Chapter 6

Ground-based Astronomy

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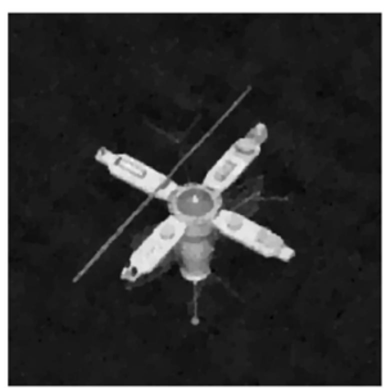
Reconstructed Telescope Image

observed image



reconstructed image

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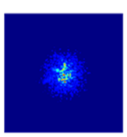
Outline

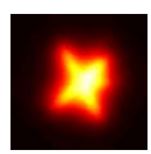
1. **Ground-based Astronomy**
2. **Models**
3. **Experiments**
4. **High-resolution Image Reconstruction**

Ground-Based Astronomy

true image
$f(x, y)$



point spread function
$k(x, y)$



observed image
$g(x, y)$

Model in Ground-Based Astronomy

$$g(x,y) = k(x,y) * f(x,y) + n(x,y)$$

where PSF $k(x,y)$ and noise $n(x,y)$ are unknown.

❑ In matrix terminologies:

$$\boxed{\mathbf{g} = K\mathbf{f} + \mathbf{n}}$$

where K is a block-Toeplitz-Toeplitz-block matrix.

❑ Standard deblurring problem if K is known.

[C. & Jin, *Iterative Toeplitz Solvers*, SIAM 2007]

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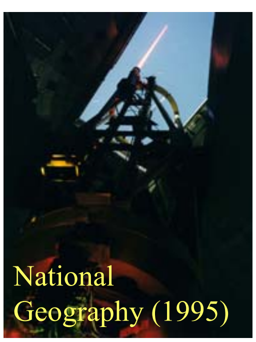
Unknown Point Spread Functions

Some well-known approaches for getting $k(x,y)$:

❑ Blind-decovolution to simultaneously obtain $k(x,y)$ and $f(x,y)$:

$$g(x,y) = k^{(i+\frac{1}{2})}(x,y) * f^{(i)}(x,y) + n(x,y),\ i = 1, 2 \ldots$$



[T. Chan & Wong IEEE TIP 98]

❑ Reconstruct $k(x,y)$ by some means (e.g. natural or artificial guide-star)

[C., Nagy, & Plemmons, SINUM 93]

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Point-spread Function Reconstruction

Planar waves change across atmospheric turbulence

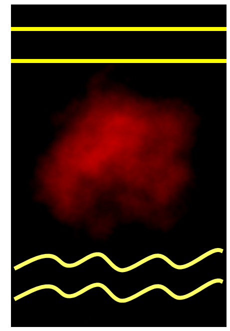
- $\phi(x,y)$: deviation from planarity is called phase error or phase

Fourier optics model:

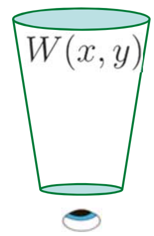
$$k(x,y) = \left|\mathcal{F}^{-1}\left\{W(x,y)e^{i\phi(x,y)}\right\}\right|^2$$

- $W(x,y)$: aperture of the telescope
- $\mathcal{F}$: Fourier transform

[Goodman 96, Bardsley SIMAX, 08]



$\phi(x,y)$



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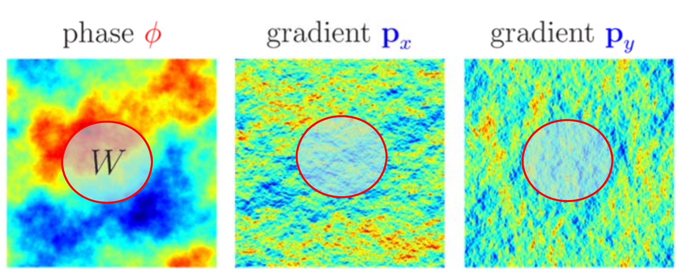
Wavefront to Wavefront Gradient

