Chapter 5

Solution to L1-L2 Models
and Parameter Estimation

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Outline

1. **Solution to L1-L2 Models**
2. **Estimating the Regularization Parameter**

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l^2-l^2 Model

Consider the ℓ^2-ℓ^2 model

$$\min_{\mathbf{x}} \frac{1}{2}\|C\mathbf{x}\|_2^2 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2.$$

- For a given α, the solution is:

$$\mathbf{x}_\alpha = \alpha\left(\alpha B^t B + C^t C\right)^{-1} B^t \mathbf{q}.$$

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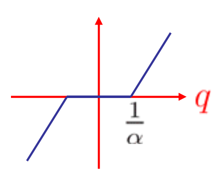
l^1-l^2 Model

Consider the ℓ^1-ℓ^2 model

$$\min_{\mathbf{x}} \|C\mathbf{x}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2$$

- Fast solution methods: ADMM, primal-dual, proximal splitting, split Bregman, ...
- Note that the problem

$$\min_{x\in\mathbb{R}} |x| + \frac{\alpha}{2}(x-q)^2$$



has explicit solution:

$$x^* = \begin{cases} \operatorname{sgn}(q)(|q| - \frac{1}{\alpha}), & \text{if } |q| > \frac{1}{\alpha}, \\ 0, & \text{if } |q| \le \frac{1}{\alpha}. \end{cases}$$

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Alternating Direction Method of Multipliers

$$\min_{\mathbf{x}} \ \|\underbrace{C\mathbf{x}}_{\mathbf{v}}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2$$ (Separation of Variables)

$$\Longleftrightarrow \quad \min_{\mathbf{x},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 \quad \text{s.t.} \quad \mathbf{v} = C\mathbf{x}$$

$$\Longleftrightarrow \quad \min_{\mathbf{p},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 - \langle \mathbf{s}, \mathbf{v} - C\mathbf{x}\rangle + \frac{\beta}{2}\|\mathbf{v} - C\mathbf{x}\|_2^2$$ (Augmented Lagrangian)

$$\Longleftrightarrow \quad \min_{\mathbf{x},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}\right\|_2^2$$

$\beta > 0$ controls the rate of convergence

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Update Variables Alternatively

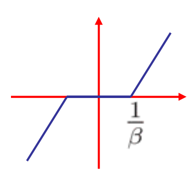
$$\min_{\mathbf{x},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}\right\|_2^2$$

- □ $$\mathbf{x}^{j+1} = \arg\min_{\mathbf{x}} \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v}^j - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}^j\right\|_2^2$$

$$\Updownarrow$$

$$\left[\alpha B^t B + \beta C^t C\right]\mathbf{x}^{j+1} = \alpha B^t \mathbf{q} + \beta C^t\left(\mathbf{v}^j - \frac{1}{\beta}\mathbf{s}^j\right)$$

- □ $$\mathbf{v}^{j+1} = \arg\min_{\mathbf{v}} \|\mathbf{v}\|_1 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x}^{j+1} - \frac{1}{\beta}\mathbf{s}^j\right\|_2^2 = \text{shrink}_{\frac{1}{\beta}}\left(C\mathbf{x}^{j+1} + \frac{1}{\beta}\mathbf{s}^j\right)$$



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Outline

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l^2-l^2 Model

Consider the ℓ^2-ℓ^2 model

$$\min_{\mathbf{x}} \frac{1}{2}\|C\mathbf{x}\|_2^2 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2.$$

- ☐ Find the best $\mathbf{x}$ and α that minimize the functional.
- ☐ For a given α, the solution is:

$$\mathbf{x}_\alpha = \alpha\left(\alpha B^t B + C^t C\right)^{-1} B^t \mathbf{q} := M_\alpha \mathbf{q}.$$

- ☐ What is the best α for the model?

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l^2-l^2 Model

- Generalized Cross Validation (GCV) [Golub, Heath, Wahba, 79]:

$$\alpha^* = \arg\min_{\alpha} \frac{mn\|B\mathbf{x}_\alpha - \mathbf{q}\|_2^2}{\text{trace}^2(I - BM_\alpha)}$$

- Discrepancy Principle [Morozov, 84]:
  Find α^* such that

$$\frac{1}{mn}\|B\mathbf{x}_\alpha - \mathbf{q}\|_2^2 = \sigma^2.$$

- L-curve [Hansen & O'Leary, 93]:
  Find α^* minimizes $\|B\mathbf{x}_\alpha - \mathbf{q}\|_2^2$ and $\|\mathbf{x}_\alpha\|_2^2$.

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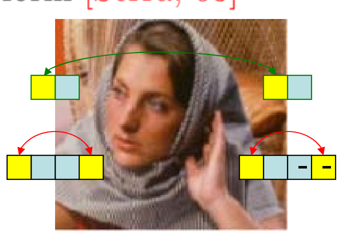
Multiple Solves

For deblurring problems $\mathbf{x}_\alpha = \alpha(\alpha B^t B + C^t C)^{-1} B^t \mathbf{q}$ can be computed fast:

- Periodic boundary condition: diagonalized by Fourier transform
- Symmetric boundary condition: diagonalized by cosine transform [Ng, C., Tang, 99]
- Anti-reflexive boundary condition: diagonalized by sine transform [Serra, 03]



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l^1-l^2 Model

Consider the ℓ_1-ℓ^2 model

$$\min_{\mathbf{x}} \|C\mathbf{x}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2$$

- No easy "closed-form" solution.
- Using ADMM, require solving an ℓ^2-ℓ^2 $\mathbf{x}$-subproblem in each iteraton:

$$\min_{\mathbf{x}} \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\|C\mathbf{x} - \mathbf{y}^j\|_2^2, \quad \beta > 0.$$

- How to find the best α for the ℓ^1-ℓ^2 model?

