Adaptive Filtered Schemes for first order Hamilton-Jacobi equations and applications

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Computational Methods for Inverse Problems in Imaging

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Outline

1. Introduction
   - Monotone schemes
   - High-order schemes
2. Adaptive filtered scheme
   - Filter function
   - Smoothness indicator function
   - Automatic tuning of the parameter ε^n
3. Convergence theorem
4. Numerical Experiments

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Introduction

Let us consider the level set equation related to the segmentation problem

$$\begin{cases} v_t + c(x)|\nabla v| = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{1}$$

where v_0 is a function representing the initial configuration of the front Γ_0, (i.e. changing sign on Γ_0).

Here we consider a given velocity $c(x)$ which for the segmentation typically is

$$c(x) = \frac{1}{1 + |\nabla(G_\rho * I(x))|^p)}$$

Bacteria

(Loading 30 sec)

The football player

(Loading 30 sec)

Introduction

Time dependent HJ equation

More in general, we want to get an accurate approximation of the viscosity solution of the evolutive Hamilton-Jacobi (HJ) equation:

$$\begin{cases} v_t + H(\nabla v) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2}$$

$(H1)$ $H(p)$ is continuous;

$(H2)$ $v_0(x)$ is Lipschitz continuous.

• Under these assumptions we have existence and uniqueness of the viscosity solution for (2).

Challenges and motivations

- In general, the solution is not classical ($v \notin C^1$) and can develop singularities in finite time
- we need to have a good resolution of the solution even at kinks
- high-order schemes allow the use of coarser grids
- very few convergence results for high-order schemes in literature
- several interesting applications: computer vision, optimal control, front propagation, differential games...

Some references

Several schemes have been developed:

- Finite difference schemes: Kružkov (65), Crandall-Lions(84), Sethian(88), Tarasyev (90), Osher/Shu(91), Tadmor/Lin(00).
- Semi-Lagrangian schemes: F (87, 94, 09), F-Giorgi (99), Ferretti-Carlini(03, 04,13), Capuzzo Dolcetta (83,89,90),....
- Discontinuous Galerkin approach: Hu-Shu(99), Li-Shu(05), Bokanowski-Chang-Shu(11,13,14), Cockburn(00), Guo-Zhong-Qiu (2013)....
- Finite Volume schemes: Kossioris/Makridakis/Souganidis(99), Kurganov/Tadmor(00)), Abgrall(00,01).

Monotone schemes

- **Discretization:** Let $\Delta t > 0$ denote the time step and $\Delta x > 0$ the mesh step, $t_n = n\Delta t$, $n \in [0, \dots, N]$, $N \in \mathbb{N}$ and $x_j = j\Delta x$, $j \in \mathbb{Z}$.

For a given function $u(x)$ with nodal values $u_j = u(x_j)$, let S^M be a **monotone scheme**,

Assumptions on S^M:

$(M1)$ the scheme can be written in **differenced form**

$$u_j^{n+1} \equiv S^M(u_j^n) := u_j^n - \Delta t\, h^M(D^- u_j^n, D^+ u_j^n)$$

for a function $h^M(p^-, p^+)$, with $D^{\pm} u_j^n := \pm \frac{u_{j\pm 1}^n - u_j^n}{\Delta x}$;

$(M2)$ h^M is a Lipschitz continuous function;

$(M3)$ (Consistency) $\forall v$, $h^M(v, v) = H(v)$;

$(M4)$ (Monotonicity) for any functions u, v,

$$u \le v \quad \Longrightarrow \quad S^M(u) \le S^M(v).$$

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Monotone schemes

Consistency error estimate:

For any $v \in C^2([0,T] \times \mathbb{R})$, there exists a constant $C_M \geq 0$ independent of Δx and Δt such that

$$\mathcal{E}_M(v)(t,x) := \left| \frac{v(t+\Delta t, x) - S^M(v(t,\cdot))(x)}{\Delta t} \right| \leq C_M \left(\Delta t ||v_{tt}||_\infty + \Delta x ||v_{xx}||_\infty \right).$$

Theorem (Crandall-Lions (84))

Assume that the Hamiltonian H and the initial data v_0 are Lipschitz continuous. Let the monotone finite difference scheme (1) (with numerical hamiltonian h^M) satisfy (M1)-(M4)) and define $v_j^n := v(t_n, x_j)$, where v is the exact solution of (2). Then, there is a constant C such that for any $n \leq T/\Delta t$ and $j \in \mathbb{Z}$, we have

$$|v^n(x_j) - u^n(x_j)| \leq C\sqrt{\Delta x}. \tag{3}$$

where $\Delta x = c\Delta t$ and $\Delta t \to 0$.

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Examples of monotone schemes

- For the *eikonal equation*, where $H(v_x) = |v_x|$,

$$h^M(p^-, p^+) := \max\{p^-, -p^+\};$$

- For general hamiltonians, the *Central Upwind scheme* of Kurganov et al. (2001) [4]

$$h^M(p^-, p^+) := \frac{1}{a^+ - a^-}\left[a^- H(p^+) - a^+ H(p^-) - a^+ a^- (p^+ - p^-)\right],$$

with $a^+ = \max\{H_p(p^\pm), 0\}$ and $a^- = \min\{H_p(p^\pm), 0\}$;

- and the *Lax-Friedrichs scheme*

$$h^M(p^-, p^+) := H\left(\frac{p^- + p^-}{2}\right) - \frac{\theta}{2}(p^+ - p^-)$$

where $\theta > 0$ is a constant. The scheme is monotone under the restrictions $\max_{x,p} |H_p(p)| < \theta$ and $\theta\lambda \leq 1$.

High-order schemes

Let S^A denote a **high-order** (possibly unstable) scheme,

Assumptions on S^A:

$(A1)$ the scheme can be written in **differenced form**

$$u_j^{n+1} = S^A(u^n)_j := u_j^n - \Delta t h^A(D^{k,-}u_j, \dots, D^- u_j^n, D^+ u_j^n, \dots, D^{k,+}u_j^n),$$

for some function $h^A(p^-, p^+)$ (in short), with $D^{k,\pm}u_j^n := \pm \frac{u_{j\pm k}^n - u_j^n}{k\Delta x}$;

$(A2)$ h^A is a Lipschitz continuous function.

$(A3)$ (*high-order consistency*) Fix $k \geq 2$ order of the scheme, then for all $l = 1, \dots, k$ and for all functions $v \in C^{l+1}$, there exists a constant $C_{A,l} \geq 0$ such that

$$\begin{aligned}\mathcal{E}_A(v)(t,x) &:= \left| \frac{v(t+\Delta t, x) - S^A(v(t,\cdot))(x)}{\Delta t} \right| \\ &\leq C_{A,l} \left(\Delta t^l ||\partial_t^{l+1} v||_\infty + \Delta x^l ||\partial_x^{l+1} v||_\infty \right).\end{aligned}$$

High-order schemes

Lemma

Let v be the solution of (2). Then, if $v \in C^r\left(\Omega_{(t,x)}\right)$, $r \geq 2$, where $\Omega_{(t,x)}$ is a neighborhood of a point $(t,x) \in \Omega := [0,T] \times \mathbb{R}$, we have

$$\frac{\partial^k v(t,x)}{\partial t^k} = (-1)^k \frac{\partial^{k-2}}{\partial x^{k-2}} \left(H_p^k(v_x(t,x)) v_{xx}(t,x) \right)$$

for $k = 2, \ldots, r$.