Phase $\phi(x,y)$ cannot be directly measured, only its gradients by wavefront sensors:

- $\mathbf{p}_x = D_x\phi(x,y)$: horizontal wavefront gradient
- $\mathbf{p}_y = D_y\phi(x,y)$: vertical wavefront gradient

D_i: 1st-order derivative operator modeling the sensor



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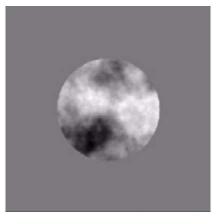
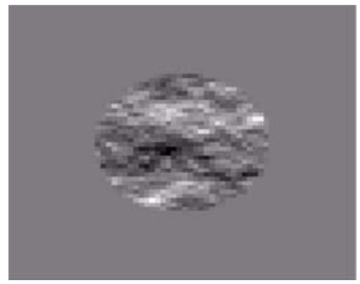
The Problem

- Wavefront sensors collect wavefront gradients $D_i\phi(x, y)$, not the phase $\phi(x, y)$
- $D_i\phi(x, y)$ are collected on coarse grids



Phase $\phi(x, y)$



$D_x\phi(x, y)$

- Not accurate to compute ϕ from $D_i\phi(x, y)$

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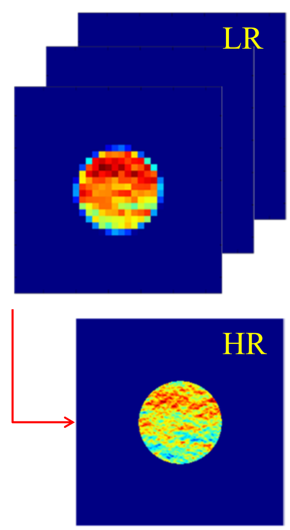
The Aim

- Use a sequences of low-resolution (LR) frames of wavefront gradients to obtain the high-resolution (HR) wavefront gradients $\mathbf{p}_i = D_i\phi(x, y)$
- From HR wavefront gradient $D_i\phi(x, y)$ reconstruct more accurate $\phi(x, y)$
- From $\phi(x, y)$ reconstruct $k(x, y)$
- Using $k(x, y)$, deblur $g(x, y)$ to get $f(x, y)$



$\mathbf{p}_x$

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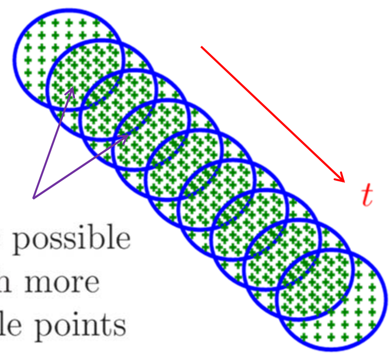
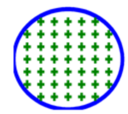
Frozen Flow Hypothesis

Move telescope to get a sequence of LR frames of wavefront gradients to reconstruct the HR wavefront gradients.



HR is possible with more sample points

A LR wavefront gradients sensor



[Jefferies & Hart, 10]

Within a short time interval, phase does not change.

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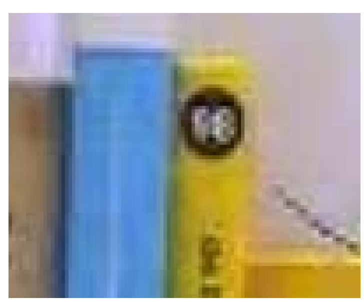
Video Enhancement

