Pattern recognition in biomedical imaging: the Hough transform

Anna Maria Massone
Dipartimento di Matematica
Università degli Studi di Genova
massone@dima.unige.it

Some preliminaries

Pixel – grey levels

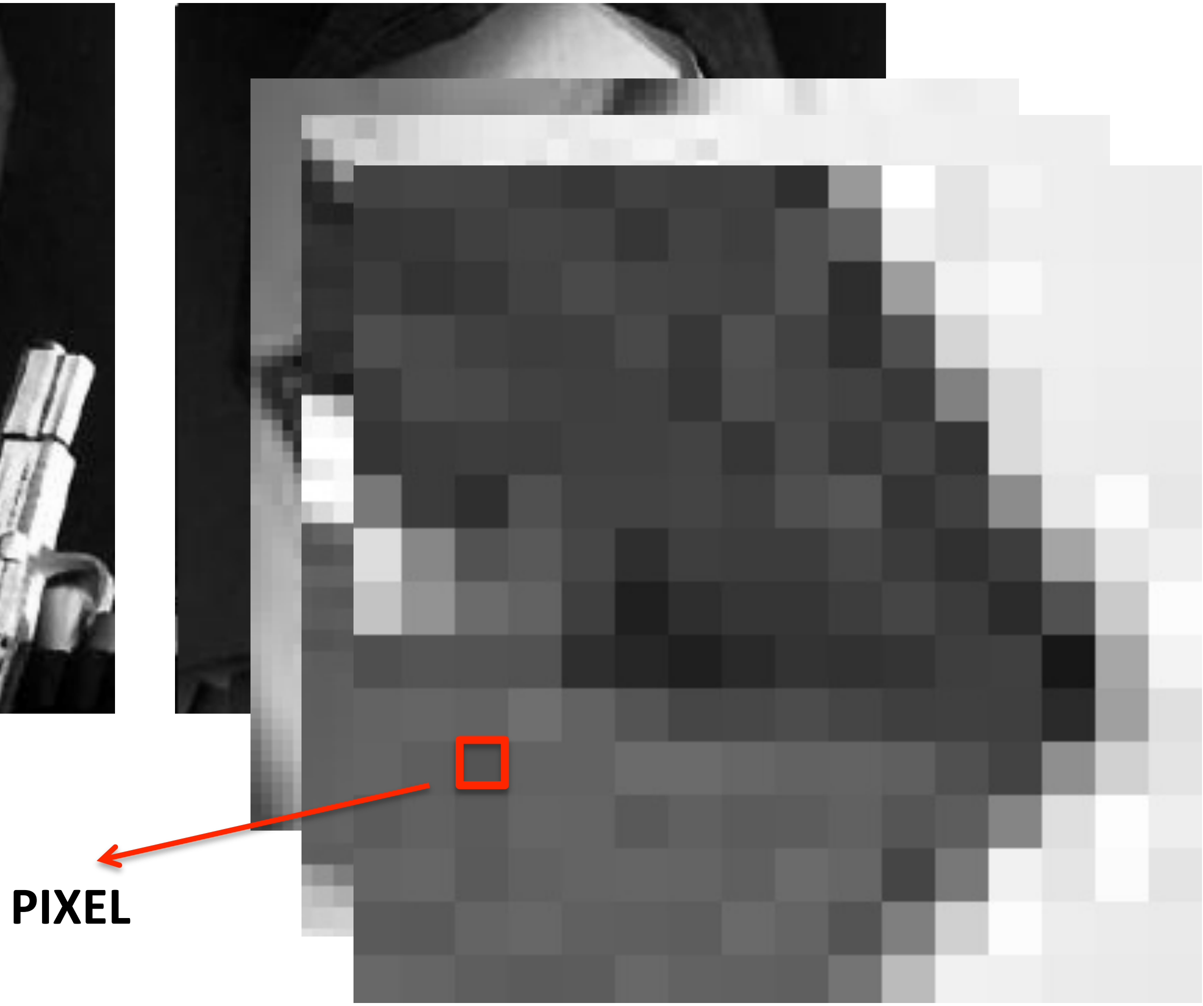
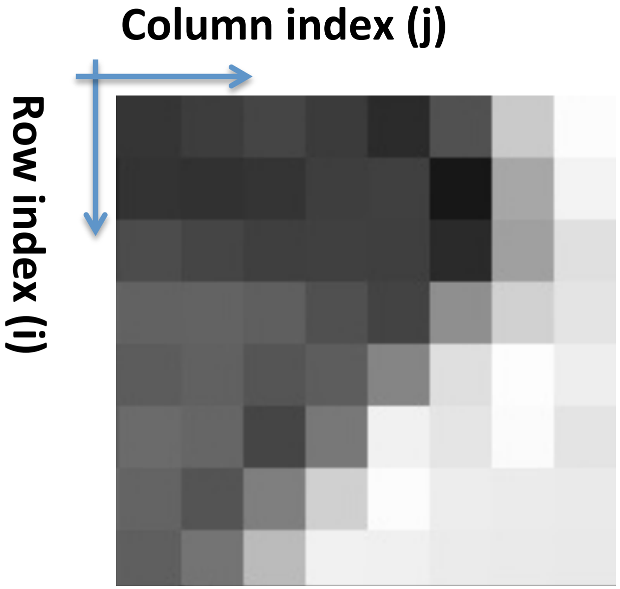
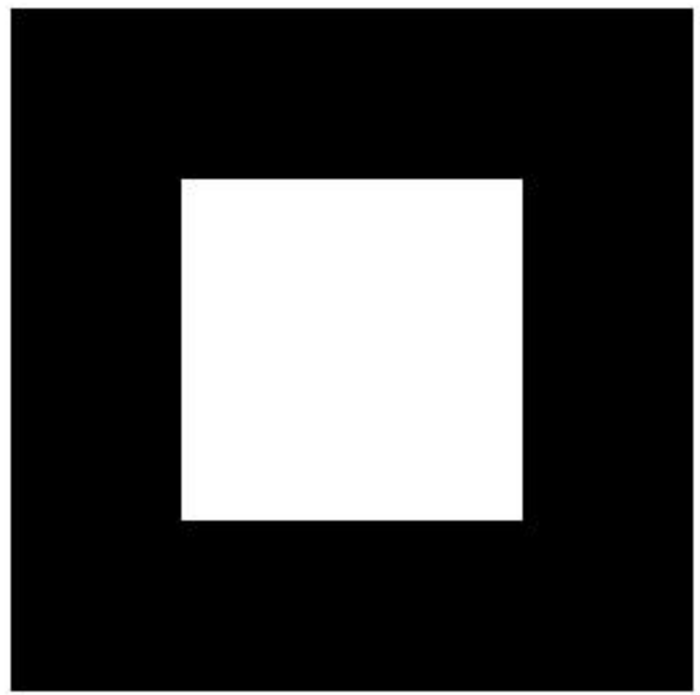


Image – Matrix

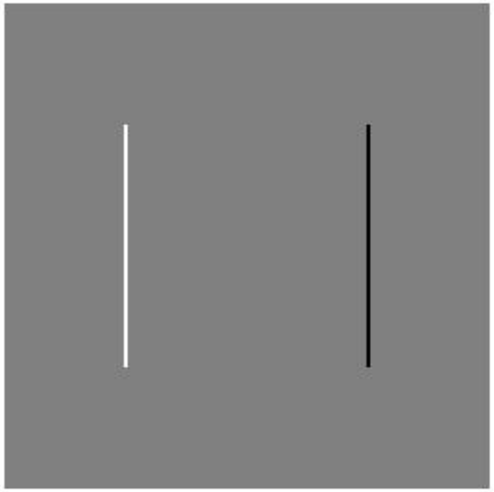


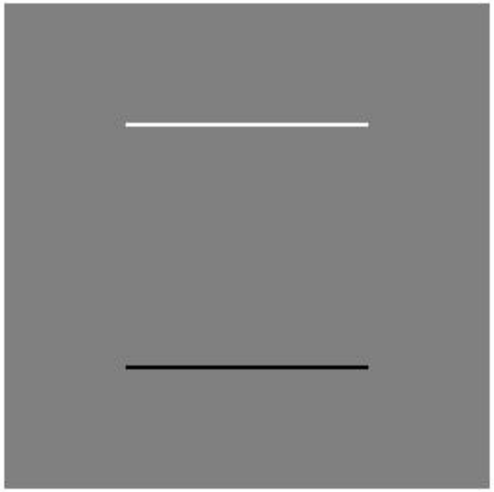
10	12	20	12	5	12	120	255
					0		
							150
						180	180
				200	180	180	180

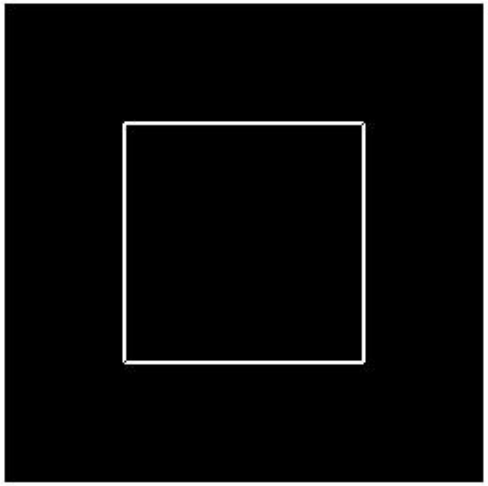
Edge detection



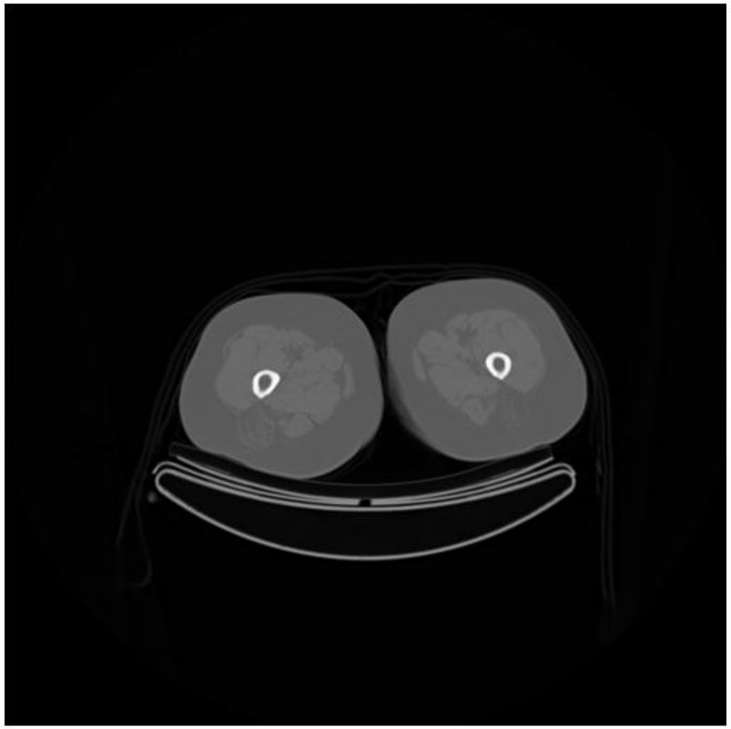
Edge detection

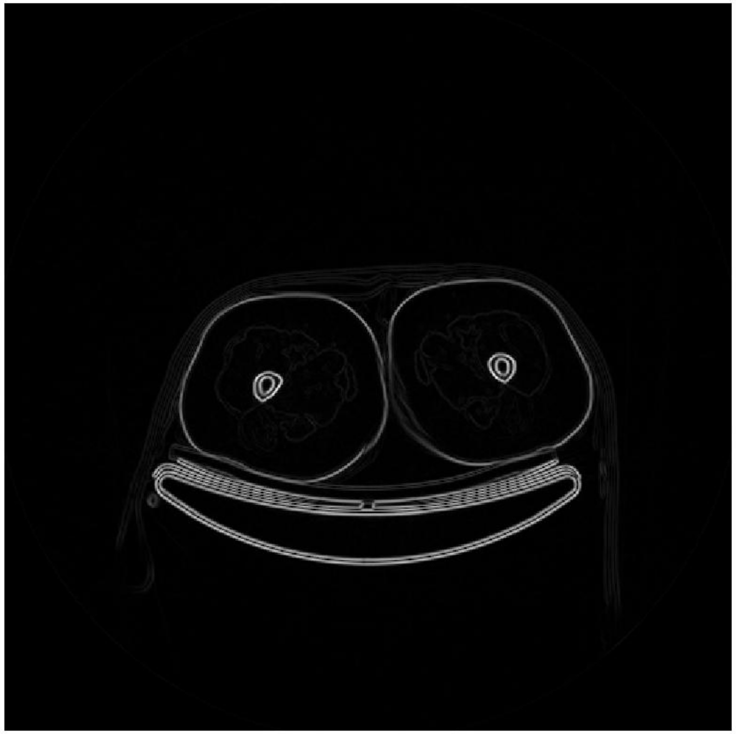






Edge detection





The Hough transform

Hough transform

The **Hough transform** (1962) is a standard pattern recognition technique for the detection of straight lines, circles, and ellipses in images.