Consistency error estimate:

Assuming (A1)-(A2), the consistency property is equivalent to require that for $l = 2, \ldots, k$, and for all $v \in C^{l+1}$,

$$\begin{aligned} \mathcal{E}_{S^A}(v)(t,x) = & \left| h^A(D^- v, D^+ v) - H(v_x) + \tfrac{\Delta t}{2} H_p^2(v_x) v_{xx} \right| \\ & \leq C_{A,l} \left(\Delta t^l ||\partial_t^{l+1} v||_\infty + \Delta x^l ||\partial_x^{l+1} v||_\infty \right). \end{aligned}$$

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Examples of high-order schemes

- A simple example is the *TVD-RK2* discretization in time

$$\begin{cases} u^* = u^n - \Delta t h_*^A(D^{\pm}u^n) \\ u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^* - \frac{\Delta t}{2}h_*^A(D^{\pm}u^*). \end{cases}$$

where $h_*^A(D^-v_j^n, D^+v_j^n) = H(v_x(t_n, x_j)) + O(\Delta x^2)$ is a second-order in space numerical hamiltonian, e.g. the simple *centered* approximation

$$h_*^A(D^-u_j^n, D^+u_j^n) = H\left(\frac{D^+u_j^n + D^-u_j^n}{2}\right).$$

Then, $h^A(D^{\pm}u^n) = \frac{1}{2}\left[h_*^A(D^{\pm}u^n) + h_*^A(D^{\pm}u^*)\right]$;

- or the *Lax-Wendroff scheme* for HJ equations

$$h^A(p^-, p^+) = \frac{1}{2}\left\{H\left(p^+\right) + H\left(p^-\right) - \frac{\Delta t}{\Delta x}H_p\left(\frac{p^- + p^+}{2}\right)\left[H\left(p^+\right) - H\left(p^-\right)\right]\right\},$$

and the *Richtmyer* version

$$h^A(p^-, p^+) = H\left(\frac{p^- + p^+}{2} - \frac{\Delta t}{2\Delta x}\left[H\left(p^+\right) - H\left(p^-\right)\right]\right).$$

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4th order Lax-Wendroff scheme

Using the previous lemma we can easily construct Lax-Wendroff scheme of order $r > 2$ with very compact stencils. For example with $r = 4$, if we define

- $H_1 = H\left(\frac{u_{j-2}-8u_{j-1}+8u_{j+1}-u_{j+2}}{12\Delta x}\right)$,
- $H_2 = H_p^2\left(\frac{u_{j-2}-8u_{j-1}+8u_{j+1}-u_{j+2}}{12\Delta x}\right)\left(\frac{-u_{j-2}+16u_{j-1}-30u_j+16u_{j+1}-u_{j+2}}{12\Delta x^2}\right)$,
- $H_3 = \frac{1}{2\Delta x}\left[H_p^3\left(\frac{u_{j+2}-u_j}{2\Delta x}\right)\left(\frac{u_{j+2}-2u_{j+1}+u_j}{\Delta x^2}\right) - H_p^3\left(\frac{u_j-u_{j-2}}{2\Delta x}\right)\left(\frac{u_j-2u_{j-1}+u_{j-2}}{\Delta x^2}\right)\right]$,
- $H_4 = \frac{1}{\Delta x^2}\left[H_p^4\left(\frac{u_{j+2}-u_j}{2\Delta x}\right)\left(\frac{u_{j+2}-2u_{j+1}+u_j}{\Delta x^2}\right) - 2H_p^4\left(\frac{u_{j+1}-u_{j-1}}{2\Delta x}\right)\left(\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2}\right) + H_p^4\left(\frac{u_j-u_{j-2}}{2\Delta x}\right)\left(\frac{u_j-2u_{j-1}+u_{j-2}}{\Delta x^2}\right)\right]$,

and then compute

$$h^A(D^-u_j^n, D^+u_j^n) = H_1 - \frac{\Delta t}{2}\left[H_2 - \frac{\Delta t}{3}\left(H_3 - \frac{\Delta t}{4}H_4\right)\right],$$

we get a fourth order scheme in both space and time with a 5-points stencil.

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Filtered scheme

It is known that a monotone scheme can be at most of first order accurate. Therefore, to get a high-order scheme, we look for non-monotone schemes.

The difficulty is to control the scheme, prove a convergence result and get an error estimate for the approximation of viscosity solution of (2).

In our approach we start from the results in Bokanowski, F. and Sahu (2016) and of Oberman and Salvador (2015) and we extend them introducing an adaptive choice of the parameter controlling the filter.

Adaptive Filtered Scheme

The adaptive scheme proposed here is

$$u_j^{n+1} \equiv S^{AF}(u_j^n) := S^M(u_j^n) + \phi_j^n \varepsilon^n \Delta t F\left(\frac{S^A(u_j^n) - S^M(u_j^n)}{\varepsilon^n \Delta t}\right) \tag{4}$$

starting from the initial condition u_j^0.

- $\varepsilon^n = \varepsilon^n(\Delta t, \Delta x) > 0$ is the switching parameter that will satisfy

$$\lim_{(\Delta t, \Delta x) \to 0} \varepsilon^n = 0;$$

- F is the *filter function*;
- ϕ_j is the *smoothness indicator function related to the cell* I_j.

Remark

For $\varepsilon^n \equiv \varepsilon \Delta x$, with $\varepsilon > 0$ and $\phi_j^n \equiv 1$, we get the standard Filtered Schemes of Bokanowski-F.-Sahu (2016), so we are generalizing that approach to exploit more carefully the local regularity of the solution at every time t_n and cell I_j.

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Filter function

In our approach the *filter function F* must satisfy

- $F(x) \approx x$ for $|x| \leq 1$ so that if $|S^A - S^M| \leq \Delta t \varepsilon^n$ and $\phi_j^n = 1 \Rightarrow S^{AF} \approx S^A$;
- $F(x) = 0$ for $|x| > 1$ so that if $|S^A - S^M| > \Delta t \varepsilon^n$ or $\phi_j^n = 0 \Rightarrow S^{AF} = S^M$;

$\Rightarrow$ Several choices for F, different for regularity properties:

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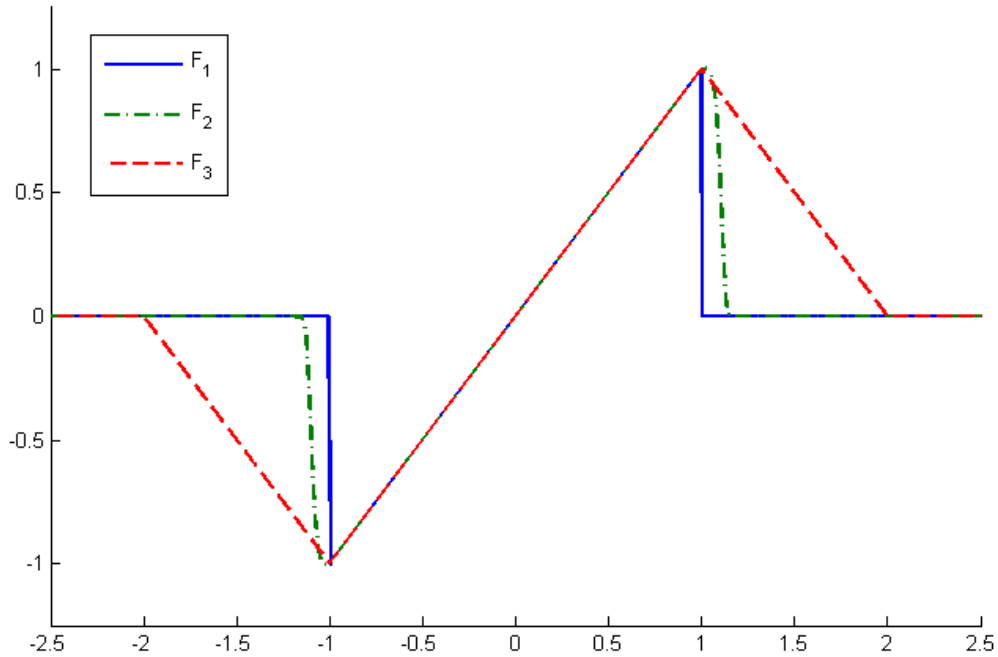