A 352-by-288 video from a video recorder

20 to 30 frames/second



Bilinear interpolation from 1 frame

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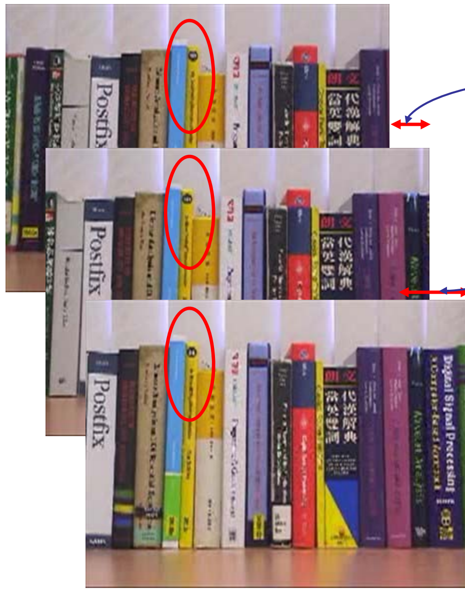
Video Enhancement

Reference frame



displacement

t

Improving resolution of LR reference frame

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Video Enhancement

A 352-by-288 video from a video recorder

Tight-frame method using 21 frames

[C., Shen, & Xia, ACHA 08]

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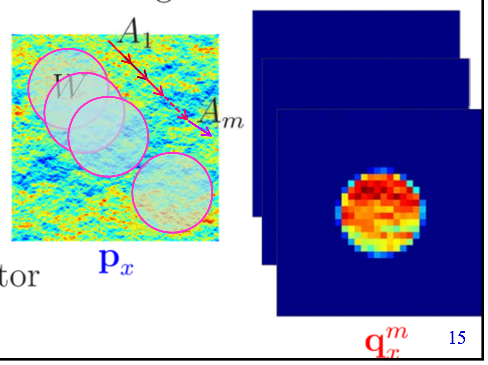
Relation between HR and LR Gradients

$$\mathbf{q}_x^i = RWA_i\mathbf{p}_x + \mathbf{n}_x^i, \quad i = 1, 2, \cdots, m$$
$$\mathbf{q}_y^i = RWA_i\mathbf{p}_y + \mathbf{n}_y^i, \quad i = 1, 2, \cdots, m$$

- ❑ $\mathbf{p}_x$, $\mathbf{p}_y$: HR wavefront gradients
- ❑ $\mathbf{q}_x^i$, $\mathbf{q}_y^i$: sequences of LR wavefront gradients
- ❑ $\mathbf{n}_x^i$, $\mathbf{n}_y^i$: noise
- ❑ A_i: motion operator
- ❑ W: aperture operator
- ❑ R: down-sampling operator



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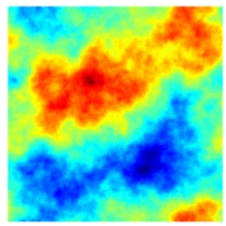
Tikhonov l^2-l^2 Model

[Nagy, Jefferies, & Chu, Maui Conf. 10, SISC 13]:

$$\min_{\mathbf{p}_x} \|\mathbf{p}_x\|_2^2 + \frac{\alpha}{2}\sum_{i=1}^{m} \|RWA_i\mathbf{p}_x - \mathbf{q}_x^i\|_2^2$$

$$\min_{\mathbf{p}_y} \|\mathbf{p}_y\|_2^2 + \frac{\alpha}{2}\sum_{i=1}^{m} \|RWA_i\mathbf{p}_y - \mathbf{q}_y^i\|_2^2$$

