An ADMM algorithm for constrained material decomposition in spectral CT

May, 23th, 2018

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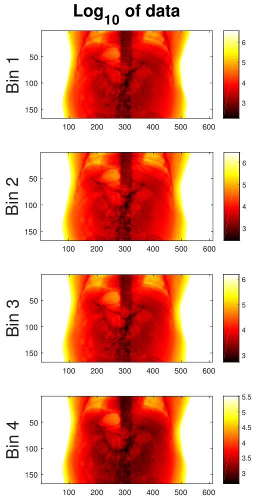
Spectral CT

Spectral CT records energy-dependant data



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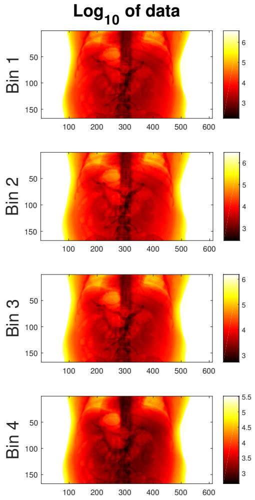
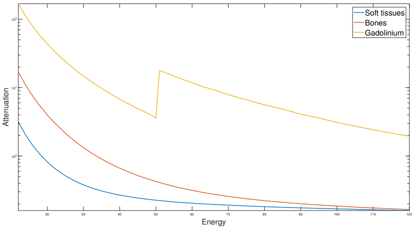


Attenuation coefficients depend on energy



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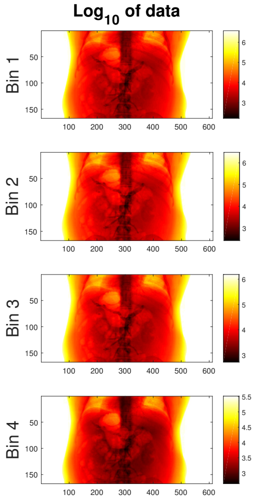
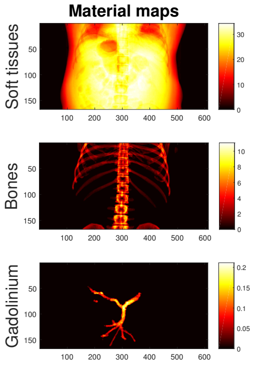


Allow decomposing materials



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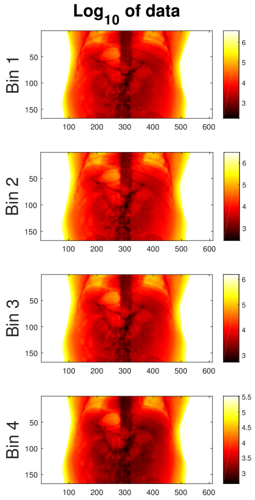
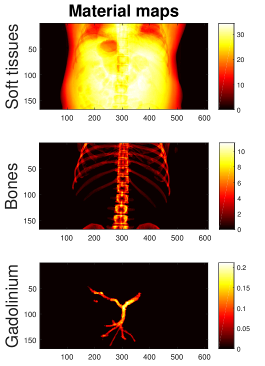


Solve the non-convex $\longrightarrow$ nonlinear problem $\mathbf{s}=\mathcal{F}(\mathbf{a})$

Allow decomposing materials

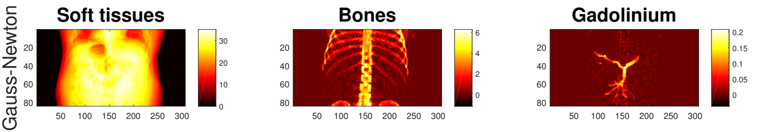


Introduction

Previous work

A Gauss-Newton algorithm was developed [1]

+ Fast convergence
+ Better decomposition error than first order algorithms
+ Spatially regularized
- Can have negative values



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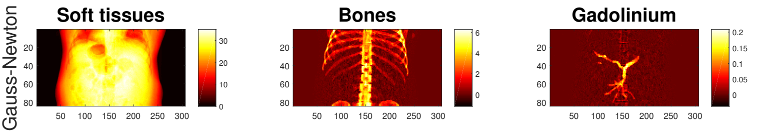