United States Patent Office

3,069,654

Patented Dec. 18, 1962

1

3,069,654
METHOD AND MEANS FOR RECOGNIZING COMPLEX PATTERNS

Paul V. C. Hough, Ann Arbor, Mich., assignor to the United States of America as represented by the United States Atomic Energy Commission

Filed Mar. 25, 1960, Ser. No. 17,715
6 Claims. (Cl. 340—146.3)

This invention relates to the recognition of complex patterns and more specifically to a method and means for machine recognition of complex lines in photographs or

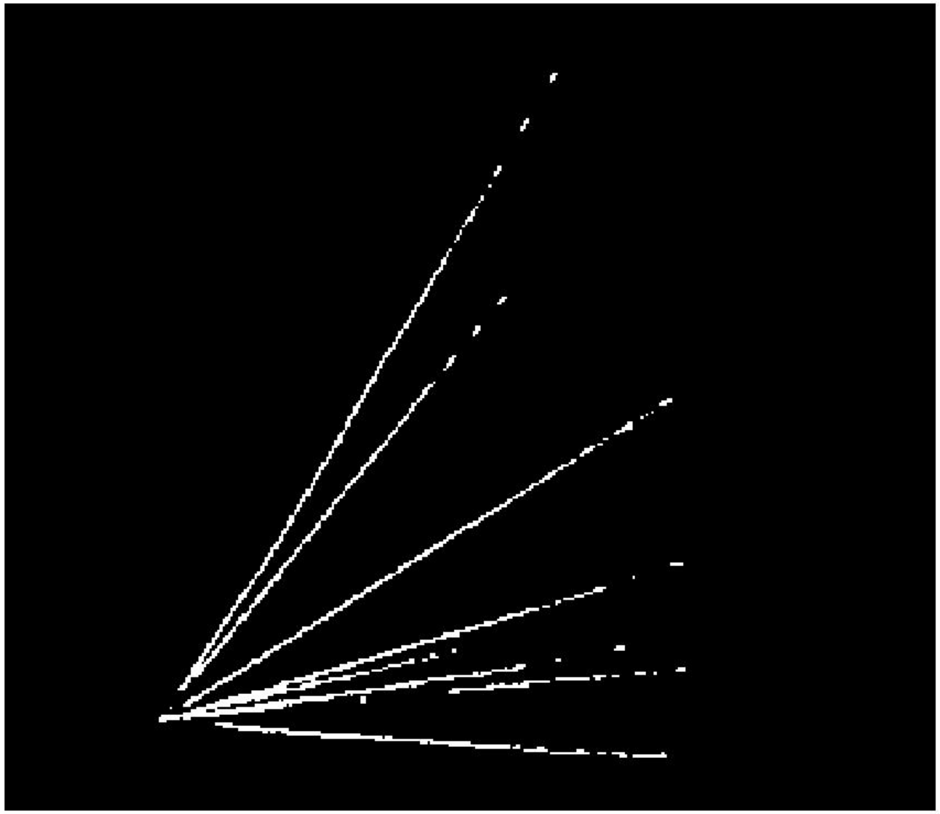
2

of the point on the line segment from the horizontal midline 109 of the framelet 108.

(3) Each line in the transformed plane is made to have an intercept with the horizontal midline 101 of the picture 100 equal to the horizontal coordinate of its respective point on the line segment in framelet 108.

Thus, for a given reference point 110 on line segment 102 a line 110A is drawn in the plane transform 102A. The reference point 110 is approximately midway between the top and the horizontal midline 109 of framelet 108 and hence the line 110A is inclined to the right at an angle to the vertical whose tangent is approximately 1/

Detection of straight lines in images

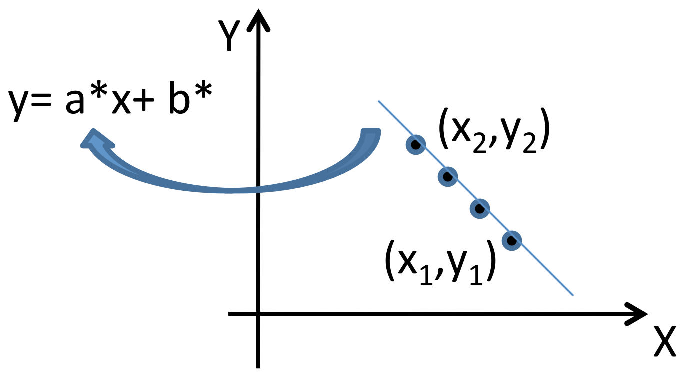


Detection of straight lines

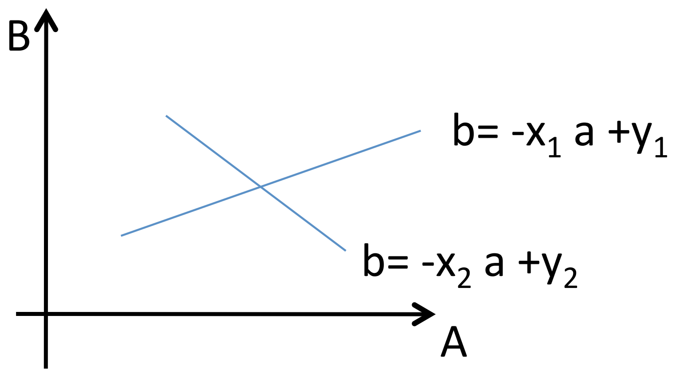
The equation of a straight line is usually written this way:

$$y = ax + b$$

Image space



Parameter space



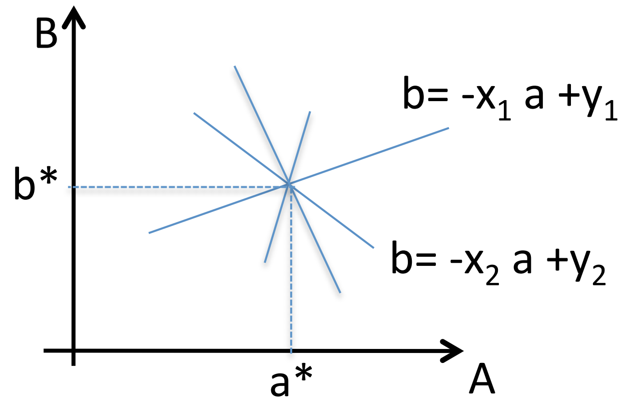
(x_1,y_1) belongs to the line: $y_1 = ax_1 + b$

➔ In the parameter space $b = -x_1 a + y_1$

(x_2,y_2) belongs to the line: $y_2 = ax_2 + b$

➔ In the parameter space $b = -x_2 a + y_2$

Lines and points duality



A point on a straight line in the image space is projected into a straight line in the parameter space, and the whole straight line in the image space is projected into a single point in the parameter space:

this is (a*; b*), i.e., the intersection point of all projected straight lines.

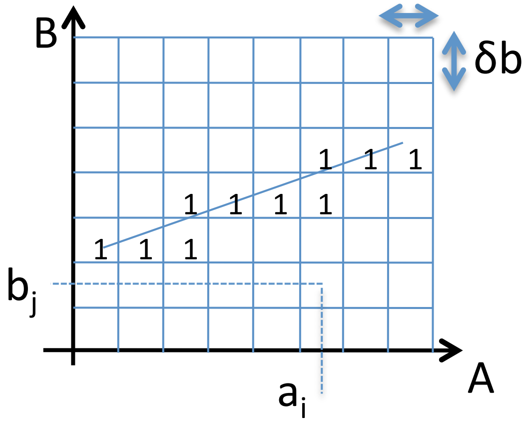
The straight lines in the parameter space are the Hough transforms of the points in the image space

The recognition algorithm

1. Discretization of the parameter space in NxM *cells:*

   $a_i = a_0 + i\ \delta a$ where $i=1,..,N$

   $b_j = b_0 + j\ \delta b$ where $j=1,..,M$



2. Define an **accumulator matrix** $H=H_{MxN}=(H_{ji})$ such that

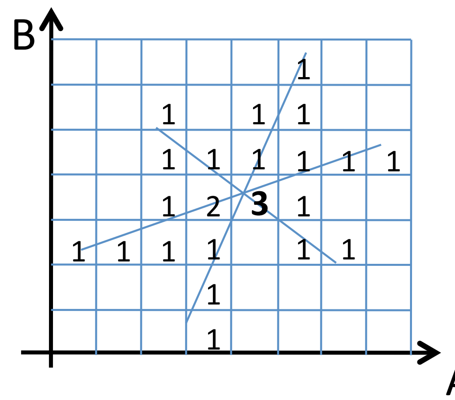
$$H_{ji} = 0 \qquad \forall i, j$$

3. Project point (x_P,y_P) into the straight line $b= -x_P\ a + y_P$ and look for the cells crossed by the line

4. Update the accumulator matrix entries corresponding to the cells crossed by the line

$$H_{ji} = \begin{cases} H_{ji} + 1 & \text{If cell } ji \text{ is crossed} \\ H_{ji} & \text{otherwise} \end{cases}$$

The recognition algorithm



5. Repeat for each point in the image:

6. Local maxima in the accumulator matrix identify the intersection points of the projected straight lines

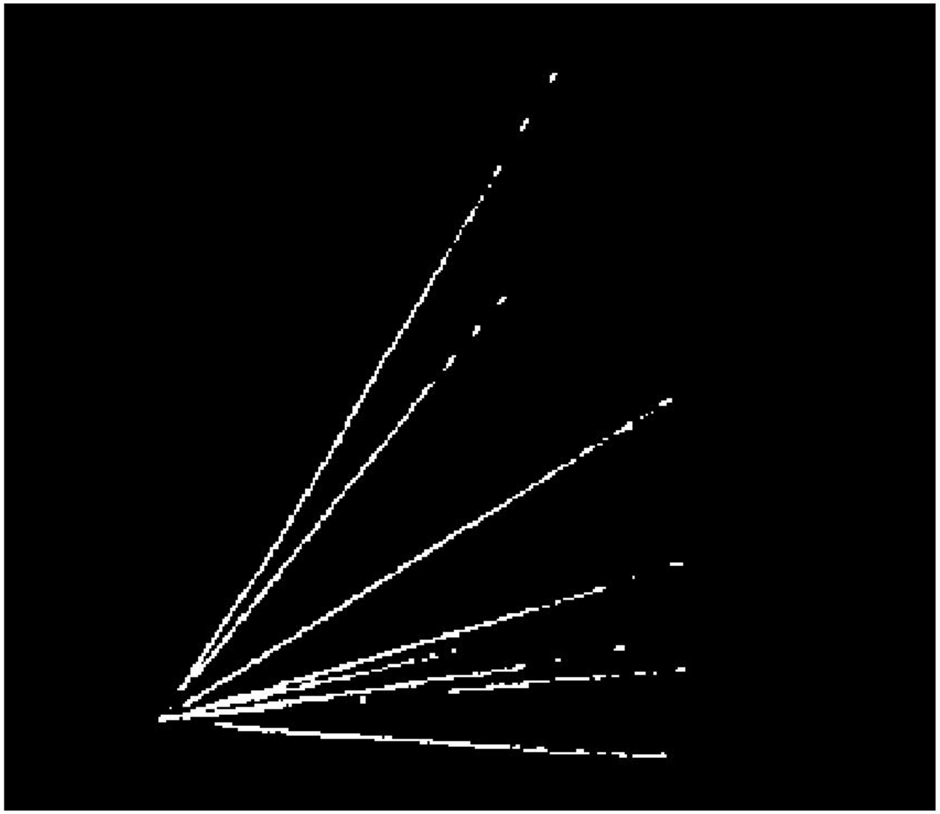
7. Local maxima in the accumulator matrix identify collinear points in the input image