- ☐ linear solve with $\left[I + \alpha \sum_i (RWA_i)^T RWA_i\right]$
- ☐ $\|\mathbf{p}_x\|_2^2, \|\mathbf{p}_y\|_2^2 \approx \|\nabla\phi\|_2^2$
- ☐ may smooth the edges in ϕ



phase ϕ

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l^1-l^2 Model

- ☐ $\|\mathbf{p}_x\|_1, \|\mathbf{p}_y\|_1 \approx \|\nabla\phi\|_1$: total-variation of ϕ
- ☐ can avoid over-smoothing the edges in ϕ
- ☐ [C., Zhang, & Yuan, J. Amer. Optics Soc. A, 12]

$$\min_{\mathbf{p}_x} \|\mathbf{p}_x\|_1 + \frac{\alpha}{2}\sum_{i=1}^{m} \|RWA_i\mathbf{p}_x - \mathbf{q}_x^i\|_2^2$$

$$\min_{\mathbf{p}_y} \|\mathbf{p}_y\|_1 + \frac{\alpha}{2}\sum_{i=1}^{m} \|RWA_i\mathbf{p}_y - \mathbf{q}_y^i\|_2^2$$

- ☐ ℓ^1-ℓ^2 model on $\mathbf{p}_x$ and $\mathbf{p}_y$

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Combined Model for the Phase

- Note that:

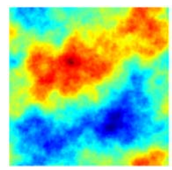
$$\begin{cases} \mathbf{q}_x^i = RWA_iD_x\phi + \mathbf{n}_x^i, \\ \mathbf{q}_y^i = RWA_iD_y\phi + \mathbf{n}_y^i, \end{cases} \quad i = 1, 2, \cdots, m.$$

- Treat ϕ as an "image" and regularize it:

$$\min_{\phi} \|C\phi\|_1 + \frac{\alpha}{2}\sum_{i=1}^{m}\left\| \begin{bmatrix} RWA_iD_x \\ RWA_iD_y \end{bmatrix} \phi - \begin{bmatrix} \mathbf{q}_x^i \\ \mathbf{q}_y^i \end{bmatrix} \right\|_2^2.$$

- Regularizer C can be TV, wavelet, tight-frame, fractional, ...

[C., Yuan, & Zhang, Science China, 13]



ϕ

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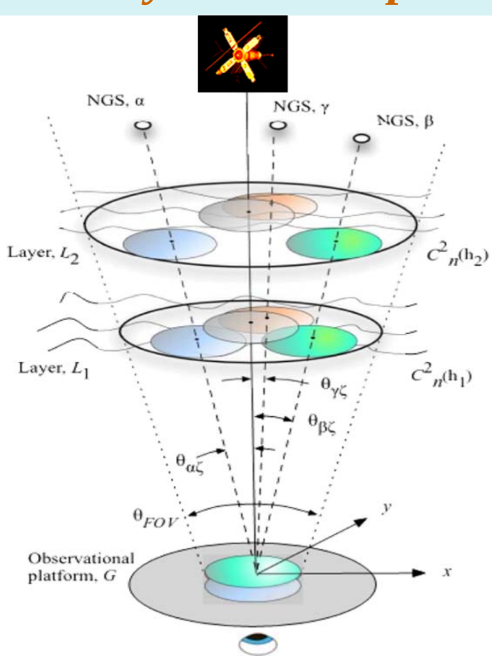
Multi-layered Atmosphere



phase $\phi = \sum_{j=1}^{l} \phi_j$

$\phi_2(x, y)$

$\phi_1(x, y)$

S.J. Weddell, "Optical wavefront prediction with reservoir computing".

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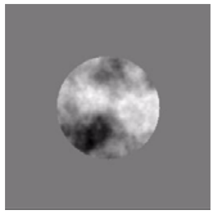
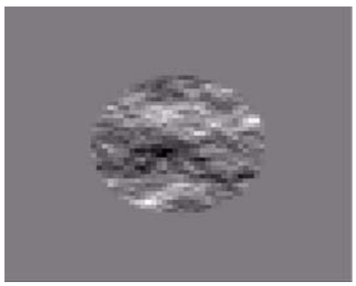
Multi-layered Phase



One-layer ϕ

Multi-layer $\{\phi_l\}$



LR Horizontal $\mathbf{q}_x$

LR Horizontal $\mathbf{q}_x$

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Multi-layered Astronomical Imaging System

