 & 2 & -1 \\ -2 & 4 & -2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\frac{\sqrt{2}}{16}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} \quad \frac{1}{8}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \frac{\sqrt{2}}{16}\begin{bmatrix} -1 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\frac{1}{16}\begin{bmatrix} -1 & -2 & -1 \\ 2 & 4 & 2 \\ -1 & -2 & -1 \end{bmatrix} \quad \frac{\sqrt{2}}{16}\begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{bmatrix} \quad \frac{1}{16}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{matrix} H_{0,0} & H_{0,1} & H_{0,2} \\ H_{1,0} & H_{1,1} & H_{1,2} \\ H_{2,0} & H_{2,1} & H_{2,2} \end{matrix}$$

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1D Piecewise Cubic Tight Frame

1. Start with cubic B-spline $\widehat{\phi}(\omega) = \frac{\sin^4(\omega/2)}{(\omega/2)^4}$.

2. Define framelets

$$\widehat{\psi}_i(\omega) = \widehat{h}_i(\omega/2)\widehat{\phi}(\omega/2)$$

with framelet masks

$$\widehat{h}_i(\omega) = \sqrt{\binom{4}{i}} \sin^i(\omega/4) \cos^{4-i}(\omega/4),$$

for $1 \leq i \leq 4$.

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1D Piecewise Cubic Tight Frame

3. The system

$$\mathcal{X} = \{2^{k/2}\psi_i(2^k \cdot -j) : k,\ j \in \mathbb{Z}; i = 1, \ldots, 4\}$$

is the *piecewise cubic tight framelet system*.

4. The filters are: $h_0 = \frac{1}{16}[1, 4, 6, 4, 1]$,

$$h_1 = \frac{1}{8}[1, 2, 0, -2, -1], \qquad h_2 = \frac{\sqrt{6}}{16}[-1, 0, 2, 0, -1],$$

$$h_3 = \frac{1}{8}[-1, 2, 0, -2, 1], \qquad h_4 = \frac{1}{16}[1, -4, 6, -4, 1].$$

5. Again we still have $\mathcal{A}^\top \mathcal{A} = \mathcal{I}$.

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Tight Frames

- Tight frames—redundant bases
  - preserve the unitary property of the analysis and synthesis operators
  - sacrifice orthogonality and linear independence to get more flexibility
- Robust signal representation—errors in signals can be reduced when represented by a redundant system
- Discrete Fourier transform frames applied successfully to many fields

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Spline Framelet Systems

- Spline framelet systems: piecewise linear or cubic tight frames
- Can be constructed from the unitary extension principle of Ron and Shen (JFA, 97)
- Either symmetric or anti-symmetric
- Have small supports for a given smoothness order —good time-frequency localization
- There are 2D tightframes not form tensor product and smaller numbers of filters

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