8. If *j*i** is the pair of indices corresponding to the global maximum in the accumulator matrix, then

$$y = a_{i*}\, x + b_{j*}$$

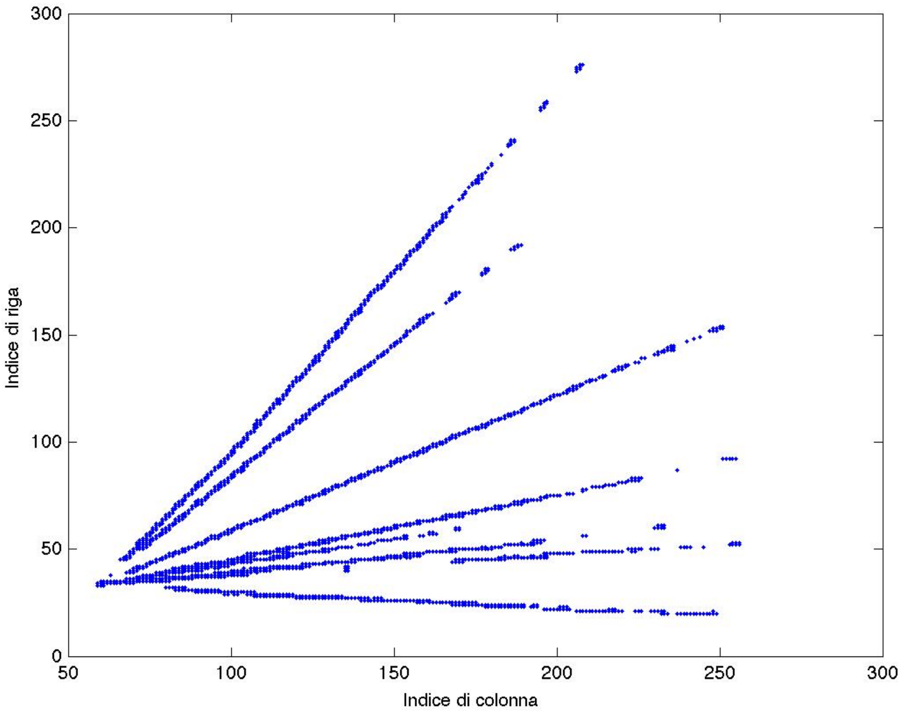
is the straight line in the image space we are looking for

Example

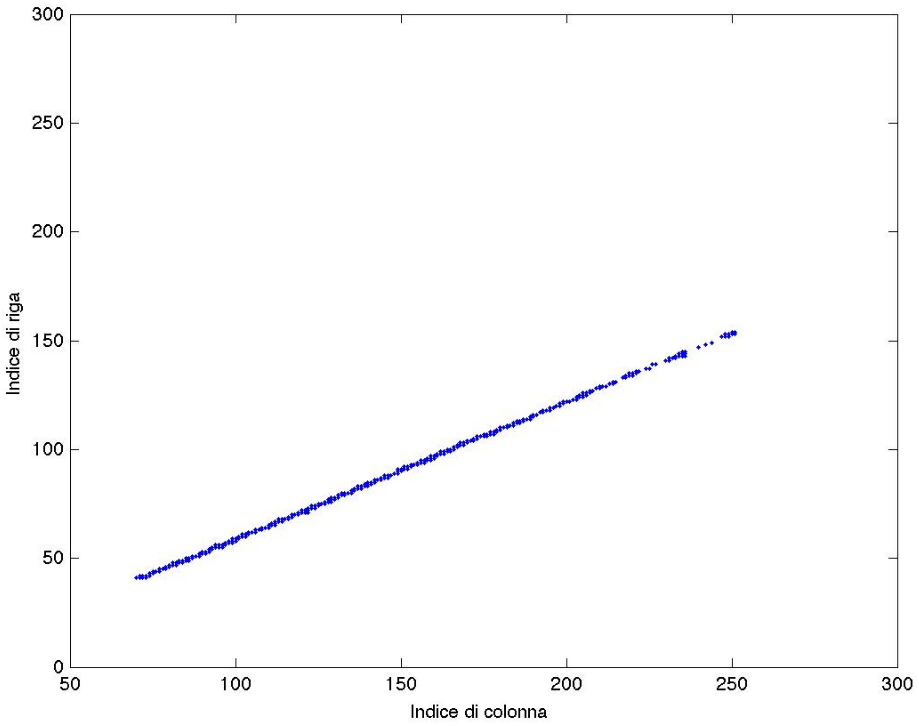


a) Input image
(303x350 = 106050 pixels)

Selected points from the image

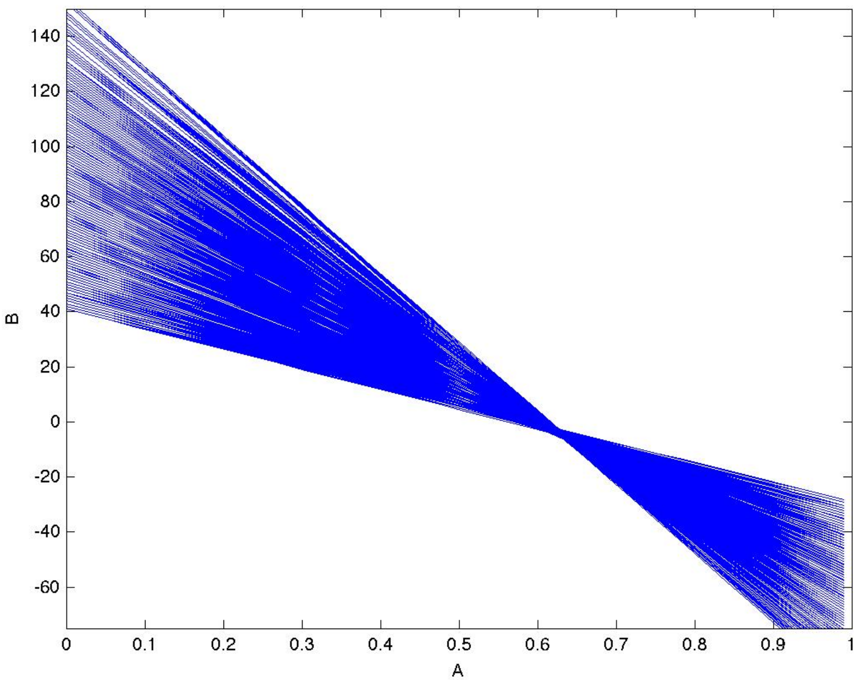


b) Extraction of points to be transformed (1882)

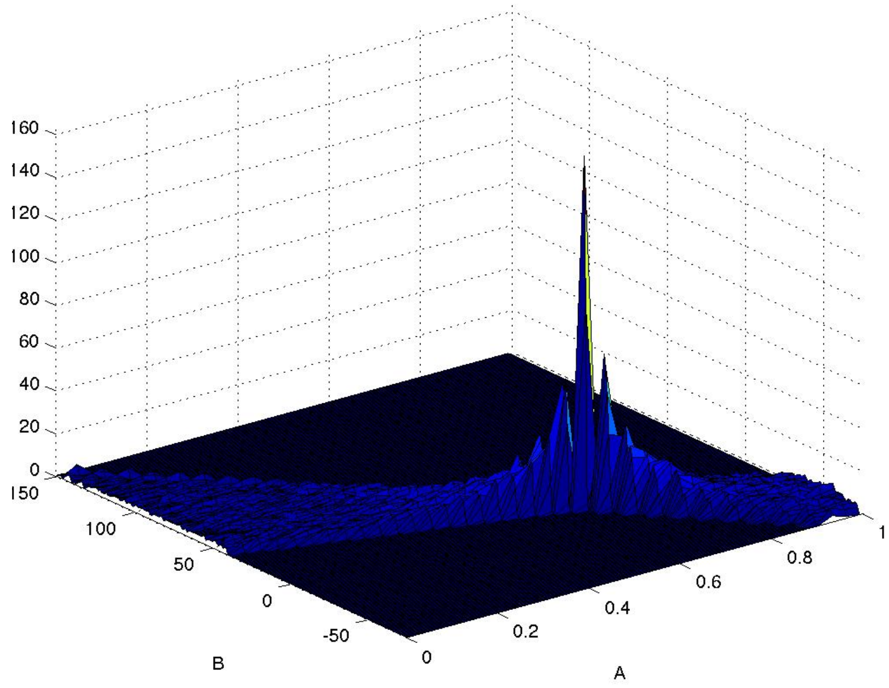


c) Points corresponding to a single straight line (306)

Parameter space

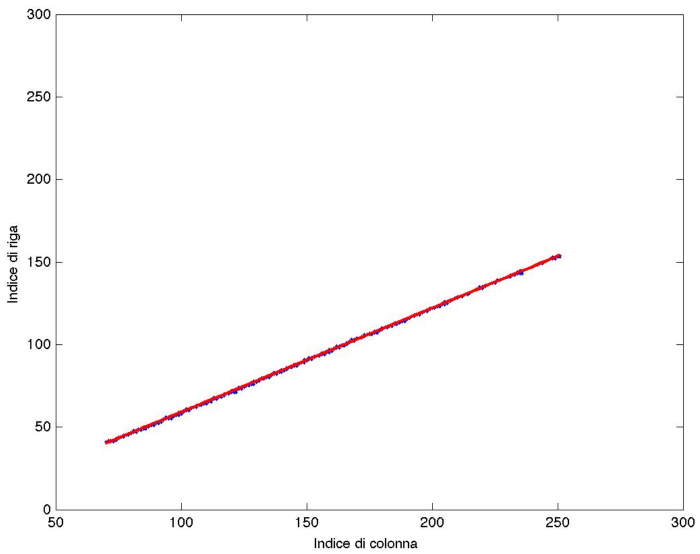


d) Hough transforms of the points in d)