The LR wavefront gradients and phases are related by

$$\mathbf{q}_x^i = RWA_{1i}D_x\phi_1 + RWA_{2i}D_x\phi_2 + \cdots + RWA_{li}D_x\phi_l + \mathbf{n}_x^i$$

$$\mathbf{q}_y^i = RWA_{1i}D_y\phi_1 + RWA_{2i}D_y\phi_2 + \cdots + RWA_{li}D_y\phi_l + \mathbf{n}_x^i$$

where

- ϕ_j: phase at the j-th layer
- A_{ji}: motion matrix at the i-th frame of ϕ_j

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Model for Multi-layered System

The minimization model is:

$$\min_{\{\phi_i\}} \sum_{i=1}^{l} \|C\phi_i\|_1 + \frac{\alpha}{2}\sum_{i=1}^{m}\left\| \begin{bmatrix} RWA_{1i}D_x & \cdots & RWA_{li}D_x \\ RWA_{1i}D_y & \cdots & RWA_{li}D_y \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_l \end{bmatrix} - \begin{bmatrix} \mathbf{q}_x^i \\ \mathbf{q}_y^i \end{bmatrix} \right\|_2^2$$

- An ℓ^1-ℓ^2 model on $\{\phi_i\}$. Solved by ADMM.
- ϕ: the phase $\phi = \sum_{j=1}^{l} \phi_j$
- $k(x,y) = \left|\mathcal{F}^{-1}\left\{W(x,y)e^{i\phi(x,y)}\right\}\right|^2$

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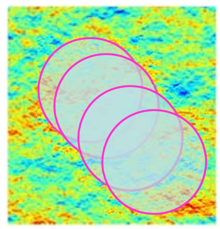
Experiment Setup

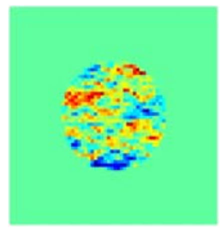
- generate true 256-by-256 $\phi^\star$ [Nagy *et al.*, 10, 13]
- generate HR $\mathbf{p}_x^\star = D_x\phi^\star$ and $\mathbf{p}_y^\star = D_y\phi^\star$
- generate LR $\mathbf{q}_j^i$ with 1% Gaussian noise by

$$\mathbf{q}_x^i = RWA_i\mathbf{p}_x^\star + \mathbf{n}_x^i, \quad i = 1, 2, \cdots, m$$
$$\mathbf{q}_y^i = RWA_i\mathbf{p}_y^\star + \mathbf{n}_y^i, \quad i = 1, 2, \cdots, m$$