Smoothness indicator function

In order to construct a function ϕ such that

$$\phi^n(x_j) = \begin{cases} 1 & \text{if the solution } u^n \text{ is regular in } I_j, \\ 0 & \text{if } I_j \text{ contains a point of singularity,} \end{cases}$$

where $I_j = (x_{j-1}, x_{j+1})$,

we make use of the smoothness indicators for WENO schemes for (2) (Jiang-Peng (2000)):

$$\beta_k = \sum_{l=2}^{r} \int_{x_{j-1}}^{x_j} \Delta x^{2l-3} \left(P_k^{(l)}(x) \right)^2 dx, \qquad k = 0, \dots, r-1,$$

where $P_k(x)$ is the Lagrange polynomial of degree r on the stencil $\mathcal{S}_{j+k} = \{x_{j+k-r}, \dots, x_{j+k}\}$.

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Tuning of the parameter ε^n

If we want our scheme

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to switch to the high-order scheme when some regularity is detected, we have to choose ε^n such that

$$\left|\frac{S^A(v^n)_j - S^M(v^n)_j}{\varepsilon^n \Delta t}\right| = \left|\frac{h^A(\cdot) - h^M(\cdot)}{\varepsilon^n}\right| \leq 1, \qquad \text{for } (\Delta t, \Delta x) \to 0,$$

in the region of regularity at time t_n

$$\mathcal{R}^n := \{x_j : \phi^n(\omega(x_j)) = 1\}$$

Remark

We added ϕ_j^n to force the monotone scheme, if necessary. This gives a more stable scheme with the same properties of consistency and convergence.

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Tuning of the parameter ε^n

Then we can compute directly by Taylor expansions,

- for the monotone scheme

$$D^{\pm} v_j^n = v_x(t_n, x_j)) \pm \frac{\Delta x}{2} v_{xx}(t_n, x_j) + O(\Delta x^2)$$

$$h^M(D^- v_j^n, D^+ v_j^n) = H(v_x^n(x_j)) + \frac{\Delta x}{2} v_{xx}^n(x_j) \left(\partial_{p_+} h_j^M - \partial_{p_-} h_j^M \right) + O(\Delta x^2),$$

- for the high-order scheme, by the consistency property,

$$h^A(D^- v_j^n, D^+ v_j^n) = H(v_x^n(x_j)) - \frac{\Delta t}{2} H_p^2(v_x^n) v_{xx} + O(\Delta t^2) + O(\Delta x^2).$$

Therefore, we obtain the following lower bound

$$\varepsilon^n \geq \left| \frac{\Delta x}{2} v_{xx} \left(\partial_{p_+} h_j^M - \partial_{p_-} h_j^M + \lambda H_p^2(v_x) \right) + O(\Delta t^2) + O(\Delta x^2) \right|$$

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$$D^{\pm} v_j^n = v_x(t_n, x_j)) \pm \frac{\Delta x}{2} v_{xx}(t_n, x_j) + O(\Delta x^2)$$

$$h^M(D^- v_j^n, D^+ v_j^n) = H(v_x^n(x_j)) + \frac{\Delta x}{2} v_{xx}^n(x_j) \left(\partial_{p_+} h_j^M - \partial_{p_-} h_j^M \right) + O(\Delta x^2),$$

- for the high-order scheme, by the consistency property,

$$h^A(D^- v_j^n, D^+ v_j^n) = H(v_x^n(x_j)) - \frac{\Delta t}{2} H_p^2(v_x^n) v_{xx} + O(\Delta t^2) + O(\Delta x^2).$$

Therefore, we obtain the following lower bound

$$\varepsilon^n \geq \left| \frac{\Delta x}{2} v_{xx} \left(\partial_{p_+} h_j^M - \partial_{p_-} h_j^M + \lambda H_p^2(v_x) \right) + O(\Delta t^2) + O(\Delta x^2) \right|$$

Tuning of the parameter ε^n

Then we can compute directly by Taylor expansions,

- for the monotone scheme

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Therefore, we obtain the following lower bound

$$\varepsilon^n \geq \left| \frac{\Delta x}{2} v_{xx} \left(\partial_{p_+} h_j^M - \partial_{p_-} h_j^M + \lambda H_p^2(v_x) \right) + O(\Delta t^2) + O(\Delta x^2) \right|$$

Tuning of the parameter ε^n

To compute ε^n we then use the following

Formula for ε^n

$$\varepsilon^n = \max_{x_j \in \mathcal{R}^n} K \Big| H\left(D\, u_j^n\right) - H\left(D\, u_j^n - \lambda \left[H(D^+ u_j^n) - H(D^- u_j^n)\right]\right) + \left[h^M(D\, u_j^n, D^+ u_j^n) - h^M(D\, u_j^n, D^- u_j^n)\right] - \left[h^M(D^+ u_j^n, D\, u_j^n) - h^M(D^- u_j^n, D\, u_j^n)\right]\Big| . \quad (5)$$

for $K > \frac{1}{2}$, with $\lambda = \frac{\Delta t}{\Delta x}$ and $D\, u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$.

In this way the tuning is automatic.
Note that we only need to compute h^M and H to get ϵ^n.

Outline

1. Introduction
   - Monotone schemes
   - High-order schemes
2. Adaptive filtered scheme
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3. Convergence theorem
4. Numerical Experiments

Convergence theorem

Definition (ε-monotonicity)

The filtered scheme is ε-monotone i.e. for any functions u, v,

$$u \leq v \Longrightarrow S(u) \leq S(v) + C\varepsilon\Delta t,$$

where C is constant and $\varepsilon \to 0$ as $\Delta = (\Delta t, \Delta x) \to 0$.

Proposition

Let u^n be the solution obtained by the scheme (4)-(5) and assume that v_0 and H are Lipschitz continuous functions. Assume also a CFL condition for $\lambda = \Delta t/\Delta x$. Then, ε^n is well defined and u^n satisfies, for any i and j, the discrete bound

$$\frac{|u_i^n - u_j^n|}{\Delta x} \leq L$$

for some constant $L > 0$, for $0 \leq n \leq T/\Delta t$. Moreover, there exists a constant $C > 0$ such that

$$\varepsilon^n \leq C\Delta x \quad \textit{(upper bound)}$$

Convergence theorem

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Convergence theorem

Theorem

Let (M1)-(M4), (A1)-(A3) be satisfied and assume that v_0 and H are Lipschitz continuous functions. Moreover, assume that u_j^{n+1} is computed by (4)-(5), with $K > 1/2$ and that a CFL condition of the form $\lambda = \Delta t/\Delta x \leq C$ is satisfied. Denote by $v_j^n = v(t^n, x_j)$ the values of the viscosity solution on the nodes of the grid.
Then,

i) *the Adaptive Filtered scheme (4) satisfies the Crandall-Lions estimate*

$$||u^n - v^n||_\infty \leq C_1\sqrt{\Delta x}, \quad \forall\, n = 0, \ldots, N,$$

for some constant $C_1 > 0$ independent of Δx;

Convergence theorem

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Let (M1)-(M4), (A1)-(A3) be satisfied and assume that v_0 and H are Lipschitz continuous functions. Moreover, assume that u_j^{n+1} is computed by (4)-(5), with $K > 1/2$ and that a CFL condition of the form $\lambda = \Delta t/\Delta x \leq C$ is satisfied. Denote by $v_j^n = v(t^n, x_j)$ the values of the viscosity solution on the nodes of the grid.
Then,