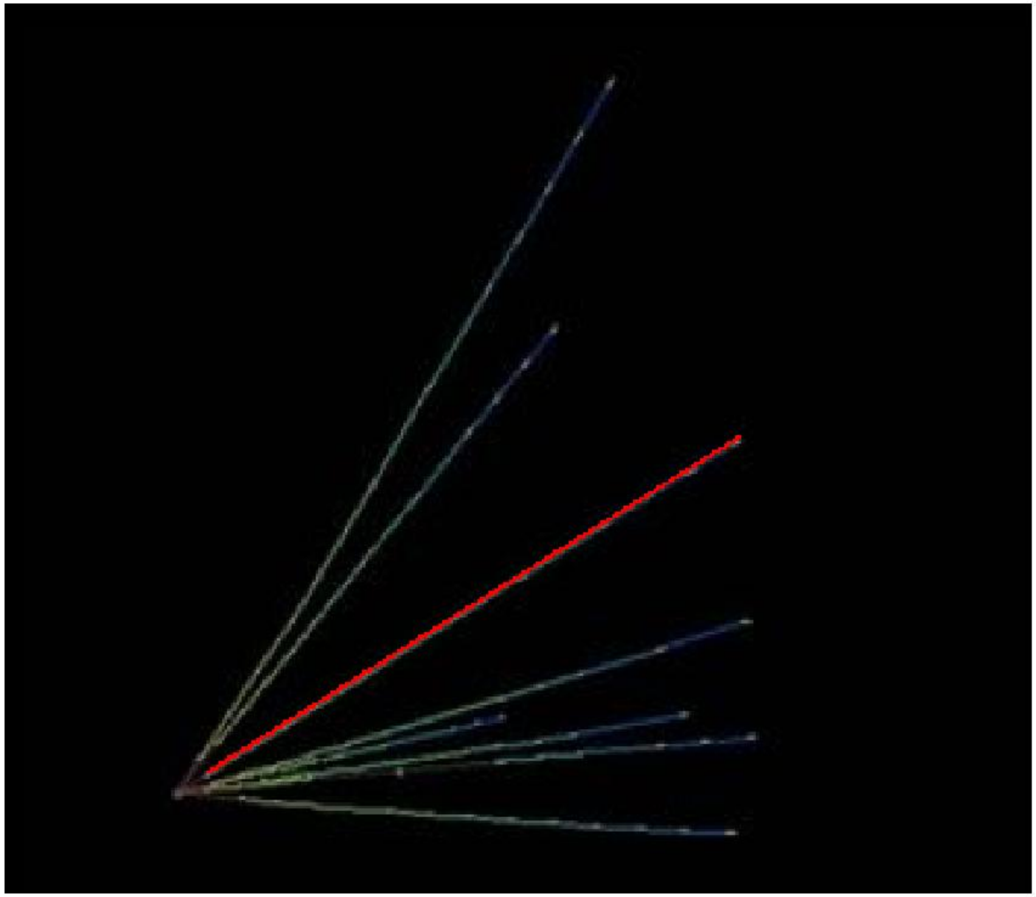


e) Accumulator matrix

Image space

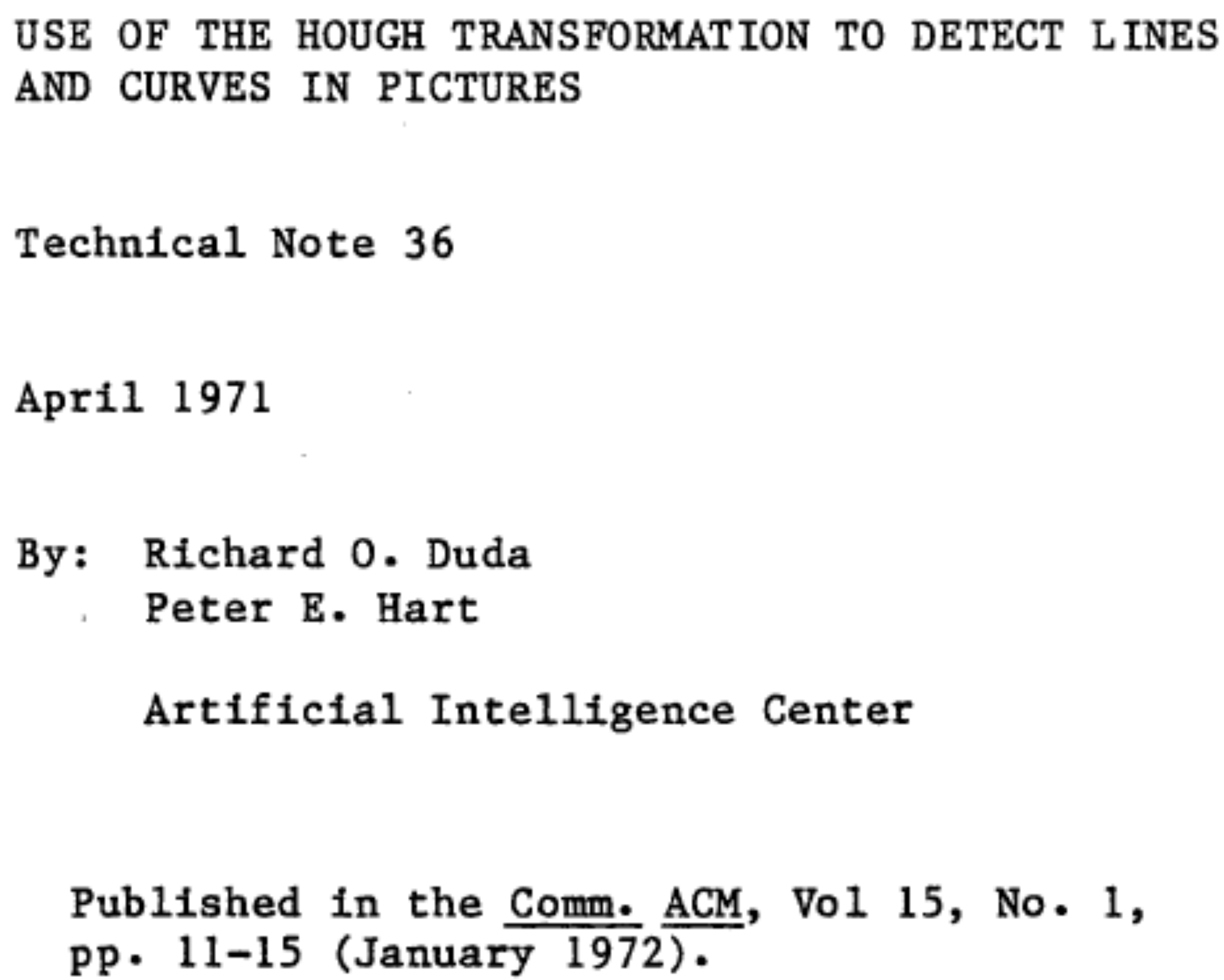


f) Selected points and detected straight line



g) Recognized straight line superimposed to the input image

Duda – Hart (1971-1972)


USE OF THE HOUGH TRANSFORMATION TO DETECT LINES
AND CURVES IN PICTURES

Technical Note 36

April 1971

By: Richard O. Duda
Peter E. Hart

Artificial Intelligence Center

Published in the Comm. ACM, Vol 15, No. 1,
pp. 11-15 (January 1972).

Duda – Hart (1971-1972)

ABSTRACT

Hough has proposed an interesting and computationally efficient procedure for detecting lines in pictures. In this paper we point out that the use of angle-radius rather than slope-intercept parameters simplifies the computation further. We also show how the method can be used for more general curve fitting, and give alternative interpretations that explain the source of its efficiency.

The general transform approach can be extended to curves other than straight lines.
The only thing you need is a a
parametric representation for the family of curves you are looking for

Duda – Hart (1971-1972) - Circles

If , as a parametric representation , we describe a **circle** in the image space by

$$(x-a)^2+(y-b)^2=r^2$$

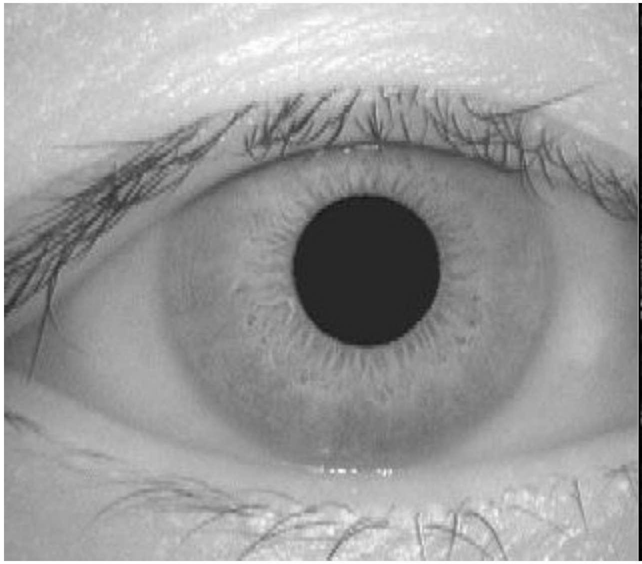
then an arbitrary point (x_P,y_P) will be transformed into a surface in the <A,B,R> parameter space defined by

$$(A-x_P)^2+(B-y_P)^2=R^2$$

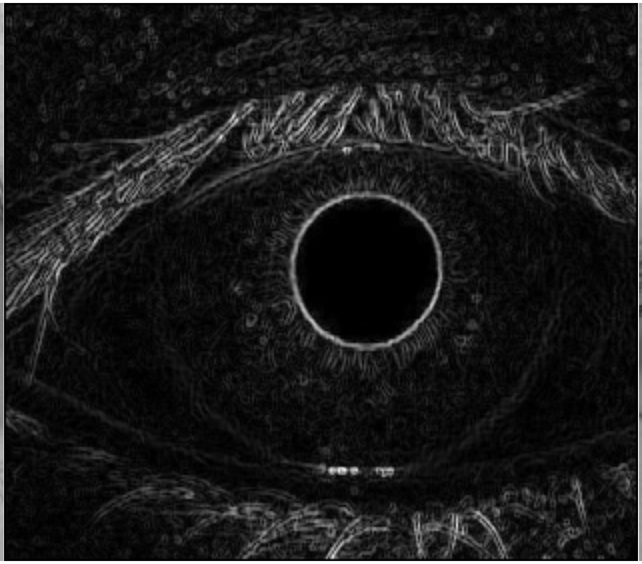
- Each point is then transformed into a right circular cone in a three-dimensional parameter space.
- If the cones corresponding to many points intersect at a single point , say the point (a*,b*,r*), then all these points lie on the circle defined by those three parameters

The process needs the use of **a three-dimensional accumulator** representing the three-dimensional parameter space thus increasing the computational burden

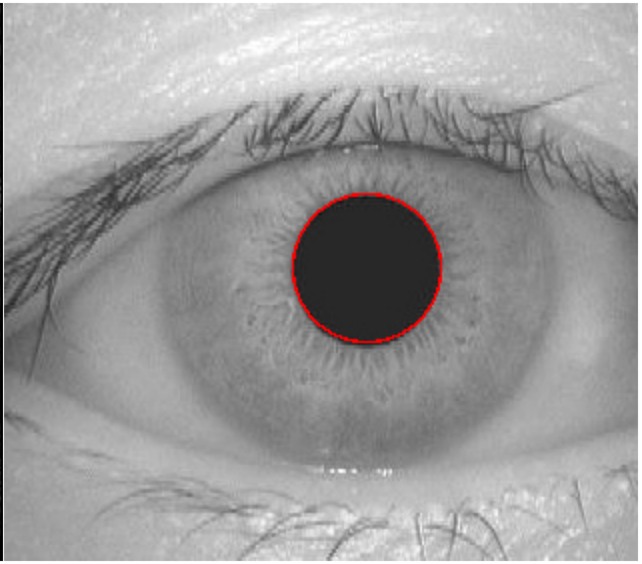
Iris/pupil boundary



a) Input image (317x279 = 88443 pixels)



b) Edge detection



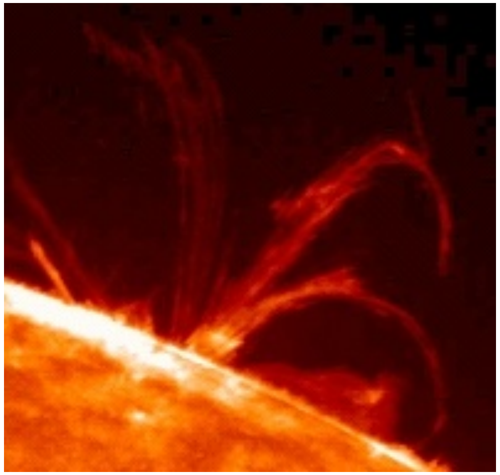
c) Circle corresponding to the global maximum in the accumulator matrix

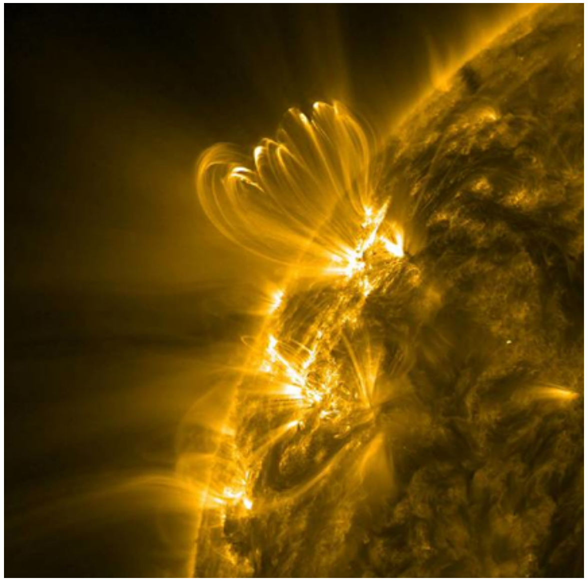
Hough transform for special classes of algebraic curves

Old idea, new developments and applications

Is it possible a formal definition of the Hough transform?

Is it possible an effective application to the processing of astronomical imaging?

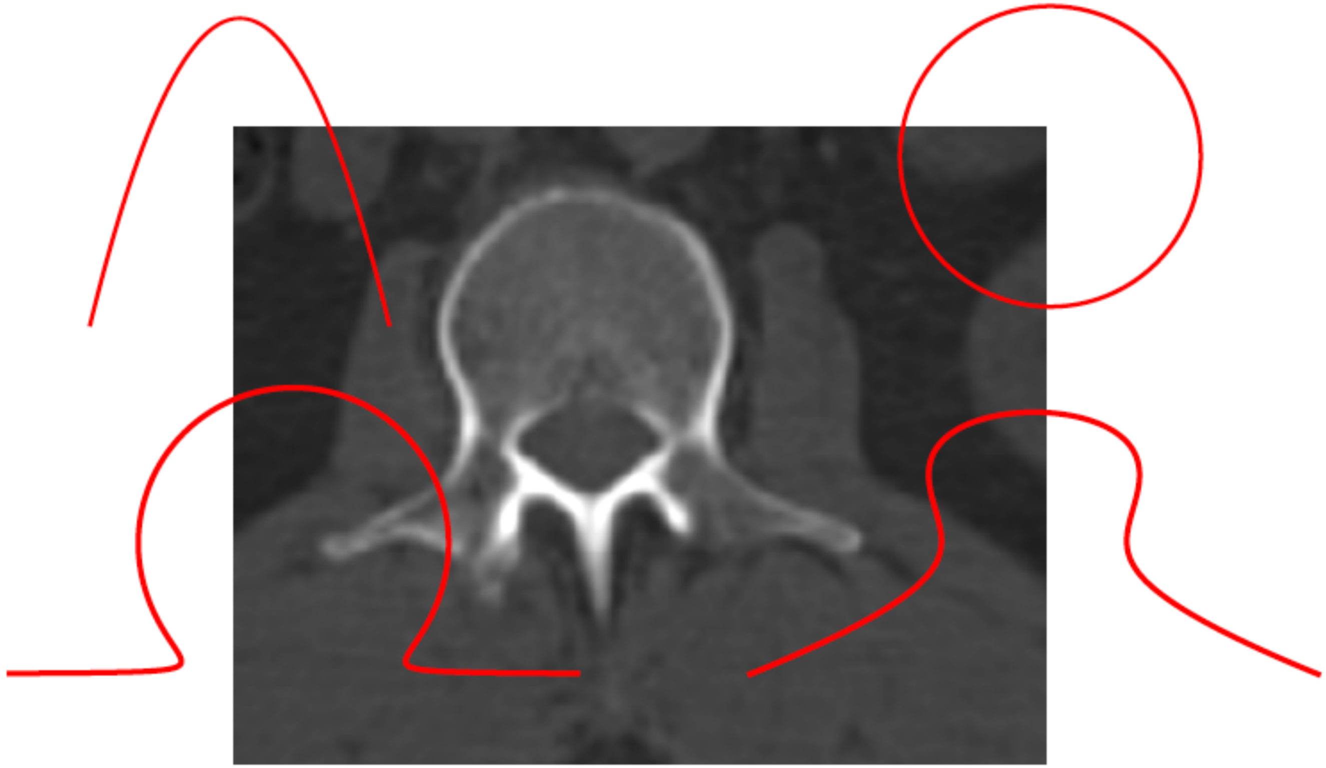




Is it possible an effective application to the processing of biomedical imaging?