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l^1-l^2 Model

- Find α by trial and error such that $\mathbf{x}_\alpha$ that minimizes MSE or maximizes PSNR.
- Majorization-minimization approach [Oliveira, Bioucas-Dias & Figueiredo, 09]
- SURE-based [Lin, Wohlberg & Guo, 10]
- GCV methods [Liao & Ng, 11] ...
- Require multiple solves of ℓ_1-ℓ_2 problem.
- Can we just solve ℓ_1-ℓ_2 problem once?

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l^1-l^2 Model

Consider the ℓ^1-ℓ^2 model

$$\min_{\mathbf{x}} \|C\mathbf{x}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2$$

Using the substitution $\mathbf{v} = C\mathbf{x}$, we solve

$$\min_{\mathbf{x},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}\right\|_2^2$$

alternatively between $\mathbf{v}$ and $\mathbf{x}$.

Update Variables Alternatively

$$\min_{\mathbf{x},\mathbf{v}} \ \|\mathbf{v}\|_1 + \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}\right\|_2^2$$

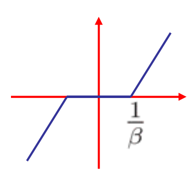
- $\mathbf{x}^{j+1} = \arg\min_{\mathbf{x}} \frac{\alpha}{2}\|B\mathbf{x} - \mathbf{q}\|_2^2 + \frac{\beta}{2}\left\|\mathbf{v}^j - C\mathbf{x} - \frac{1}{\beta}\mathbf{s}^j\right\|_2^2$

$$\Updownarrow$$

$$\left[\alpha B^t B + \beta C^t C\right]\mathbf{x}^{j+1} = \alpha B^t \mathbf{q} + \beta C^t(\mathbf{v}^j - \frac{1}{\beta}\mathbf{s}^j)$$

- $\mathbf{v}^{j+1} = \arg\min_{\mathbf{v}} \|\mathbf{v}\|_1 + \frac{\beta}{2}\left\|\mathbf{v} - C\mathbf{x}^{j+1} - \frac{1}{\beta}\mathbf{s}^j\right\|_2^2$

$$= \text{shrink}_{\frac{1}{\beta}}\left(C\mathbf{x}^{j+1} + \frac{1}{\beta}\mathbf{s}^j\right)$$



Our Idea

- Using ADMM, $\mathbf{x}$-subproblem in each iteraton is:

$$\min_{\mathbf{x}} \frac{\alpha}{2}\|B\mathbf{x}-\mathbf{q}\|_2^2 + \frac{\beta}{2}\|C\mathbf{x}-\mathbf{y}^j\|_2^2, \quad \beta > 0.$$

- "Closed-form" solution is:

$$\mathbf{x}_\alpha = (\alpha B^t B + \beta C^t C)^{-1}(\alpha B^t \mathbf{q} + \beta C^t \mathbf{y}^j).$$

- Find best α^j e.g. by discrepancy principle:

$$\frac{1}{mn}\|B\mathbf{x}_\alpha - \mathbf{q}\|_2^2 = \sigma^2.$$

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Our Idea

- After finding best α^j, define $\mathbf{x}^j := \mathbf{x}_{\alpha^j}$.
- When $\mathbf{x}^j$ converges to $\mathbf{x}^*$, α^j converges to α^*?
- What is α^*?
- α^* is the Lagrange multiplier for the constrained problem:

$$\min \|C\mathbf{x}\|_1$$
$$\text{s.t. } \|B\mathbf{x}-\mathbf{q}\|_2^2 \le \sigma^2$$

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What have been done?

- Proven when solution method is primal-dual [Wen & C., 12]
- Proven when solution method is ADMM [He, Hu, Zhang, & Shi, 14]
- Extended to I-divergence model (Poisson and multiplicative noise) [Teuber, Steidl, & C., 13; Carlavan & Blanc-Feraud, 12]:

$$\min_{\mathbf{x}} \|C\mathbf{x}\|_1 + \sum_j ([B\mathbf{x}]_j - [\mathbf{q}]_j \log[B\mathbf{x}]_j).$$

- Noise unknown: determine λ^j by other methods such as GCV in subproblem?

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Multiple Solves of l^2-l^2 Subproblem

- Compute many times for different λ:

$$\mathbf{x}_\alpha = \left(\alpha B^t B + \beta C^t C\right)^{-1} \mathbf{b}^j$$

- Expensive for phase reconstruction problem:

$$\left[\alpha \sum_{i=1}^{m} (RWA_i)^t RWA_i + \beta C^t C\right] \mathbf{x}_\alpha = \mathbf{b}^j$$

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Concluding Remarks

- ℓ^1-ℓ^2 problems are common in image processing
- Solution requires repeatedly ℓ^2-ℓ^2 solve
- ℓ^2-ℓ^2 parameter estimation methods could be used for ℓ^1-ℓ^2 problems
- still need an estimation of the noise level σ
- have found a way to apply ℓ^2-ℓ^2 GCV formula for ℓ^2-ℓ^1 problem. [Wen & Chan, submittd]

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