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Convergence theorem

Theorem (Cont.)

ii) *(First order convergence for regular solutions) if moreover* $v \in C^2([0,T] \times \mathbb{R})$*, then*

$$||u^n - v^n||_\infty \leq C_2 \Delta x, \quad \forall\, n = 0, \ldots, N,$$

for some constant $C_2 > 0$ *independent of* Δx*;*

iii) *(High-order local consistency) let* $k \geq 2$ *order of the scheme* S^A*, if* $v \in C^{l+1}$ *in some neighborhood of a point* $(t,x) \in [0,T] \times \mathbb{R}$*, then for* $1 \leq l \leq k$*,*

$$\mathcal{E}_{S^{AF}}(v^n)_j = \mathcal{E}_{S^A}(v^n)_j = O(\Delta x^l) + O(\Delta t^l)$$

for $t^n - t$*,* $x_j - x$*,* Δt*,* Δx *sufficiently small.*

Convergence theorem

Theorem (Cont.)

ii) *(First order convergence for regular solutions) if moreover* $v \in C^2([0,T] \times \mathbb{R})$*, then*

$$||u^n - v^n||_\infty \leq C_2 \Delta x, \quad \forall\, n = 0, \ldots, N,$$

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Transport equation

Example 1: smooth solution.

$$\begin{cases} v_t(t,x) + v_x(t,x) = 0 & \text{in } (-2,2)\times(0,0.9) \\ v(0,x) = sin(\pi x). \end{cases}$$

Table: (Example 1.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.9$).

		F-HC ($\varepsilon = 5\Delta x$)		**AF-HC**		**AF-LW4ord**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	10	1.36e-02		1.36e-02		1.37e-04		8.02e-02	
80	20	2.56e-03	2.41	2.56e-03	2.41	8.66e-06	3.98	2.62e-02	1.62
160	40	5.76e-04	2.15	5.76e-04	2.15	5.43e-07	4.00	4.50e-03	2.54
320	80	1.40e-04	2.04	1.40e-04	2.04	3.40e-08	4.00	1.95e-04	4.52
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	10	3.58e-02		3.58e-02		3.62e-04		2.07e-01	
80	20	6.66e-03	2.43	6.66e-03	2.43	2.25e-05	4.01	4.14e-02	2.32
160	40	1.48e-03	2.17	1.48e-03	2.17	1.40e-06	4.01	5.09e-03	3.02
320	80	3.57e-04	2.05	3.57e-04	2.05	8.69e-08	4.01	3.08e-04	4.05

Transport equation

Example 2: non smooth solution.

$$\begin{cases} v_t + v_x = 0 & \text{in } (-\frac{3}{2}, \frac{7}{2}) \times (0,2) \\ v(0,x) = v_0(x), \end{cases} \qquad v_0(x) = \begin{cases} \min\{(1-x)^2, (1+x)^2\} & x \in (-1,1), \\ \sin^2(\pi(x-2)) & x \in (2,3) \\ 0 & \text{otherwise}, \end{cases}$$

Table: (Example 2.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.4$).

		F-HC ($\varepsilon = 10\Delta x$)		**AF-HC**		**AF-LW4ord**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
50	50	3.46e-01		2.98e-01		2.65e-01		3.47e-01	
100	100	1.41e-01	1.29	1.78e-01	0.75	1.56e-01	0.77	2.07e-01	0.75
200	200	9.69e-02	0.54	1.12e-01	0.66	9.08e-02	0.78	1.28e-01	0.70
400	400	7.29e-02	0.41	7.05e-02	0.67	5.06e-02	0.84	7.66e-02	0.74
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
50	50	4.34e-01		2.94e-01		2.21e-01		3.62e-01	
100	100	1.41e-01	1.63	9.77e-02	1.59	4.26e-02	2.38	1.39e-01	1.39
200	200	4.24e-02	1.73	3.06e-02	1.67	9.22e-03	2.21	3.83e-02	1.86
400	400	1.38e-02	1.62	1.01e-02	1.60	2.61e-03	1.82	8.39e-03	2.19

Transport equation

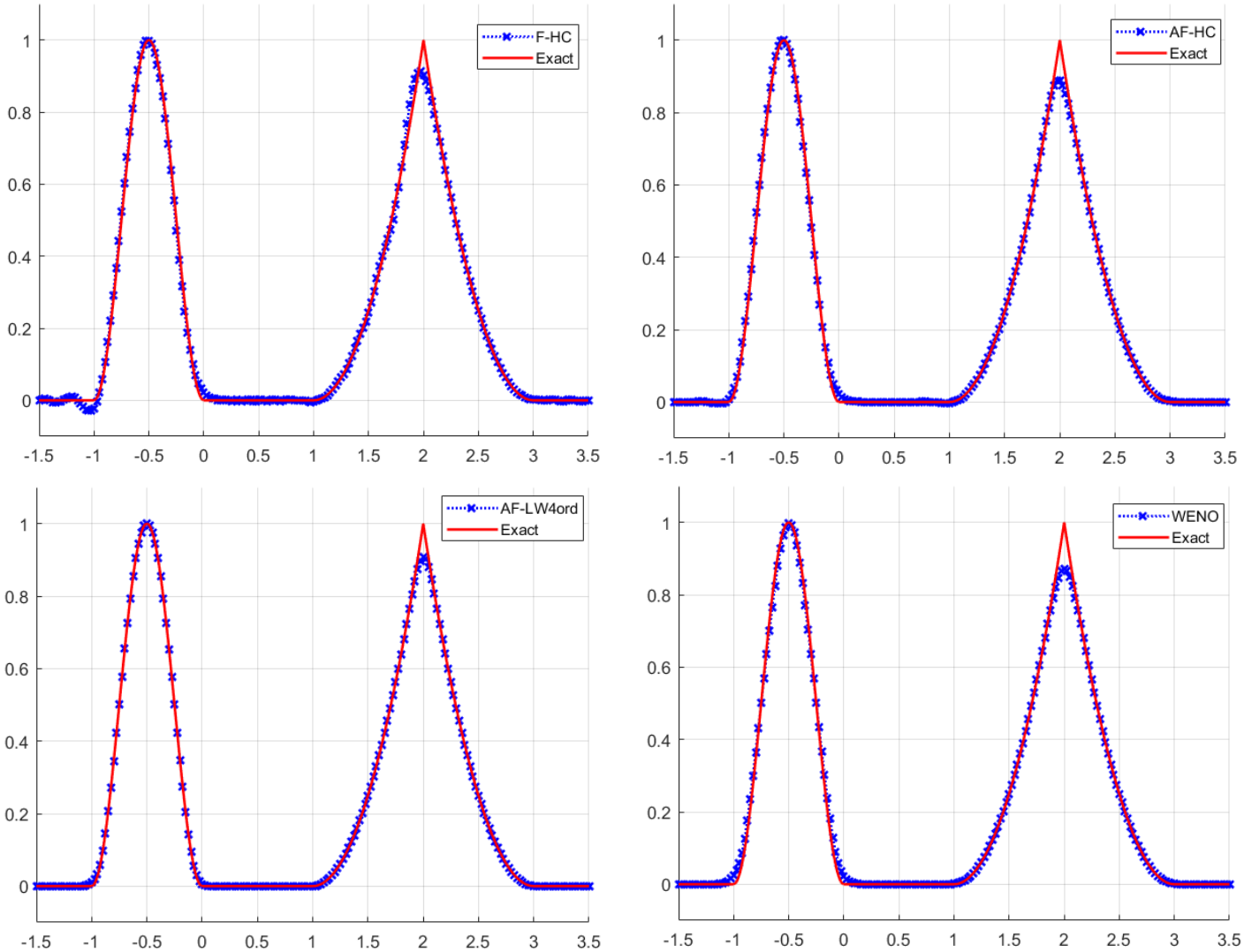


Figure: (Example 2.) Plots of the solution at time $T = 2$ with $\Delta x = 0.025$. Top: F-HC (left), AF-HC (right). Bottom: AF-LW4ord (left), WENO 2/3 (right).