- downsample by a factor of 4 (64-by-64 LR)
- use $m = 16$ frames



$\mathbf{p}_x^\star$

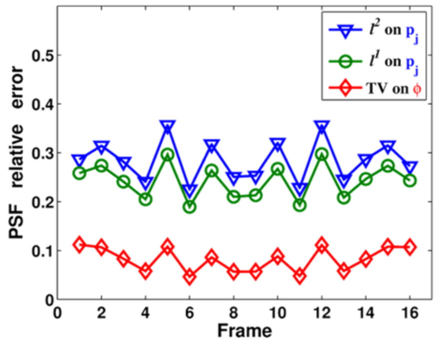


$\mathbf{q}_x^i$

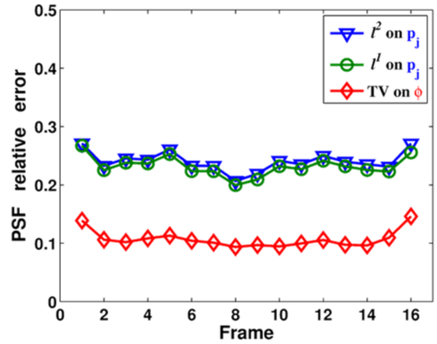
Reconstructing the PSF

- Given $\{\mathbf{q}_x^i\}_{i=1}^m$ and $\{\mathbf{q}_y^i\}_{i=1}^m$, solve
  - $\min\limits_{\mathbf{p}_x} \|\mathbf{p}_x\|_2^2 + \frac{\alpha}{2}\sum_{i=1}^m \|RWA_i\mathbf{p}_x - \mathbf{q}_x^i\|_2^2$
  - $\min\limits_{\mathbf{p}_x} \|\mathbf{p}_x\|_1 + \frac{\alpha}{2}\sum_{i=1}^m \|RWA_i\mathbf{p}_x - \mathbf{q}_x^i\|_2^2$
  - $\min\limits_{\phi} \|\nabla\phi\|_1 + \frac{\alpha}{2}\sum\limits_{i=1}^m \left\| \begin{bmatrix} RWA_iD_x \\ RWA_iD_y \end{bmatrix} \phi - \begin{bmatrix} \mathbf{q}_x^i \\ \mathbf{q}_y^i \end{bmatrix} \right\|_2^2$
- Recover ϕ from $\mathbf{p}_x = D_x\phi$ and $\mathbf{p}_y = D_y\phi$
- Recover PSF from $k(x, y) = \left|\mathcal{F}^{-1}\left\{W(x, y)e^{i\phi(x,y)}\right\}\right|^2$
- Compare computed k with true $k^\star$ from true $\phi^\star$

Relative Error in Point Spread Function



single-layer
seeing condition = 45



3-layer
seeing condition = 20

▿: [Chu, Jefferies, & Nagy, SIAM J. Sci. Comput., 13]
○: [C., Yuan, & Zhang, J. Opt. Soc. Amer. A, 12]
◇: [C., Yuan, & Zhang, Science China A., 13]

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How good is the Deblurring?

- Use true PSF $k^\star(x,y)$ to generate blurred image:

$$g(x,y) = k^\star(x,y) * f(x,y) + n(x,y)$$

with 1% Gaussian noise added.

- In matrix terminology:

$$\mathbf{g} = K^\star \mathbf{f} + \mathbf{n}$$

$g(x,y)$

- Deblur $\mathbf{g}$ with computed PSF $k(x,y)$:

$$\min_{\mathbf{f}} \|\nabla \mathbf{f}\|_1 + \frac{\mu}{2}\|K\mathbf{f} - \mathbf{g}\|_2^2$$

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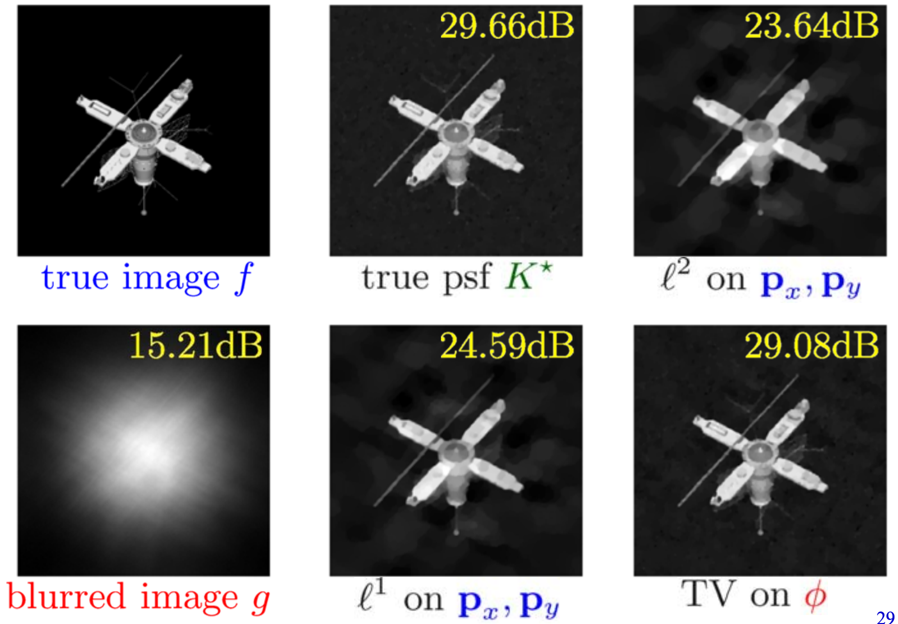
PSNR Results for 1-Layer Case