Biomedical imaging

Which are the most appropriate families of curves to detect anatomical profiles?



Notations

- (X, Y) cartesian coordinates in the affine 2-dimensional image space $\mathbb{A}^2_{(X,Y)}(\mathbb{R})$
- $\Lambda = (\Lambda_1, \ldots, \Lambda_t)$ cartesian coordinates in the affine t-dimensional parameter space $\mathbb{A}^t_{(\Lambda_1,\ldots,\Lambda_t)}(\mathbb{R})$
- <u>variables</u>: capital letters; <u>parameters</u>: lowercase letters
- $F(X, Y; \lambda)$ family of non-constant irreducible polynomials in the variables X, Y

$$F(X, Y; \lambda) = \sum_{i,j=0}^{d} g_{ij}(\lambda) X^i Y^j, \quad 0 \leq i + j \leq d, \tag{1}$$

  whose coefficients $g_{ij}(\lambda)$ are the evaluations in the parameters $\lambda = (\lambda_1, \ldots, \lambda_t)$ of polynomials $g_{ij}(\Lambda)$ in the variables $\Lambda = (\Lambda_1, \ldots, \Lambda_t)$.

- $\mathcal{F} := \{C_\lambda\}$ the corresponding family of the zero loci C_λ, defined by

$$C_\lambda \ : \ F(X, Y; \lambda) = 0$$

Hough transform for algebraic curves

(M C Beltrametti, A M Massone and M Piana, SIAM *Journal on Imaging Sciences*, 2013)

Definition

Let $\mathcal{F}$ be a family of curves C_λ as above, and let $P = (x_P, y_P)$ be a point in the image space $\mathbb{A}^2_{(X,Y)}(\mathbb{R})$. *The Hough transform of the point P with respect to the family $\mathcal{F}$ is the hypersurface defined in the affine t-dimensional parameter space $\mathbb{A}^t_{(\Lambda_1,\dots,\Lambda_t)}(\mathbb{R})$ by*

$$\Gamma_P(\Lambda) := F(x_P, y_P; \Lambda) = 0$$

Hough transform for algebraic curves

(M C Beltrametti, A M Massone and M Piana, SIAM *Journal on Imaging Sciences*, 2013)

Definition

Let $\mathcal{F}$ be a family of curves C_λ as above, and let $P = (x_P, y_P)$ be a point in the image space $\mathbb{A}^2_{(X,Y)}(\mathbb{R})$. *The Hough transform of the point P with respect to the family $\mathcal{F}$ is the hypersurface defined in the affine t-dimensional parameter space $\mathbb{A}^t_{(\Lambda_1,\dots,\Lambda_t)}(\mathbb{R})$ by*

$$\Gamma_P(\Lambda) := F(x_P, y_P; \Lambda) = 0$$

Key theorem

Let $\mathcal{C}_\lambda$ be a curve of the family $\mathcal{F}$. Then, when P varies on $\mathcal{C}_\lambda$, one has:

1. The Hough transforms $\Gamma_P(\Lambda)$ all pass through the point $\lambda = (\lambda_1, \dots, \lambda_t)$.
2. Assume that the Hough transforms $\Gamma_P(\Lambda)$ have a point in common other than λ, say $\lambda' = (\lambda'_1, \dots, \lambda'_t)$. Then $\mathcal{C}_\lambda = \mathcal{C}_{\lambda'}$.

Hough transform for algebraic curves

(M C Beltrametti, A M Massone and M Piana, SIAM *Journal on Imaging Sciences*, 2013)

Definition

Let $\mathcal{F}$ be a family of curves C_λ as above, and let $P = (x_P, y_P)$ be a point in the image space $\mathbb{A}^2_{(X,Y)}(\mathbb{R})$. *The Hough transform of the point P with respect to the family $\mathcal{F}$ is the hypersurface defined in the affine t-dimensional parameter space $\mathbb{A}^t_{(\Lambda_1,\dots,\Lambda_t)}(\mathbb{R})$ by*

$$\Gamma_P(\Lambda) := F(x_P, y_P; \Lambda) = 0$$

Key theorem

Let $\mathcal{C}_\lambda$ be a curve of the family $\mathcal{F}$. Then, when P varies on $\mathcal{C}_\lambda$, one has:

1. The Hough transforms $\Gamma_P(\Lambda)$ all pass through the point $\lambda = (\lambda_1, \dots, \lambda_t)$.
2. Assume that the Hough transforms $\Gamma_P(\Lambda)$ have a point in common other than λ, say $\lambda' = (\lambda'_1, \dots, \lambda'_t)$. Then $\mathcal{C}_\lambda = \mathcal{C}_{\lambda'}$.

Key lemma - Hough–regularity

The following conditions are equivalent:

1. for any curves $\mathcal{C}_\lambda$, $\mathcal{C}_{\lambda'}$ in $\mathcal{F}$: $\quad \mathcal{C}_\lambda = \mathcal{C}_{\lambda'} \Rightarrow \lambda = \lambda'$
2. for each curve $\mathcal{C}_\lambda$ in $\mathcal{F}$: $\quad \cap_{P \in \mathcal{C}_\lambda} \Gamma_P(\Lambda) = \lambda$

The recognition algorithm

1. Choose a set of points of interest, say P_j, $j = 1, \ldots, \nu$ in the image space (edge detection)
2. Consider a discretization in a region $\mathcal{T}$ of the parameter space
   - sampling points $\lambda_{\mathbf{n}} = (\lambda_{1,n_1}, \lambda_{2,n_2}, \ldots, \lambda_{t,n_t})$
   - cells
     $\mathbf{C}_{\mathbf{n}} := \{\Lambda \in \mathcal{T} \mid \lambda_k \in [\lambda_{k,n_k} - \frac{d_k}{2}, \lambda_{k,n_k} + \frac{d_k}{2}),\ k = 1, \ldots, t,\ n_k = 1, \ldots, N_k\}$
     where $\mathbf{n}$ denotes the multi-index $(n_1, n_2, \ldots, n_t)$, d_k the sampling distance and N_k the number of samples with respect to the component k
3. Define an *accumulator matrix* $H = (H_{\mathbf{n}})$

$$H_{\mathbf{n}} = H_{n_1,n_2,\cdots,n_t} := \#\{P_j \mid \Gamma_{P_j}(\Lambda) \cap \mathbf{C}_{\mathbf{n}} \neq \emptyset,\ 1 \leq j \leq \nu\},$$

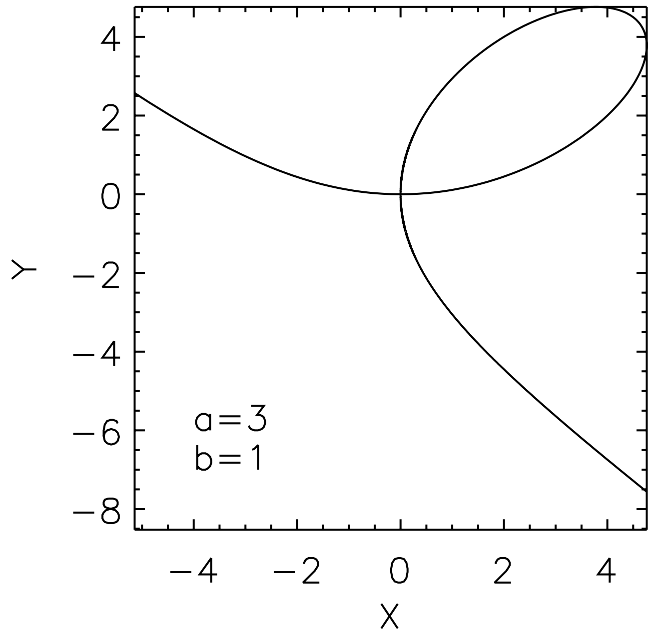
4. Optimize H

$$\mathbf{n}^* := \operatorname{argmax}_{\mathbf{n}} H_{\mathbf{n}}$$

5. Identify the set of optimal parameters $\lambda^* := \lambda_{\mathbf{n}^*} = (\lambda_{1,n_1^*}, \lambda_{2,n_2^*}, \ldots, \lambda_{t,n_t^*})$
6. Characterize the equation of the seeked curve $\mathcal{C}_{\lambda^*}$

Descartes Folium (cubic rational curve)

$$C_{a,b}: \quad 3axy - x^3 - by^3 = 0$$

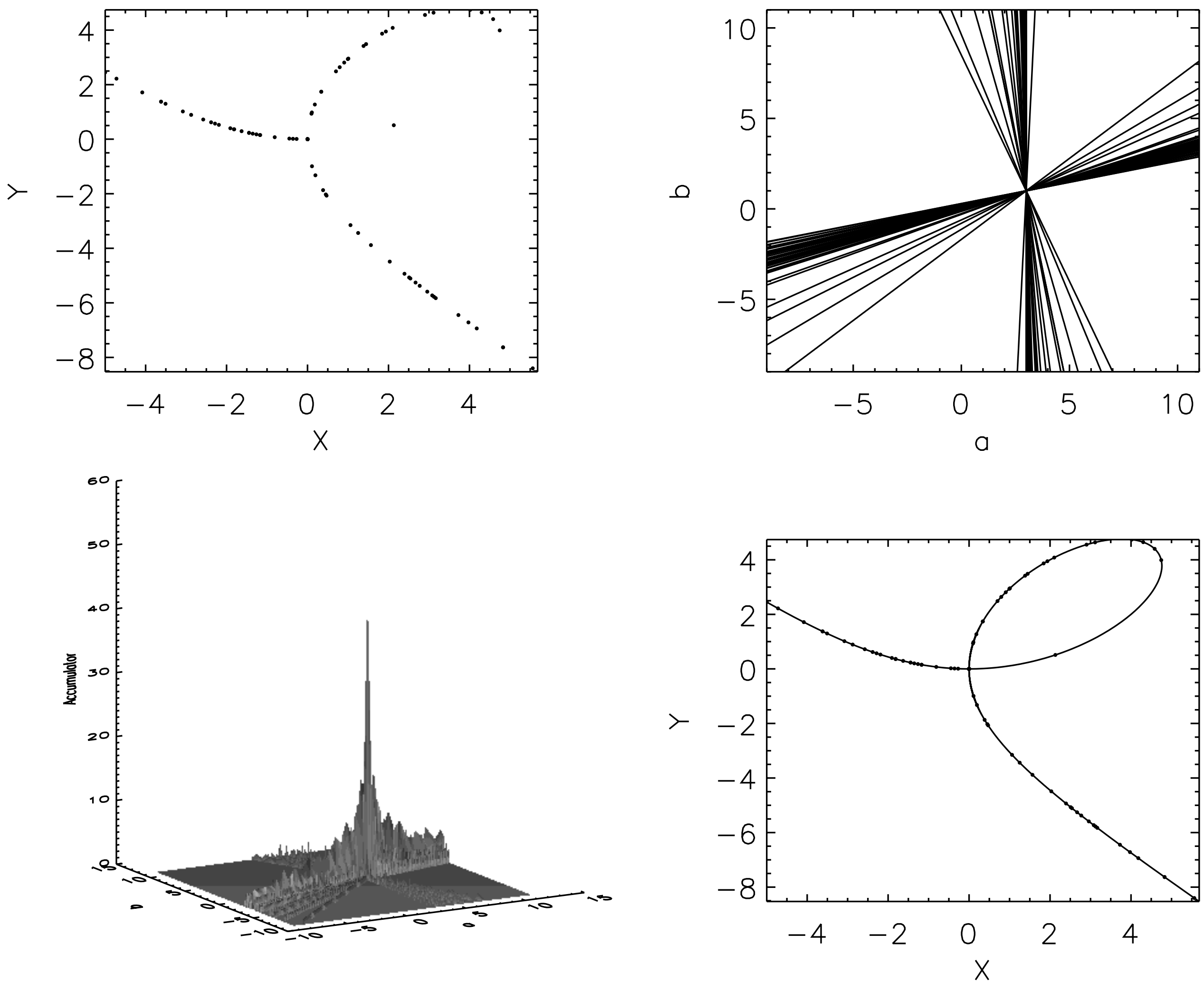


For any point $P(x_P, y_P) \in C_{a,b}$ the Hough transform is the straight line in the parameter plane defined by