Eikonal equation

Example 3.

$$\begin{cases} v_t(t,x) + |v_x(t,x)| = 0 & \text{in } (-2,2)\times(0,0.3) \\ v_0(x) = -\max\{1-x^2,0\}^4. \end{cases}$$

Table: (Example 3.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.375$).

		F-LWR ($\varepsilon = 5\Delta x$)		**AF-LWR**		**AF-LW4ord**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	8	1.91e-02		1.40e-02		1.63e-02		2.33e-02	
80	16	9.24e-03	1.04	3.37e-03	2.06	7.51e-03	1.11	1.02e-02	1.19
160	32	5.77e-03	0.68	1.58e-03	1.09	2.14e-03	1.81	4.10e-03	1.32
320	64	3.46e-03	0.74	7.09e-04	1.16	6.92e-04	1.63	1.22e-03	1.75
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	8	2.38e-02		2.01e-02		1.29e-02		2.96e-02	
80	16	8.48e-03	1.49	5.70e-03	1.82	2.05e-03	2.65	7.04e-03	2.07
160	32	3.41e-03	1.32	1.82e-03	1.65	3.20e-04	2.68	1.43e-03	2.30
320	64	1.52e-03	1.17	5.84e-04	1.64	6.38e-05	2.33	2.82e-04	2.34

Eikonal equation

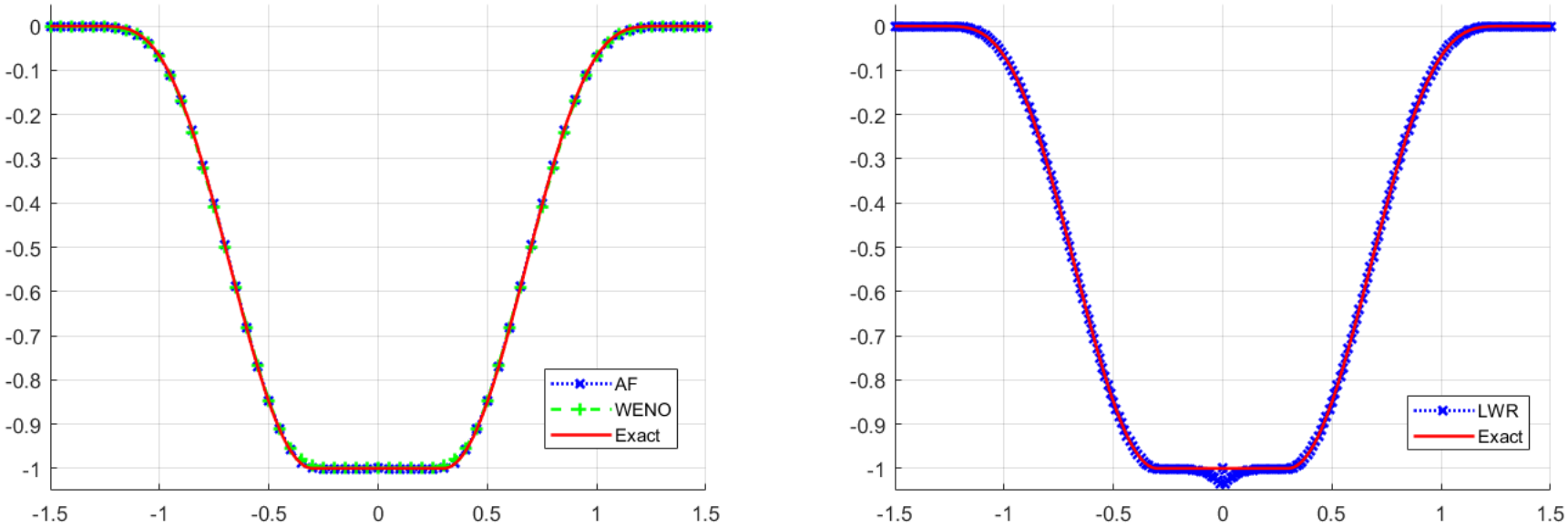


Figure: (Example 3.) Plots at time $T = 0.3$ with the AF-LWR and WENO schemes for $\Delta x = 0.05$ (left) and LWR scheme for $\Delta x = 0.0125$ (right).

Burgers' equation

Example 4.

$$\begin{cases} v_t(t,x) + \frac{1}{2}(v_x(t,x)+1)^2 = 0 & \text{in } (0,T)\times(0,2), \\ v_0(x) = -\cos(\pi x), \end{cases}$$

for $T = 5/(4\pi^2)$ (before singularity) and $T = 3/(2\pi^2)$ (after singularity)

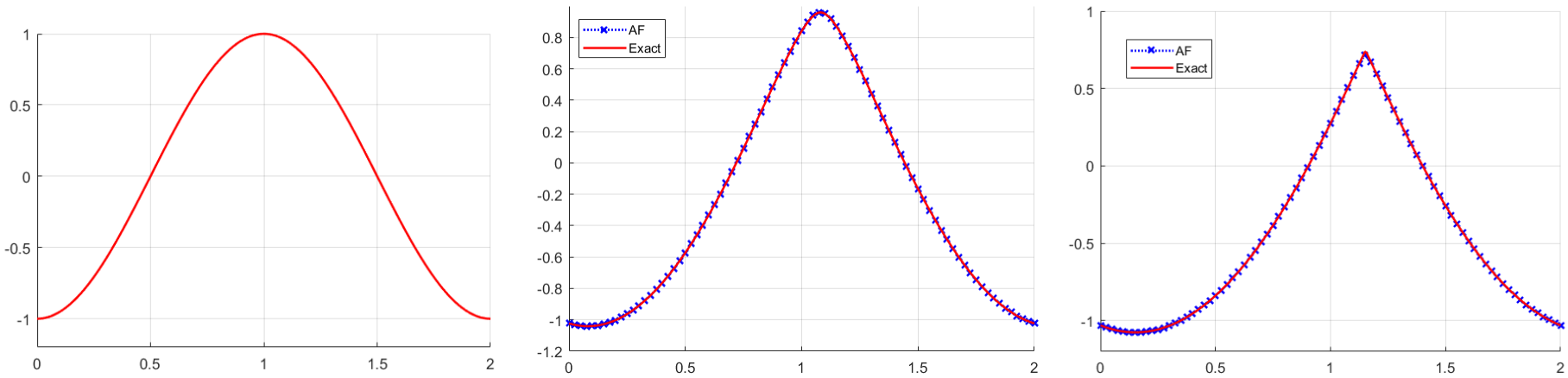


Figure: (Example 4.) From left to right: initial data and plots of the solution with the AF-LWR scheme at time $T = 4/(5\pi^2)$ and $T = 3/(2\pi^2)$ for $\Delta x = 0.025$.

Burgers' equation

Table: (Example 4.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.2$).