Constrained algorithms

Positivity constraints were added for materials decomposition

- Long and Fessler 2014 [2]
- Noh and Fessler 2009 [3]
- Sidky and Pan 2014 [4]
- Ding and Long 2017 [5]

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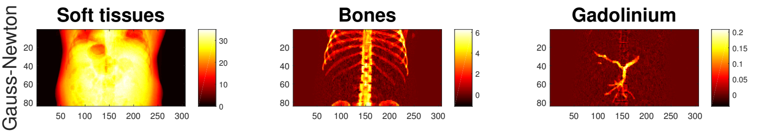

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} First oder algorithms in projection and image domain

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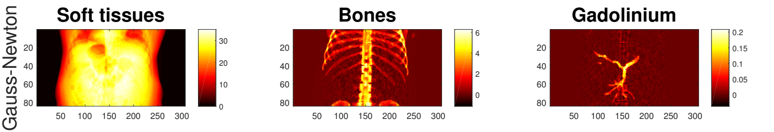

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Proposed method

Positivity / Inequality

- We propose to enforce the positivity of the materials maps

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Equality

- Quantity of injected marker is known
- Should be retrieved in the decompositions

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Algorithm

- Use of an alternating direction method of multipliers (ADMM) : update of $\mathbf{a}$ with a second order algorithm

Forward problem

Denoting $\mathbf{s} \in \mathbb{R}^{IP}$ the data vector and $\mathbf{a} \in \mathbb{R}^{MP}$ the materials maps, we have :

$$\mathbf{s} = [s_{1,1}, \ldots, s_{I,1}, \ldots, s_{I,P}]^{\mathrm{T}}$$

$$\mathbf{a} = [a_{1,1},\ \ldots,\ a_{m,1},\ \ldots,\ a_{M,P}]^{\mathrm{T}}$$

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Using the standard spectral CT forward model is :

$$\mathbf{s}_{i,p} = \int_{\mathbf{R}} n_i^0(E) \exp\left(-\sum_{m=1}^{M} a_{m,p}\tau_m(E)\right) \mathrm{d}E$$

with $n_i^0(E)$ the effective spectrum, $\tau_m(E)$ a function representing the material attenuation and $a_{m,p}$ the projected mass of material m on pixel p.

Inverse problem

- We propose to solve :

$$\min_{\mathbf{a}} \mathcal{C}(\mathbf{a}, \mathbf{s}) \quad \text{s.t} \quad \begin{cases} \mathbf{a} \geq \mathbf{0} \\ \sum_p a_{m,p} = c_m \end{cases} \tag{1}$$

with c_m is the quantity of mth material.

- We chose

$$\mathcal{C}(\mathbf{a}, \mathbf{s}) = \mathcal{D}(\mathbf{a}, \mathbf{s}) + \alpha_{\mathrm{R}} \mathcal{R}(\mathbf{a})$$

where

$$\mathcal{D}(\mathbf{a}, \mathbf{s}) = ||\mathbf{s} - \mathcal{F}(\mathbf{a})||^2_{\mathbf{W}}$$
$$\mathcal{R}(\mathbf{a}) = ||\Delta \mathbf{a}_{\text{soft}}||^2_2 + ||\nabla \mathbf{a}_{\text{bone}}||_1 + ||\nabla \mathbf{a}_{\text{Gd}}||_1$$

ADMM

The Lagrangian function is minimized by an ADMM algorithm :

$$\begin{aligned}\mathcal{L}(\mathbf{a},\mathbf{b},\boldsymbol{\alpha}_{\mathrm{I}},\alpha_{\mathrm{E}},\mathbf{s}) = {} & \mathcal{D}(\mathbf{a},\mathbf{s}) + \alpha_{\mathrm{R}}\mathcal{R}(\mathbf{a}) \\ & + \mathcal{H}_{\mathrm{E}}(\mathbf{a},\alpha_{\mathrm{E}}) + \mathcal{G}_{\mathrm{E}}(\mathbf{a}) \\ & + \mathcal{H}_{\mathrm{I}}(\mathbf{a},\mathbf{b},\boldsymbol{\alpha}_{\mathrm{I}}) + \mathcal{G}_{\mathrm{I}}(\mathbf{a},\mathbf{b}) + \mathbf{1}(\mathbf{b})\end{aligned}$$