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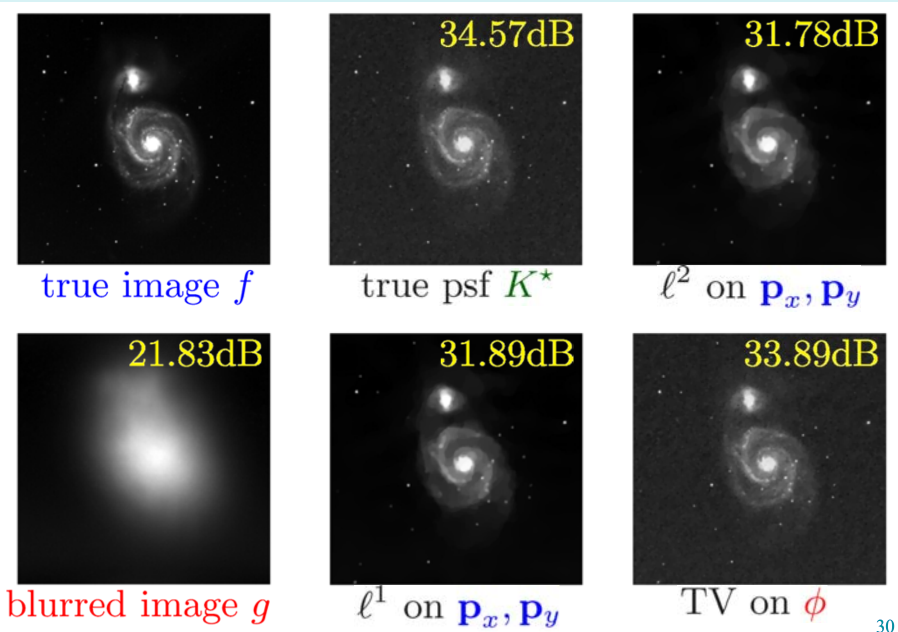
PSNR Results for 3-Layer Case



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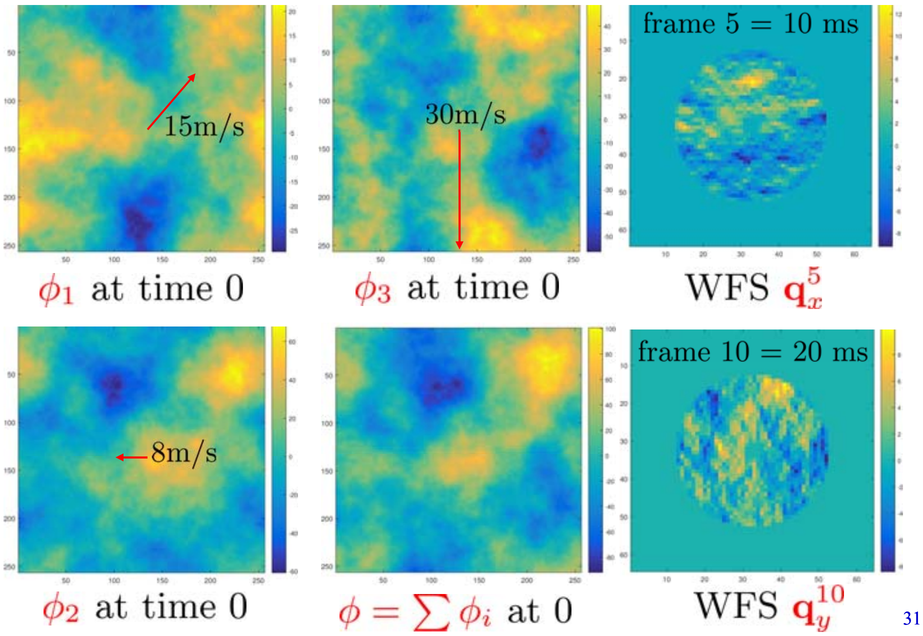
Moving Phase: 3-Layer Case