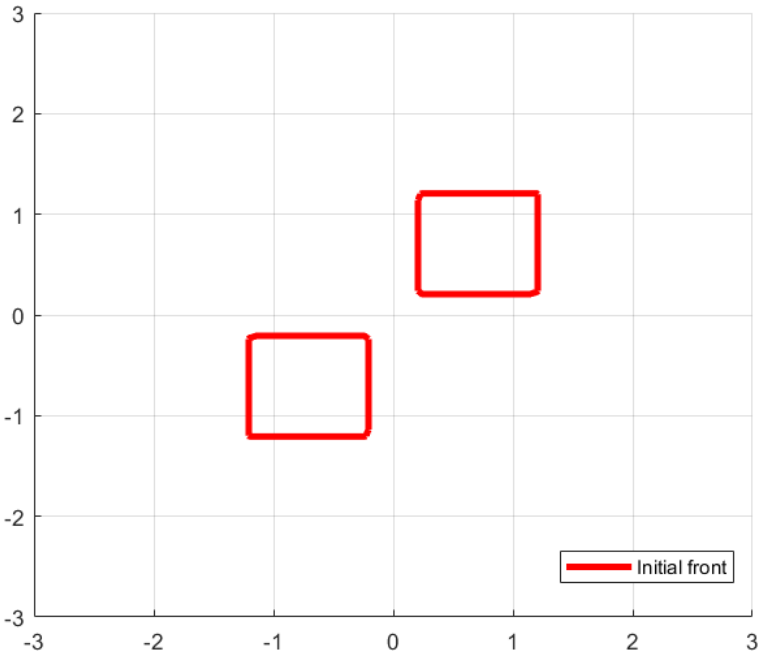
$T = 4/(5\pi^2)$		**F-LWR** ($\varepsilon = 10\Delta x$)		**AF-LWR**		**AF-LW4ord**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	8	1.30e-02		9.61e-03		1.89e-03		1.04e-02	
80	16	8.67e-03	0.59	2.77e-03	1.79	2.84e-04	2.73	2.12e-03	2.30
160	32	5.07e-03	0.77	7.24e-04	1.94	2.68e-05	3.41	1.82e-04	3.54
320	64	2.66e-03	0.93	1.83e-04	1.99	1.89e-06	3.83	2.05e-05	3.15
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	8	3.76e-03		3.30e-03		2.76e-04		3.67e-03	
80	16	1.29e-03	1.54	8.20e-04	2.01	1.97e-05	3.81	6.57e-04	2.48
160	32	4.49e-04	1.52	2.04e-04	2.01	1.50e-06	3.71	5.43e-05	3.60
320	64	1.82e-04	1.30	5.09e-05	2.00	1.04e-07	3.86	2.98e-06	4.19
$T = 3/(2\pi^2)$									
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	16	4.88e-02		5.30e-02		6.31e-02		3.89e-02	
80	32	2.47e-02	0.98	2.47e-02	1.10	2.87e-02	1.13	1.61e-02	1.27
160	64	9.81e-03	1.33	9.95e-03	1.31	1.03e-02	1.48	5.12e-03	1.65
320	128	2.57e-03	1.93	2.59e-03	1.94	2.69e-03	1.94	8.40e-04	2.61
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	16	5.17e-03		5.28e-03		3.83e-03		3.69e-03	
80	32	1.26e-03	2.03	1.27e-03	2.06	8.89e-04	2.11	6.94e-04	2.41
160	64	2.86e-04	2.14	2.87e-04	2.14	1.43e-04	2.64	8.67e-05	3.00
320	128	5.68e-05	2.33	5.68e-05	2.34	1.82e-05	2.97	6.40e-06	3.76

Front propagation

Example 6.

$$\begin{cases} v_t + \sqrt{v_x^2 + v_y^2} = 0 & \text{in } (0,T)\times\Omega, \\ v(0,x,y) = v_0(x,y), \end{cases} \qquad \begin{aligned} v_0(x,y) &= \min\{f_-, f_+, 0.2\}, \\ f_\pm &= \max\left\{\left|x \pm \tfrac{\sqrt{2}}{2}\right|, \left|y \pm \tfrac{\sqrt{2}}{2}\right|\right\}, \end{aligned}$$

where $\Omega = [-3,3]^2$ and $T = 0.6$.



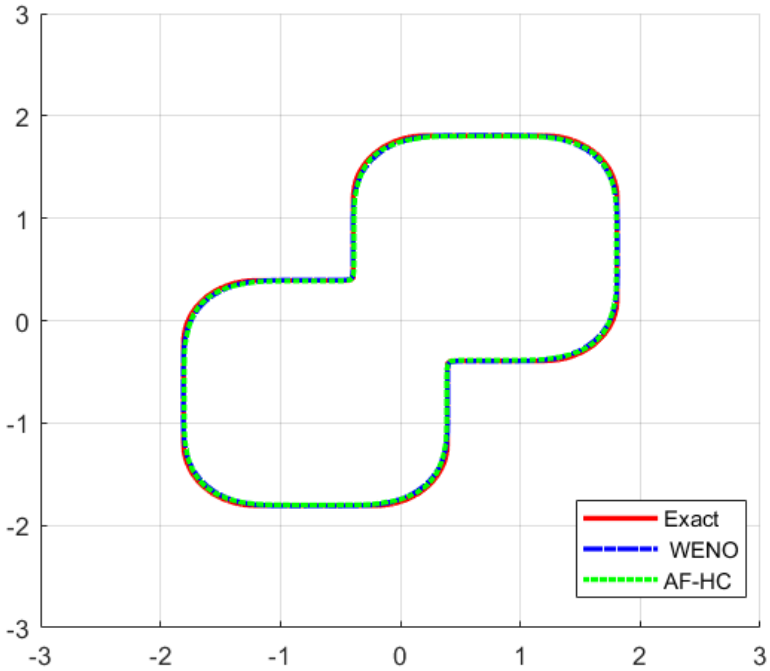


Figure: (Example 6.) Initial front (left) and fronts at $T = 0.6$ using WENO and AF-HC scheme (right).