$$\Gamma_P(A,B): \quad 3Ax_P y_P - x_P^3 - By_P^3 = 0$$

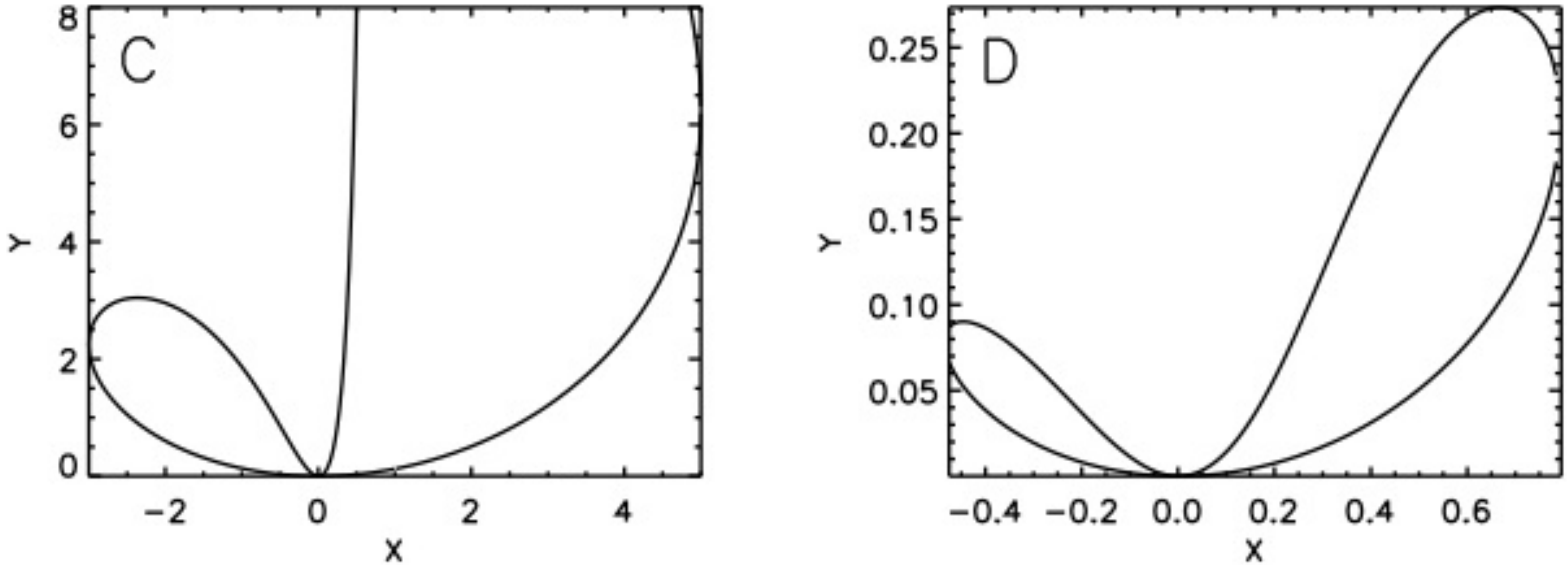
Descartes Folium

$$C_{a,b}: \quad 3axy - x^3 - by^3 = 0$$



Quartic curve with a tacnode

$$C_{a,b}: \quad y^2(x-a)^2 - byx^2 + bx^4 = 0$$



Quartic curve with a tacnode (closed loops)

$$C_{a,b,m}: \quad y^2(x-a)^2 + mx^2y(y-b) + x^4 = 0$$

Quartic curve with a tacnode (single closed loop and a loop closed at the infinity)

$$C_{a,b}: \quad y^2(x-a)^2 - byx^2 + bx^4 = 0$$

For any point $P(x_P, y_P) \in C_{a,b}$ the Hough transform is the **parabola** in the parameter plane defined by

$$\Gamma_P(A,B): \quad A^2x_P^2 - 2Ay_P^2x_P - Bx_P^2y_P + x_P^2y_P^2 + x_P^4 = 0$$

Quartic curve with a tacnode (two closed loops)

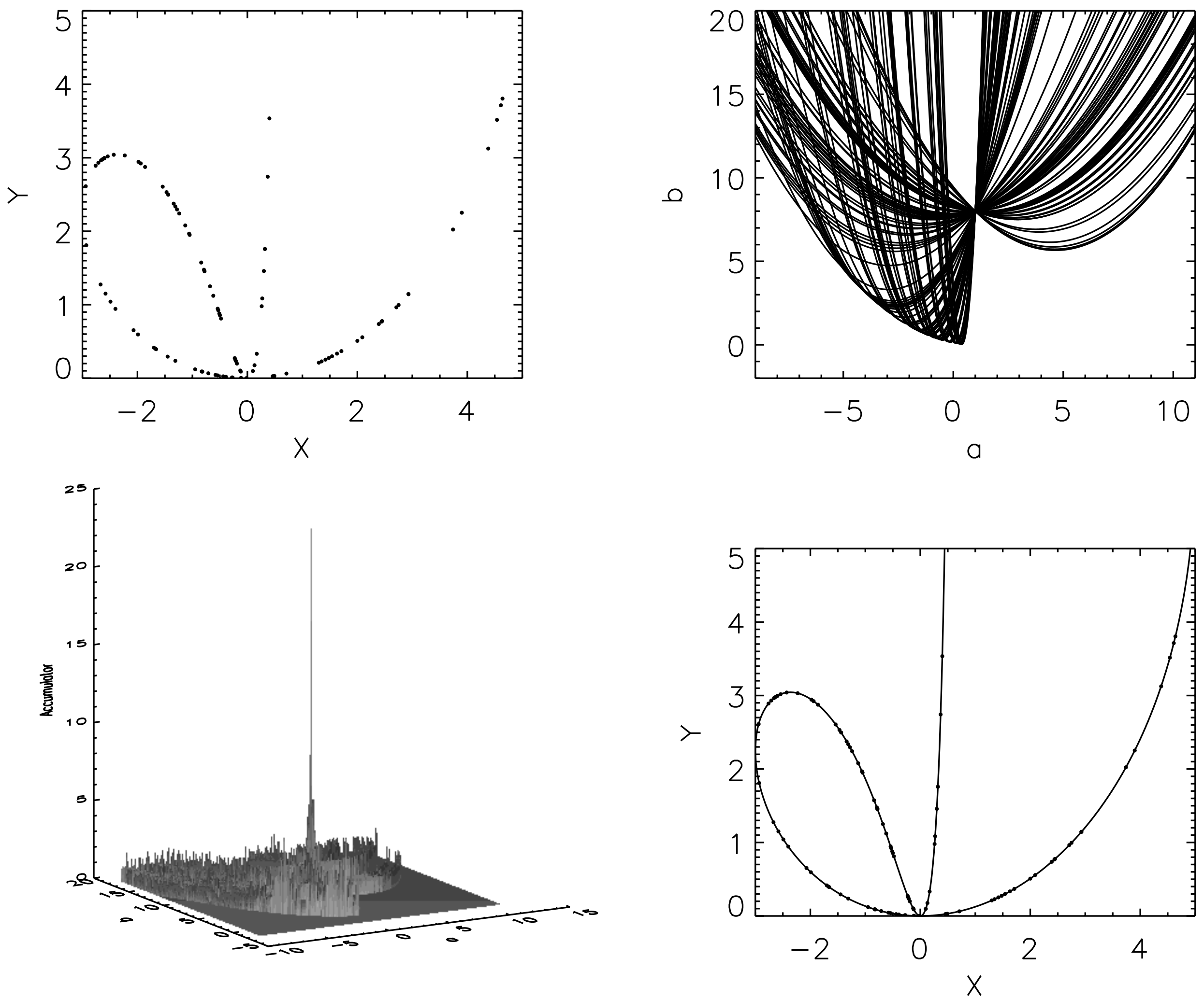
$$C_{a,b,m}: \quad y^2(x-a)^2 + mx^2y(y-b) + x^4 = 0$$

For any point $P(x_P, y_P) \in C_{a,b,m}$ the Hough transform is the **quadric surface** in the parameter plane defined by

$$\Gamma_P(A,B,M): \quad A^2y_P^2 - 2Ay_P^2x_P - BMx_P^2y_P + Mx_P^2y_P^2 + x_P^4 = 0$$

Quartic curve with a tacnode

$$C_{a,b}: \quad y^2(x-a)^2 - byx^2 + bx^4 = 0$$

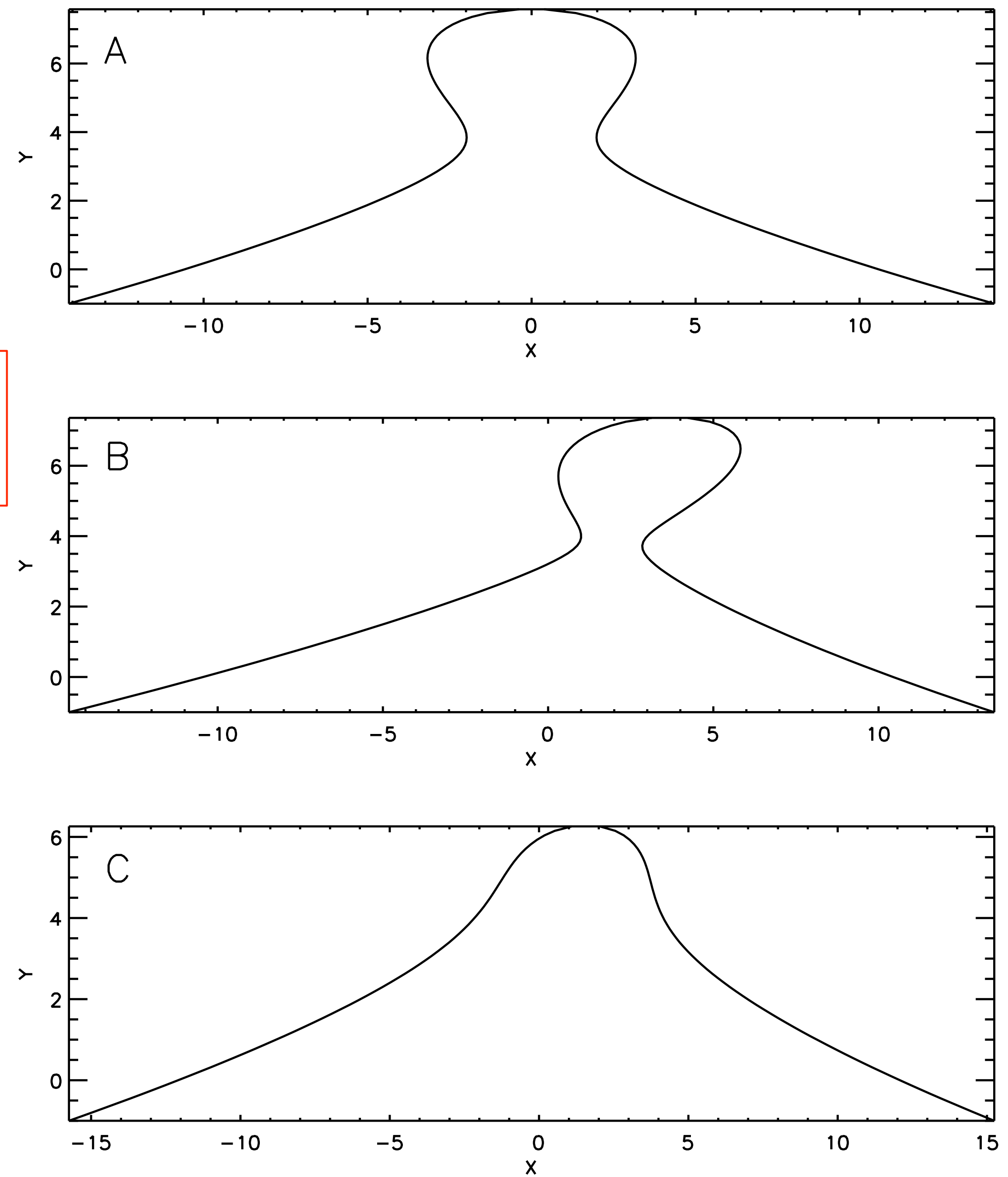


Elliptic curves in Weierstrass form

(up to the coordinate transformation $(x, y) \mapsto (-y, x - my)$)