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where

Equality : $\sum_p a_{m,p} = c_m$

$$\mathcal{H}_{\mathrm{E}}(\mathbf{a},\alpha_{\mathrm{E}}) = \alpha_{\mathrm{E}}\left(\sum_{p=1}^{P}\frac{a_{\mathrm{gd},p}}{c_{\mathrm{gd}}} - 1\right)$$

$$\mathcal{G}_{\mathrm{E}}(\mathbf{a}) = \frac{\beta_{\mathrm{E}}}{2}\left(\sum_{p=1}^{P}\frac{a_{\mathrm{gd},p}}{c_{\mathrm{gd}}} - 1\right)^2$$

with $\alpha_{\mathrm{E}} \in \mathbf{R}$ and $\beta_{\mathrm{E}} \in \mathbf{R}$.

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The Lagrangian function is minimized by an ADMM algorithm :

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where

Inequality : $\mathbf{a} \geq 0$

Inequalities need an auxiliary variable, $\mathbf{b}$, such that the constraint becomes $\mathbf{a} = \mathbf{b}$ and $\mathbf{b} \geq \mathbf{0}$.

$$\begin{aligned}\mathcal{H}_{\mathrm{I}}(\mathbf{a}, \mathbf{b}, \boldsymbol{\alpha}_{\mathrm{I}}) &= \boldsymbol{\alpha}_{\mathrm{I}}^{\mathrm{T}}(\mathbf{b} - \mathbf{a}) \\ \mathcal{G}_{\mathrm{I}}(\mathbf{a}, \mathbf{b}) &= \frac{\beta_{\mathrm{I}}}{2}||\mathbf{b} - \mathbf{a}||_2^2\end{aligned}$$

and

$$\mathbf{1}(\mathbf{b}) = \begin{cases} 0, & \text{if } \mathbf{b} \geq 0 \\ \infty, & \text{otherwise}\end{cases}$$

with $\alpha_{\mathrm{I}} \in \mathbf{R}^{MP}$ and $\beta_{\mathrm{I}} \in \mathbf{R}$.

Minimization scheme

- Minimization of $\mathcal{L}(\mathbf{a}, \mathbf{b}, \boldsymbol{\alpha}_{\mathrm{I}}, \alpha_{\mathrm{E}}, \mathbf{s})$ is done by finding the saddle point [6], through alternating updates :

Update	Algorithm used
$\mathbf{a}^{\ell+1} \in \underset{\mathbf{a}}{\operatorname{argmin}}\, \mathcal{L}(\mathbf{a}, \mathbf{b}^{\ell}, \alpha_{\mathrm{E}}^{\ell}, \boldsymbol{\alpha}_{\mathrm{I}}^{\ell})$	Iterative Gauss-Newton
$\mathbf{b}^{\ell+1} \in \underset{\mathbf{b}}{\operatorname{argmin}}\, \mathcal{L}(\mathbf{a}^{\ell+1}, \mathbf{b}, \alpha_{\mathrm{E}}^{\ell}, \boldsymbol{\alpha}_{\mathrm{I}}^{\ell})$	Proximal algorithm
$\alpha_{\mathrm{E}}^{\ell+1} \in \underset{\alpha_{\mathrm{E}}}{\operatorname{argmax}}\, \mathcal{L}(\mathbf{a}^{\ell+1}, \mathbf{b}^{\ell+1}, \alpha_{\mathrm{E}}, \boldsymbol{\alpha}_{\mathrm{I}}^{\ell})$	Ascent gradient
$\boldsymbol{\alpha}_{\mathrm{I}}^{\ell+1} \in \underset{\boldsymbol{\alpha}_{\mathrm{I}}}{\operatorname{argmax}}\, \mathcal{L}(\mathbf{a}^{\ell+1}, \mathbf{b}^{\ell+1}, \alpha_{\mathrm{E}}^{\ell+1}, \boldsymbol{\alpha}_{\mathrm{I}})$	Ascent gradient

- Moreover, at each iteration ℓ, Lagrangian parameters are increasing

$$\beta_{\mathrm{E}}^{\ell+1} = \omega \beta_{\mathrm{E}}^{\ell}$$
$$\beta_{\mathrm{I}}^{\ell+1} = \omega \beta_{\mathrm{I}}^{\ell}$$

- Until they reach the maximum allow value $\beta_{\max}$.