Front propagation

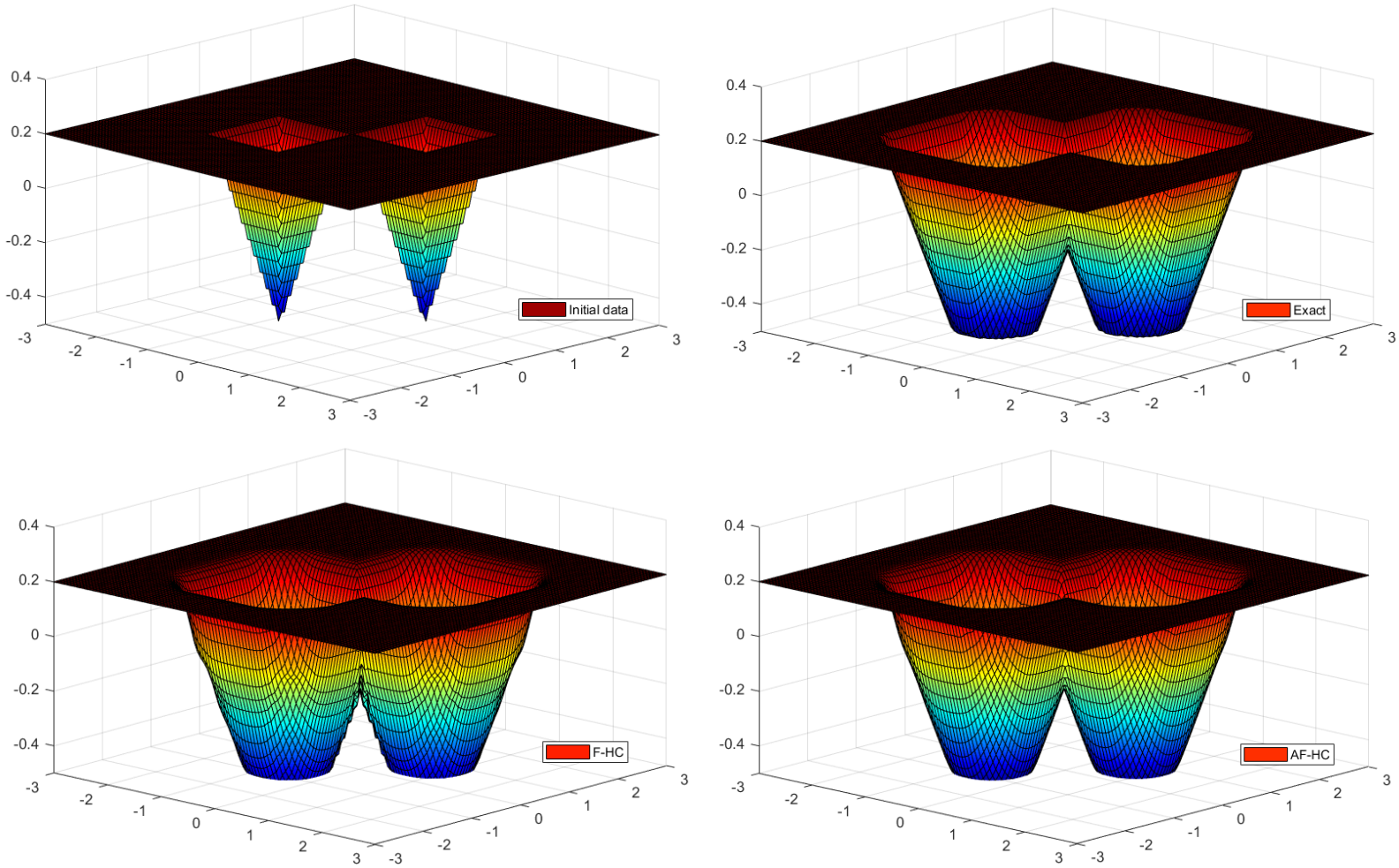


Figure: (Example 6.) Top: initial data (left) and exact solution at $T = 0.6$ (right). Bottom: plots of the solution at $T = 0.6$ computed by the F-HC scheme (left) and AF-HC scheme (right).

Front propagation

Table: (Example 6.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.25$).

		HC		**F-HC** ($\varepsilon = 20\Delta x$)		**AF-HC**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
25	15	1.56e-01		1.56e-01		1.17e-01		1.14e-01	
50	30	4.17e-01	−1.41	9.82e-02	0.67	7.05e-02	0.73	6.68e-02	0.78
100	60	1.87e+00	−2.16	4.90e-02	1.00	3.89e-02	0.86	3.95e-02	0.76
200	120	3.71e+01	−4.31	4.04e-02	0.28	2.39e-02	0.71	2.39e-02	0.72
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
25	15	7.78e-01		7.78e-01		7.93e-01		7.82e-01	
50	30	3.80e-01	1.03	3.54e-01	1.14	3.47e-01	1.19	3.52e-01	1.15
100	60	3.37e-01	0.18	1.92e-01	0.88	1.82e-01	0.93	1.54e-01	1.19
200	120	2.00e+00	−2.57	1.15e-01	0.74	9.71e-02	0.91	9.05e-02	0.77

Burgers' equation in 2D

Example 7.

$$\begin{cases} v_t + \frac{1}{2}\left(v_x^2 + v_y^2\right) = 0 & \text{in } (0,T)\times\Omega, \\ v(0,x,y) = \max\{0, 1-\left(x^2+y^2\right)\}, \end{cases}$$

where $\Omega = [-2,2]^2$ and $T = 0.5$.

Table: (Example 7.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.25$).

		Monotone		**F-HC ($\varepsilon = 10\Delta x$)**		**AF-HC**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	20	1.57e-01		6.72e-02		6.24e-02		8.59e-02	
80	40	1.01e-01	0.63	5.23e-02	0.36	4.17e-02	0.58	4.75e-02	0.86
160	80	6.26e-02	0.69	3.61e-02	0.54	2.26e-02	0.89	2.43e-02	0.96
320	160	3.76e-02	0.74	2.33e-02	0.63	1.19e-02	0.92	1.19e-02	1.03
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	20	2.00e-01		1.03e-01		1.30e-01		1.10e-01	
80	40	1.17e-01	0.77	5.55e-02	0.89	6.88e-02	0.92	5.14e-02	1.11
160	80	6.77e-02	0.79	3.16e-02	0.81	3.32e-02	1.05	2.46e-02	1.06
320	160	3.87e-02	0.81	1.96e-02	0.69	1.64e-02	1.01	1.21e-02	1.02