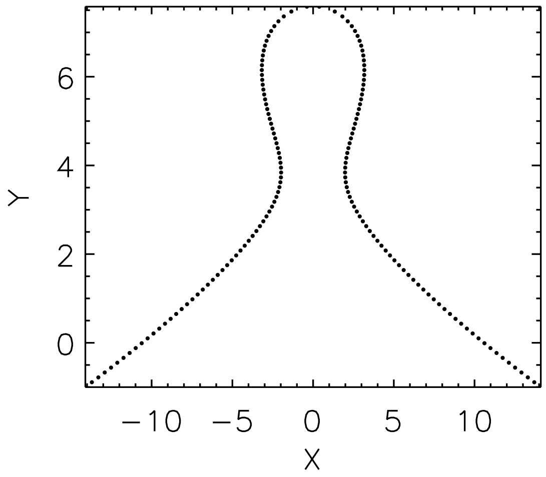
$$C_{a,b,m}: \quad (x - my)^2 = -y^3 - ay + b$$

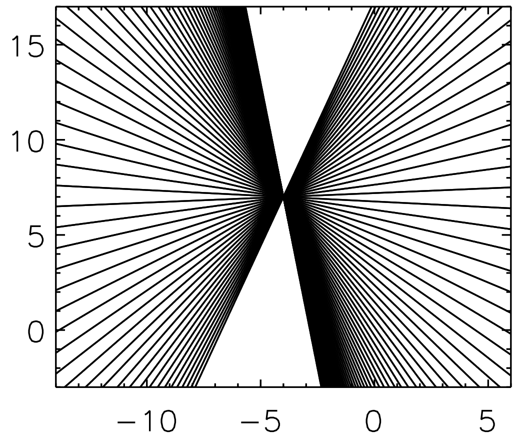
If m=0 , for any point $P(x_P, y_P) \in C_{a,b}$ the Hough transform is a line in the parameter plane

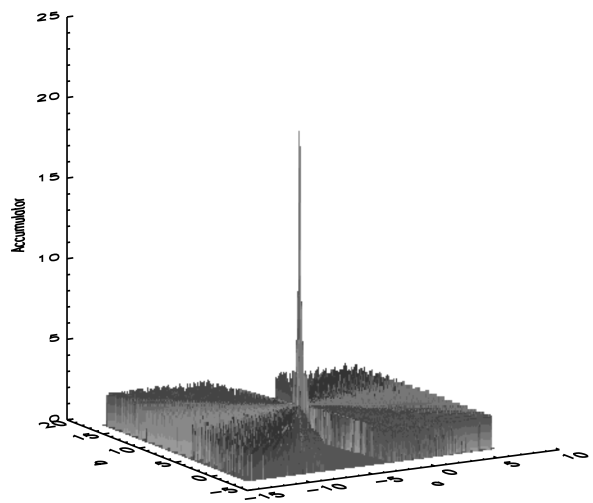


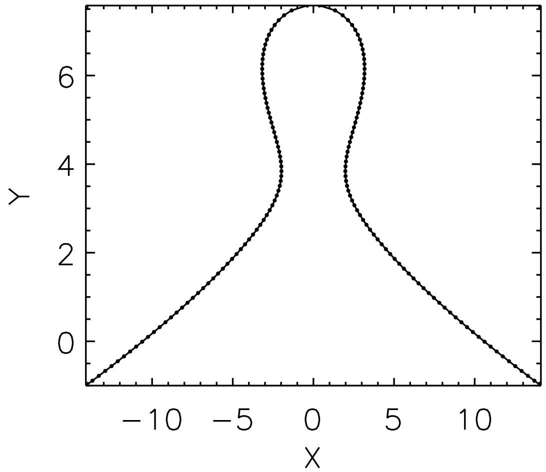
Elliptic curves

$$C_{a,b}: \quad x^2 = -y^3 - ay + b$$

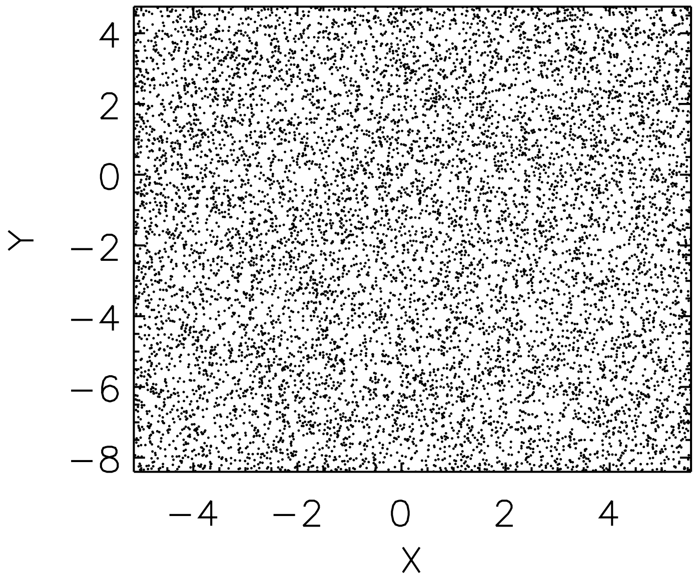


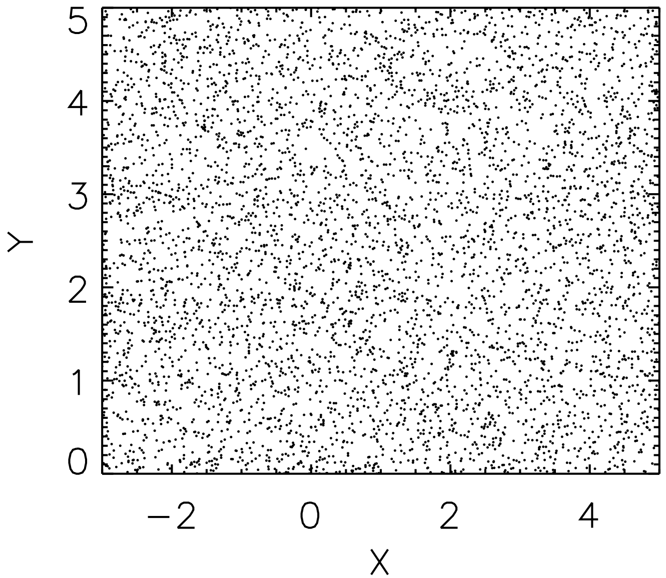




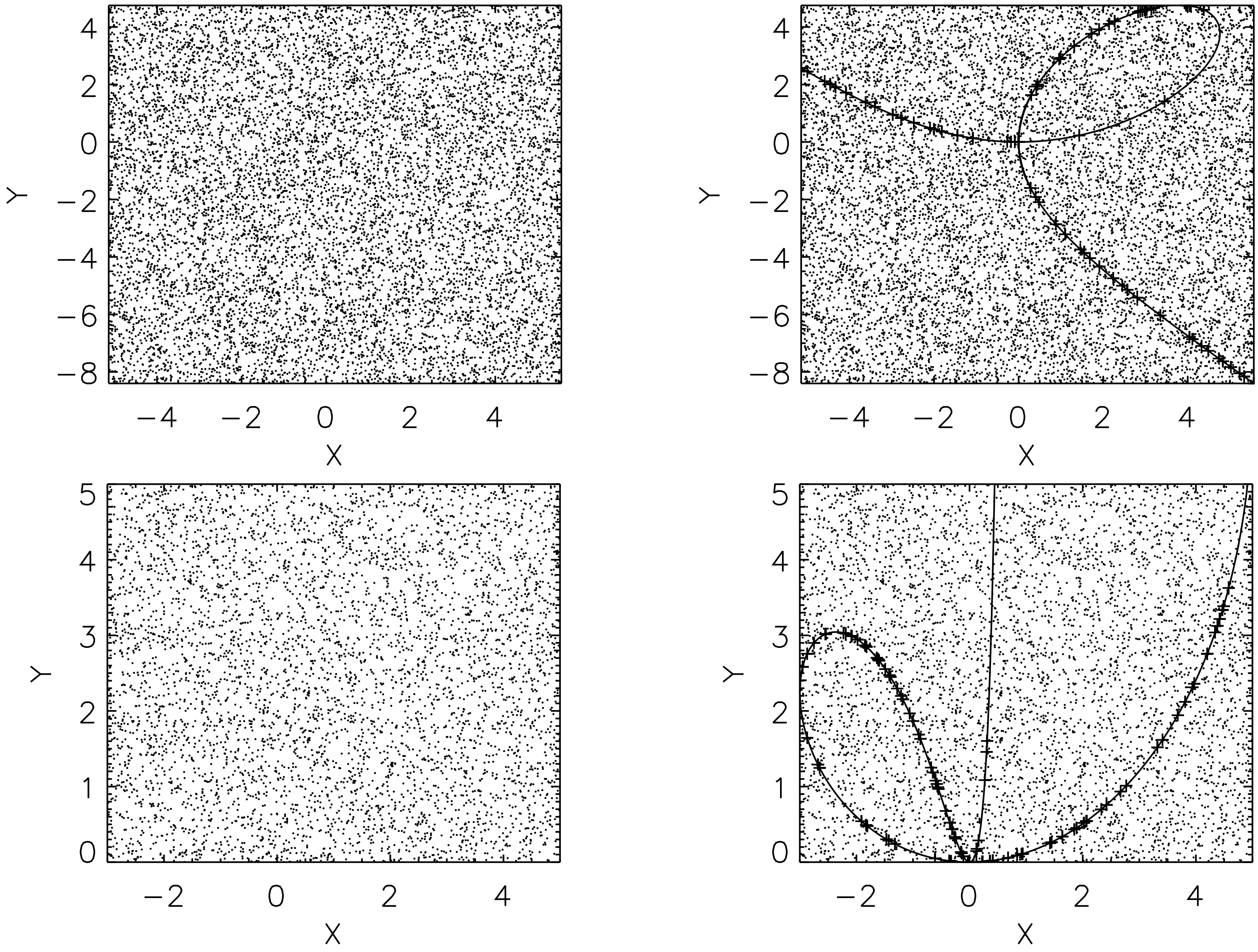


Robustness test

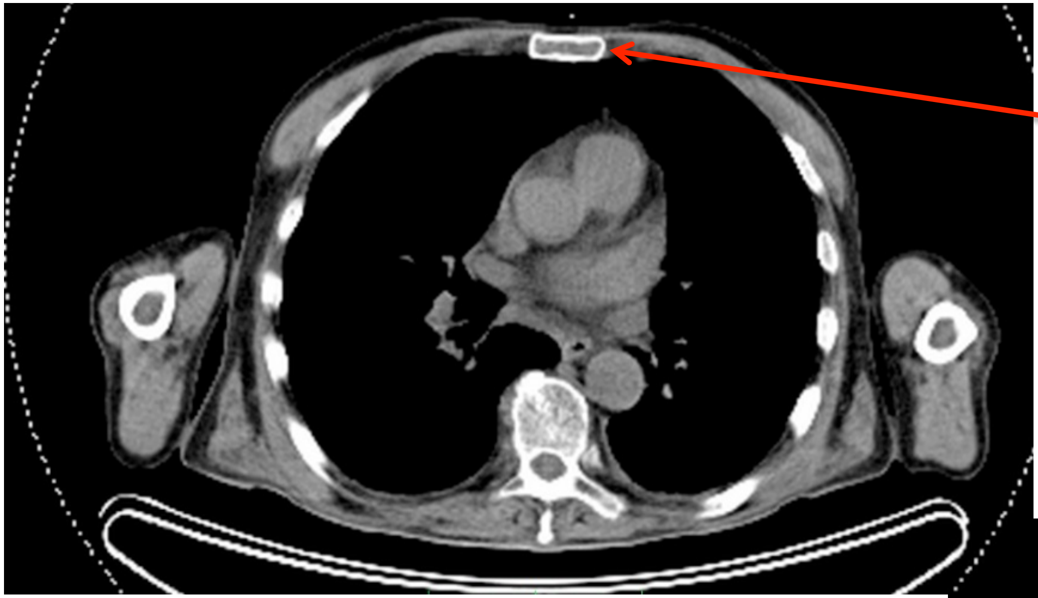




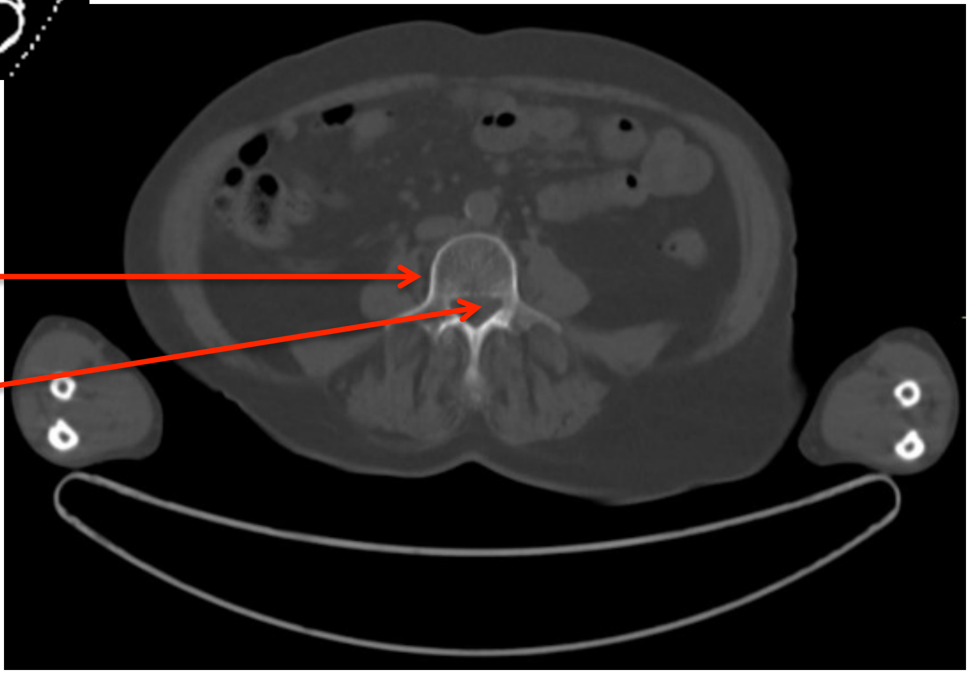
Robustness test



What about complex anatomical districts?