Parameters

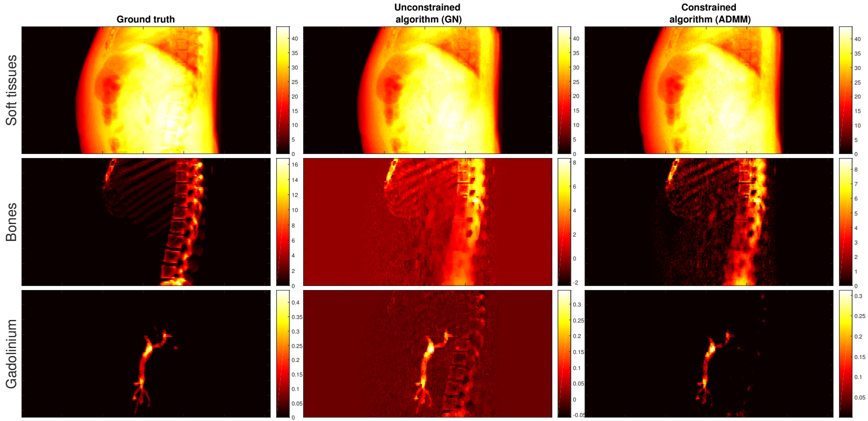
Phantom

- Thorax : 84×306 pixels
- 3 materials : soft tissues, bones and gadolinium (portal vein)
- 4 energy bins

Algorithm

- $\mathbf{a}^0 = 0\text{g.cm}^{-2}$
- $\alpha_{\mathrm{R}} = 1$, $\alpha_{\mathrm{E}}^0 = 0$, $\boldsymbol{\alpha}_{\mathrm{I}}^0 = 0$
- $\beta_{\mathrm{E}}^0 = 10^0$, $\beta_{\mathrm{I}}^0 = 10^{-2}$, $\omega = 1.5$
- $\beta_{\max} = 10^{10}$
- Stopping criteria
  - The inner loop stops if
    - the relative decrease of the cost function is too low (10^{-3})
    - it made more than 30 iterations
  - The outer loop stops if
    - the equality is respected : $\left| \left(\sum_p^P \frac{a_{\mathrm{gd,p}}^\ell}{c_{\mathrm{gd}}} - 1 \right) \right| < 10^{-3}$
    - the inequality is respected : $||\mathbf{a}^\ell - \mathbf{b}^\ell|| < 10^{-3}$
- Decomposition for $\theta = [0^\circ, 359^\circ]$

Projection domain



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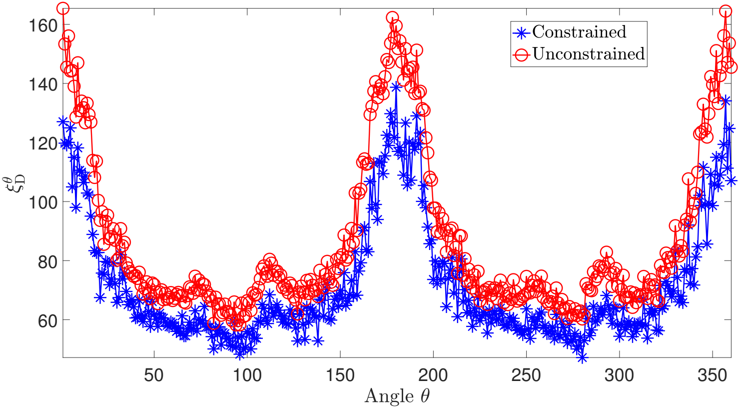
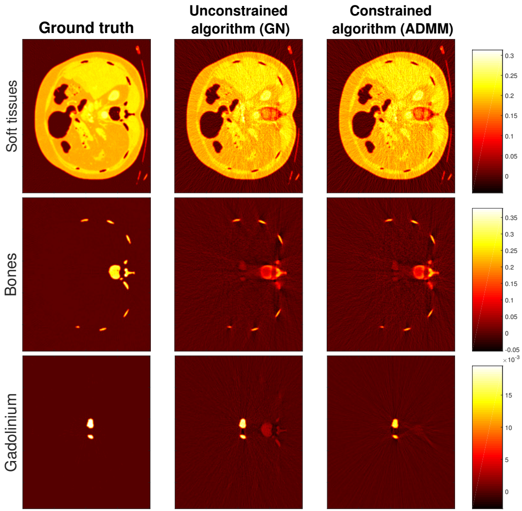