Burgers' equation in 2D

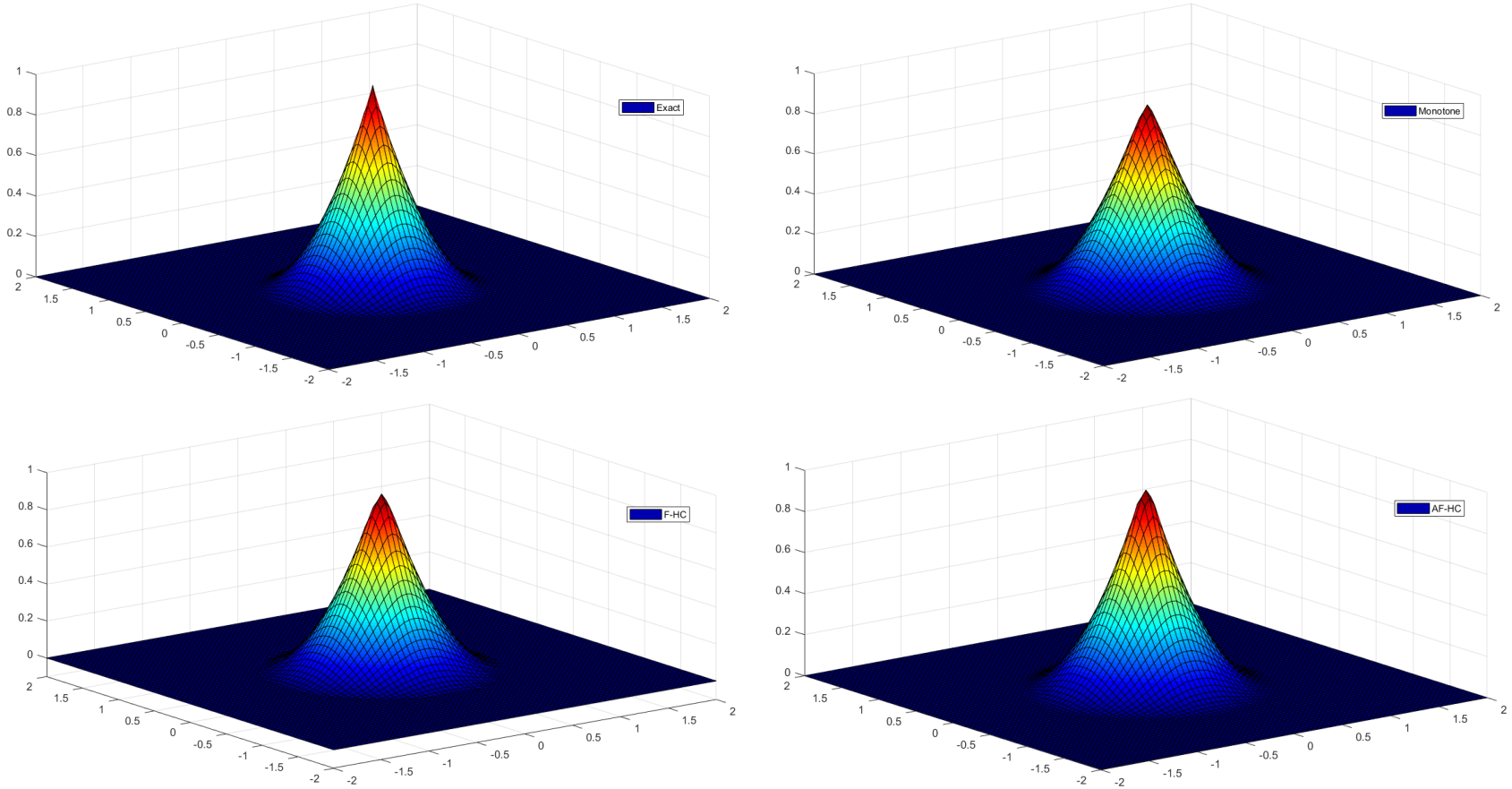


Figure: (Example 7.) Top: exact (left) and approximate solution by the monotone scheme (right) at $T = 0.5$. Bottom: plots of the solution at $T = 0.5$ computed by the F-HC scheme (left) and AF-HC scheme (right).

Burgers' equation in 2D

Example 8.

$$\begin{cases} v_t + \frac{1}{2}\left(v_x^2 + v_y^2\right) = 0 & \text{in } (0,T)\times\Omega, \\ v(0,x,y) = \min\{0, 1-\left(x^2+y^2\right)\}, \end{cases}$$

where $\Omega = [-2,2]^2$ and $T = 0.5$.

Table: (Example 8.) Errors and orders in L^∞ and L^1 norms ($\lambda = 0.25$).

		Monotone		**F-HC ($\varepsilon = 10\Delta x$)**		**AF-HC**		**WENO 2/3**	
N_x	N_t	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord	L^∞ Err	Ord
40	20	5.23e-02		5.28e-02		4.52e-02		2.09e-02	
80	40	2.62e-02	1.00	1.52e-02	1.80	1.97e-02	1.20	1.20e-02	0.80
160	80	1.31e-02	1.01	7.39e-03	1.04	1.06e-02	0.89	6.50e-03	0.88
320	160	6.50e-03	1.01	4.95e-03	0.58	5.47e-03	0.96	3.36e-03	0.95
N_x	N_t	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord	L^1 Err	Ord
40	20	2.24e-01		9.21e-02		4.31e-02		2.11e-02	
80	40	1.08e-01	1.05	2.43e-02	1.92	8.59e-03	2.33	3.79e-03	2.48
160	80	5.29e-02	1.03	6.98e-03	1.80	2.44e-03	1.81	7.62e-04	2.31
320	160	2.62e-02	1.01	2.05e-03	1.77	6.61e-04	1.89	1.95e-04	1.96

Burgers' equation in 2D

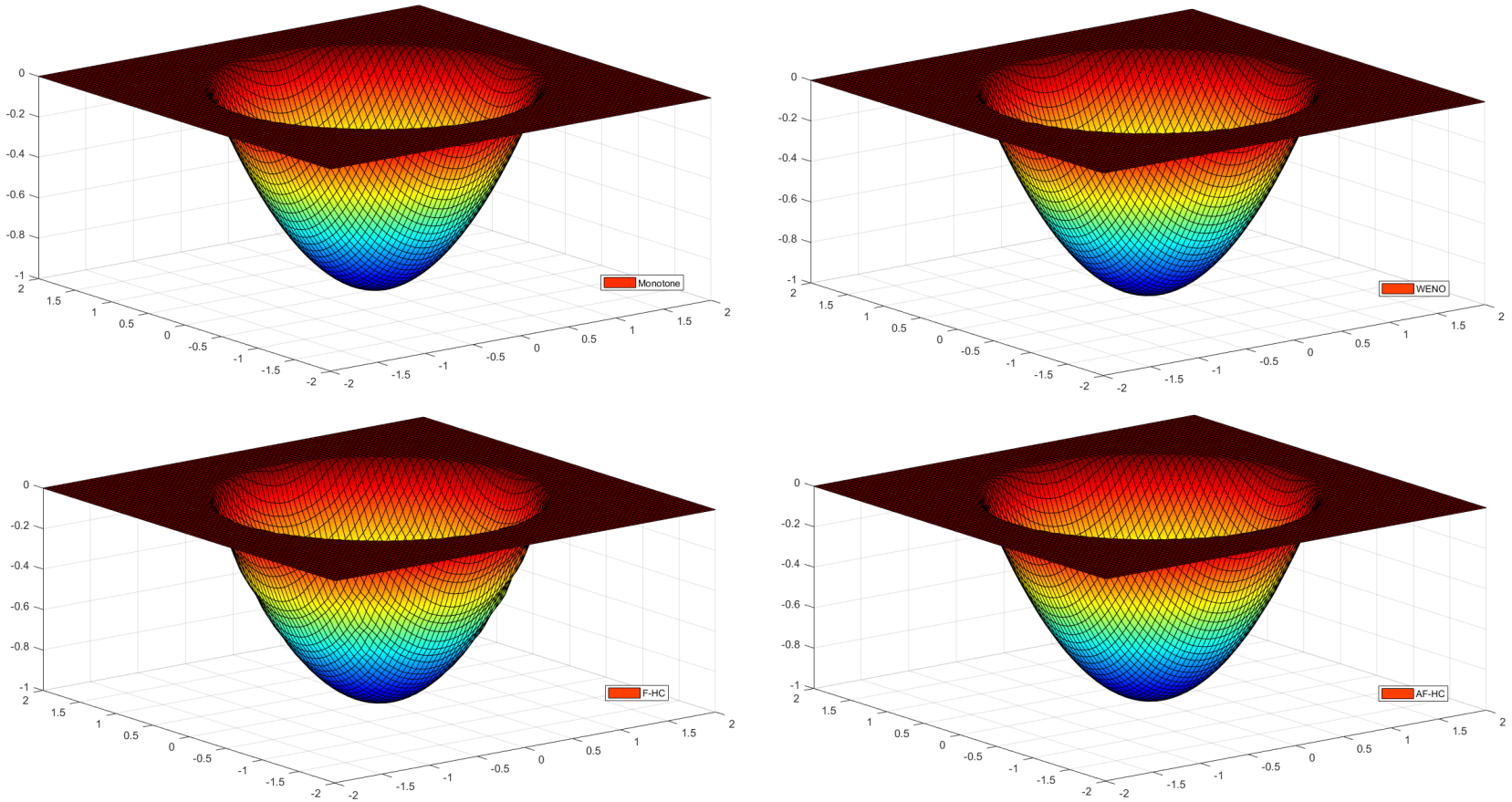


Figure: (Example 8.) Approximate solutions at $T = 0.5$. Top: Monotone scheme (left) and WENO 2/3 scheme (right). Bottom: F-HC (left) and AF-HC scheme (right).

Conclusion

- We presented a rather simple way to construct convergent schemes, which are of high-order in regions of regularity.
- Our procedure is able to stabilize an otherwise unstable (high-order) scheme, still preserving its accuracy.
- The adaptive and automatic choice of the parameter ε^n improves the scheme in $[1]$, although it is clearly more expensive. Our scheme reduces the oscillations allowed by the constant choice of ε and gives more stable results (without limiters).
- We noticed also that our scheme, with a wise choice for the high-order scheme, compares to the WENO scheme in terms of errors and seems to have a better resolution of the kinks.
- We presented a formula to compute schemes of order $r > 2$ in Lax-Wendroff form with very compact stencils. The schemes are effective in regular regions and convergent by the filtering process.

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