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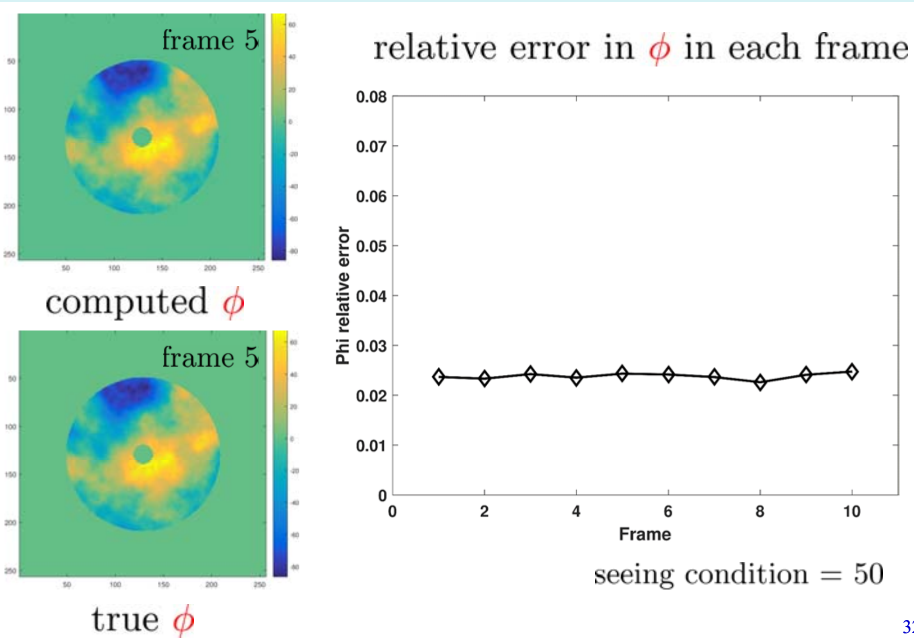
Moving Phase: 3-Layer Case



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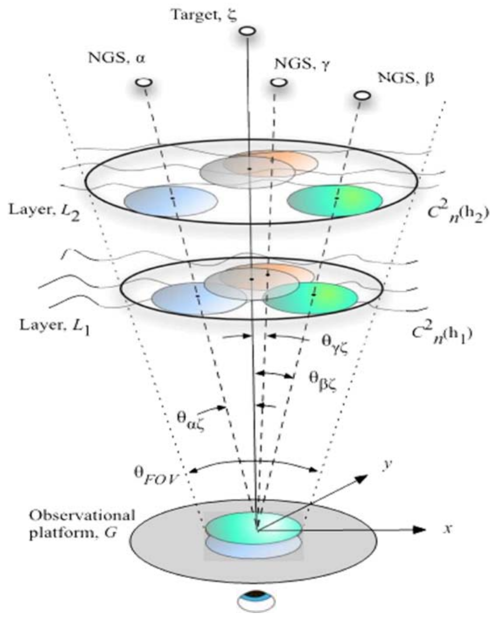
Atmospheric Tomography



$\phi_2(x, y)$

$\phi_1(x, y)$

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Concluding Remarks and Future Work

- Combined model gives better HR phase ϕ
- Fast solution by optimization method (ADMM)
- HRIR has many interesting applications

- Adapt HRIR ideas to estimating PSF for ground-based astronomy
- Application to atmospheric tomography

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