Evaluation of the decomposition with the ℓ_2-norm :

$$\xi_{\mathrm{D}}^{\theta} = \sum_{m}^{M} ||\mathbf{a}_m^{\theta} - \mathbf{a}_m^{\mathrm{truth},\theta}||_2$$

Image domain



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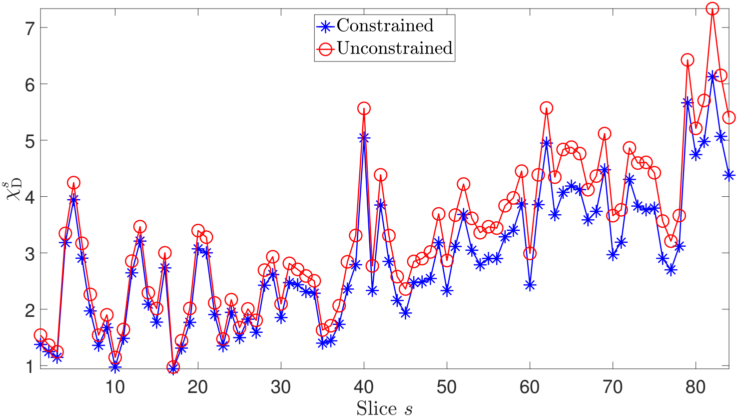

Evaluation of the reconstructed materials with the ℓ_2-norm :

$$\chi_{\mathrm{D}}^{z} = \sum_{m}^{M} ||\boldsymbol{\rho}_m^{\mathrm{z}} - \boldsymbol{\rho}_m^{\mathrm{truth,z}}||_2$$

Conclusion

- Equality and inequality constraints give a better decomposition and visualization
- Decrease cross-talk
- However, it needs more iteration

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And next :

- Application of this algorithm to real data

Thank you for your attention !

Questions ? :)

References

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Minimization scheme

while *Stopping criteria aren't met for the outer loop* **do**
  while *Stopping criteria aren't met for the inner loop* **do**
    solve $\left(\mathbf{H}^k \delta \mathbf{a}^k = -\mathbf{g}^k\right)$
    $\mathbf{a}^{k+1} = \mathbf{a}^k + \lambda^k \delta \mathbf{a}^k$
    $k = k + 1$
  end
  $\mathbf{a}^{\ell+1} = \mathbf{a}^{k-1}$

- Update of $\mathbf{a}$: Gauss-Newton algorithm

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  $\mathbf{b}^{\ell+1} = \Pi(\mathbf{a}^{\ell+1} - \frac{\boldsymbol{\alpha}_{\mathrm{I}}^{\ell}}{\beta_{\mathrm{I}}})$

- Update of **a** : Gauss-Newton algorithm
- Update of **b** : Proximal algorithm

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  $\alpha_{\mathrm{E}}^{\ell+1} = \alpha_{\mathrm{E}}^{\ell} + \beta_{\mathrm{E}} \frac{\mathcal{H}_E(\mathbf{a}_m^{\ell+1})}{\alpha_{\mathrm{E}}^{\ell}}$

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- Update of $\mathbf{a}$: Gauss-Newton algorithm
- Update of $\mathbf{b}$: Proximal algorithm
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- Update of the βs

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  $\beta_{\mathrm{E}}^{\ell+1} = \omega \beta_{\mathrm{E}}^{\ell}$
  $\beta_{\mathrm{I}}^{\ell+1} = \omega \beta_{\mathrm{I}}^{\ell}$
  if $\beta_{\mathrm{E}}^{\ell+1} > \beta_{\max}$ **then**
    $\beta_{\mathrm{E}}^{\ell+1} = \beta_{\max}$
  end
  if $\beta_{\mathrm{I}}^{\ell+1} > \beta_{\max}$ **then**
    $\beta_{\mathrm{I}}^{\ell+1} = \beta_{\max}$
  end
  $\ell = \ell + 1$
end

- Update of **a** : Gauss-Newton algorithm
- Update of **b** : Proximal algorithm
- Update of α_{E} : Ascent gradient
- Update of $\boldsymbol{\alpha}_{\mathrm{I}}$: Ascent gradient
- Update of the βs
- Make sure that the values aren't too high