Sternum



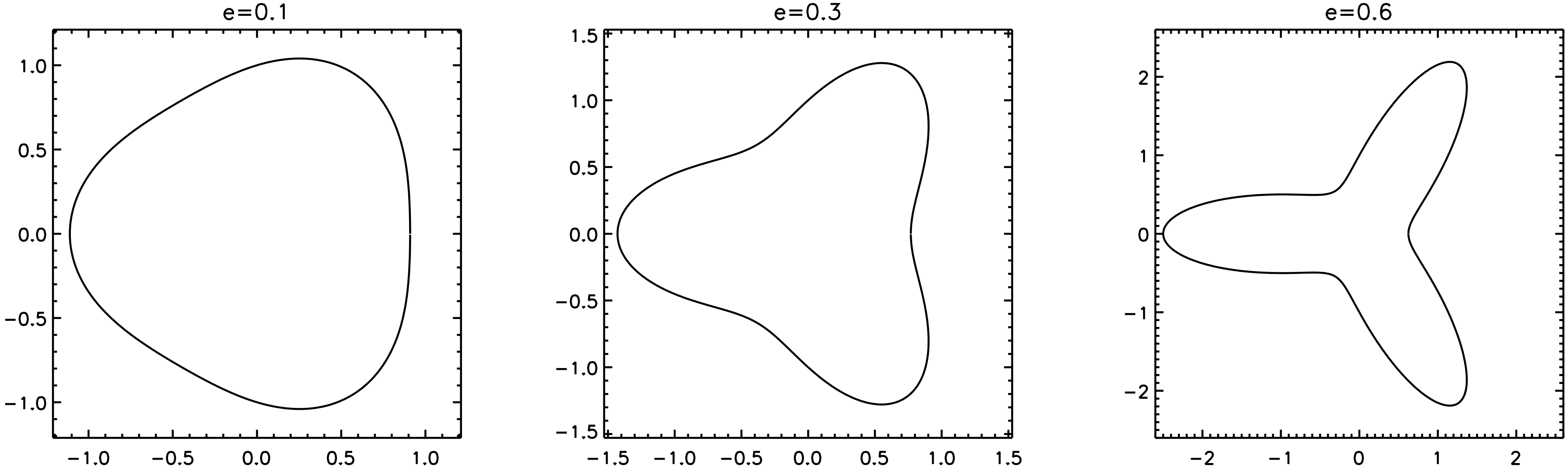
Vertebral body

Spinal canal

For today, ask just how not why!

Curve with three convexities (degree six)

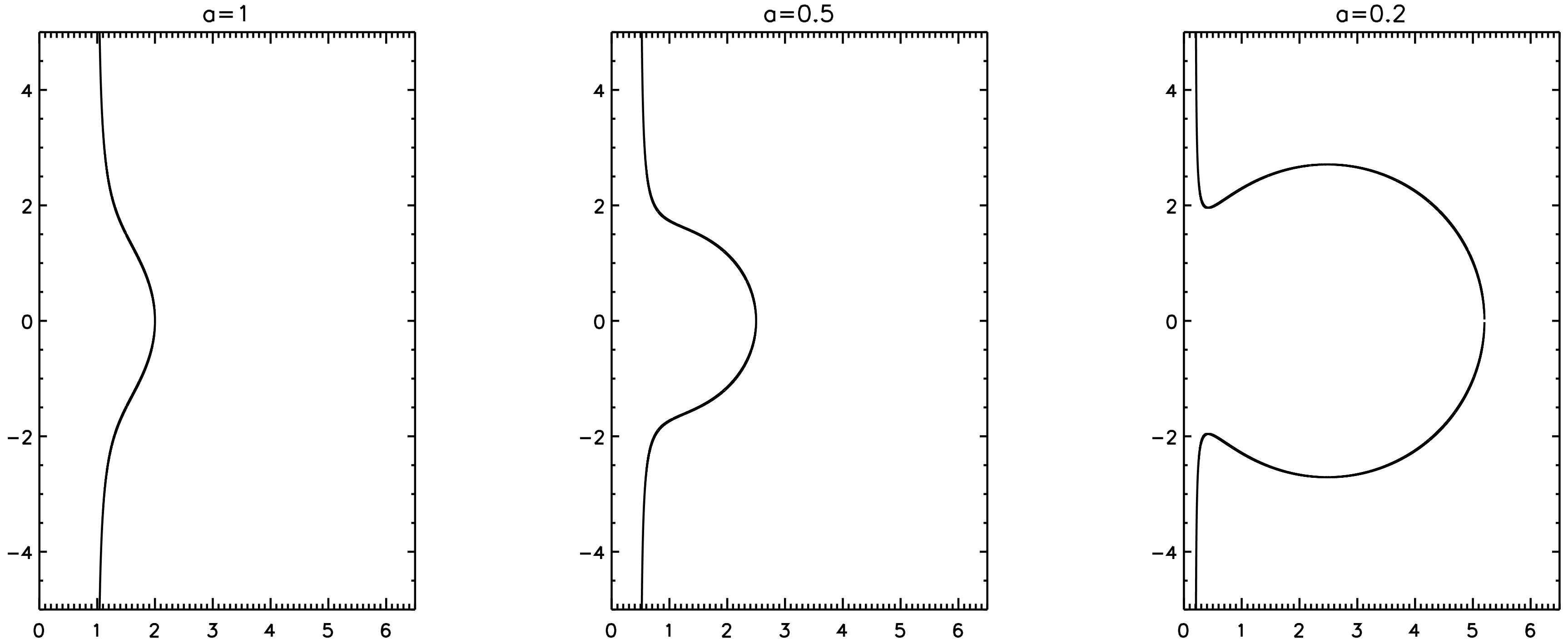
$$C_{p,e}: \quad (x^2+y^2)^3 = \left[p(x^2+y^2) - e(x^3-3xy^2) \right]^2$$



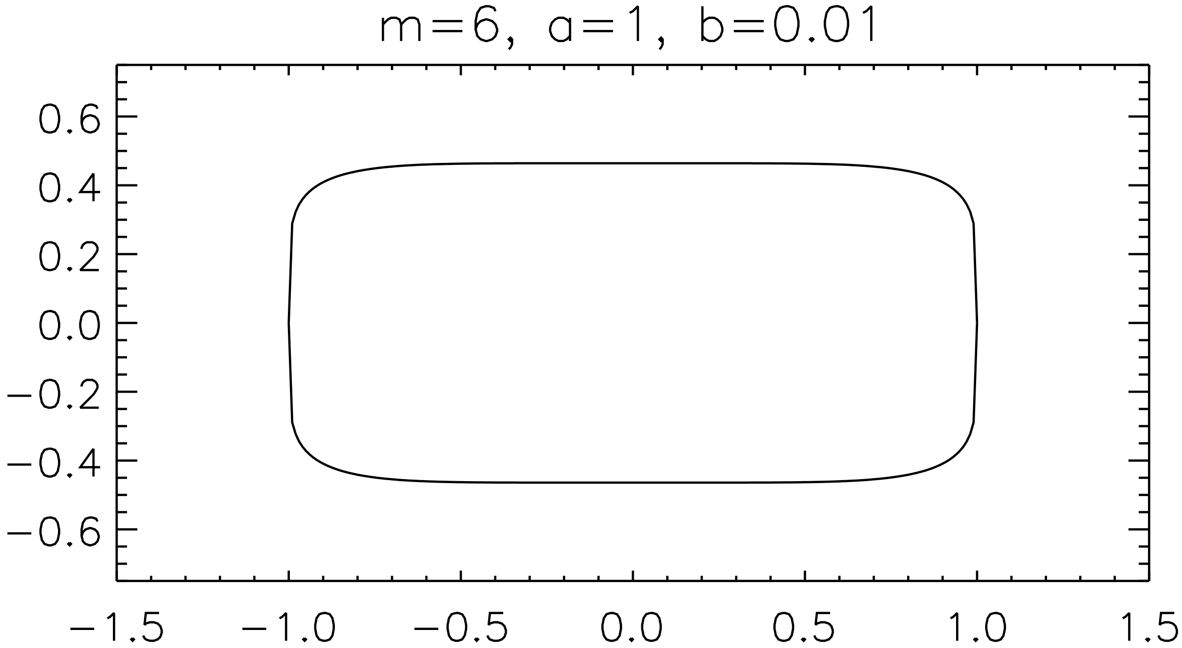
$$\Gamma_Q(P,E): \quad P(x_Q^2+y_Q^2) - E(x_Q^3 - 3x_Q y_Q^2) \mp \sqrt{(x_Q^2+y_Q^2)^3} = 0$$

Conchoid of Slüse

$$C_{a,k}: \quad a(x-a)(x^2+y^2) = k^2x^2$$

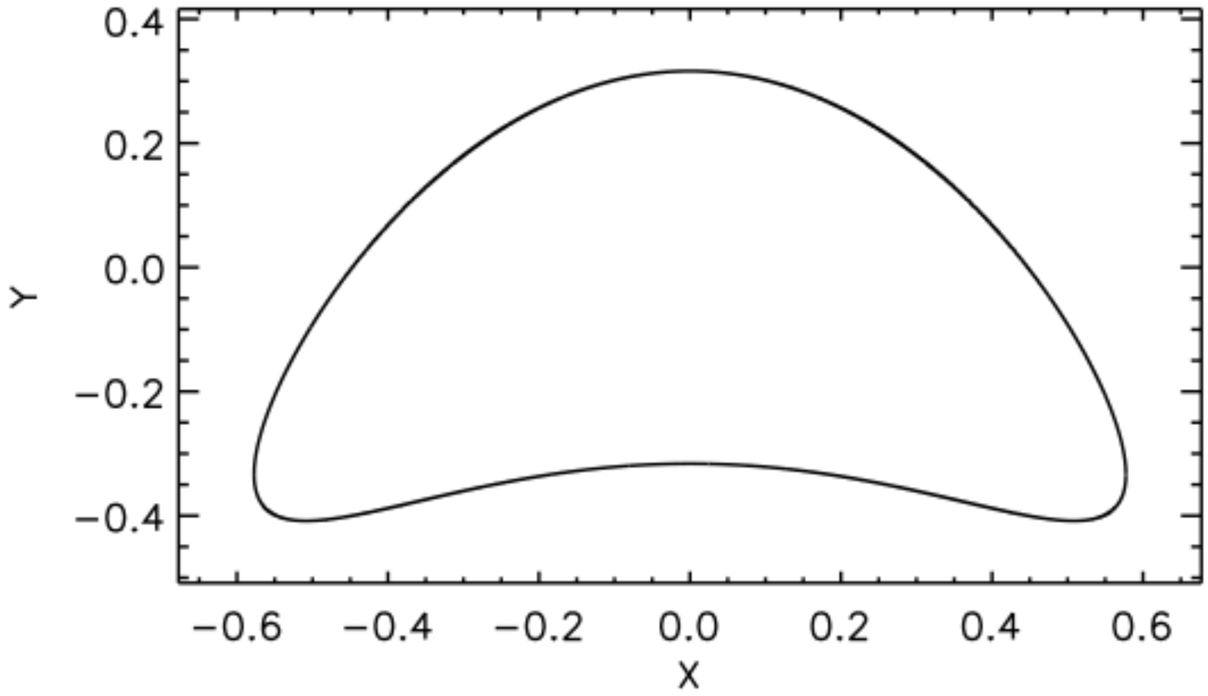


$$C_{a,b,m}: \quad \left(\frac{x}{a}\right)^{m} + \frac{y^{m}}{b} = 1$$

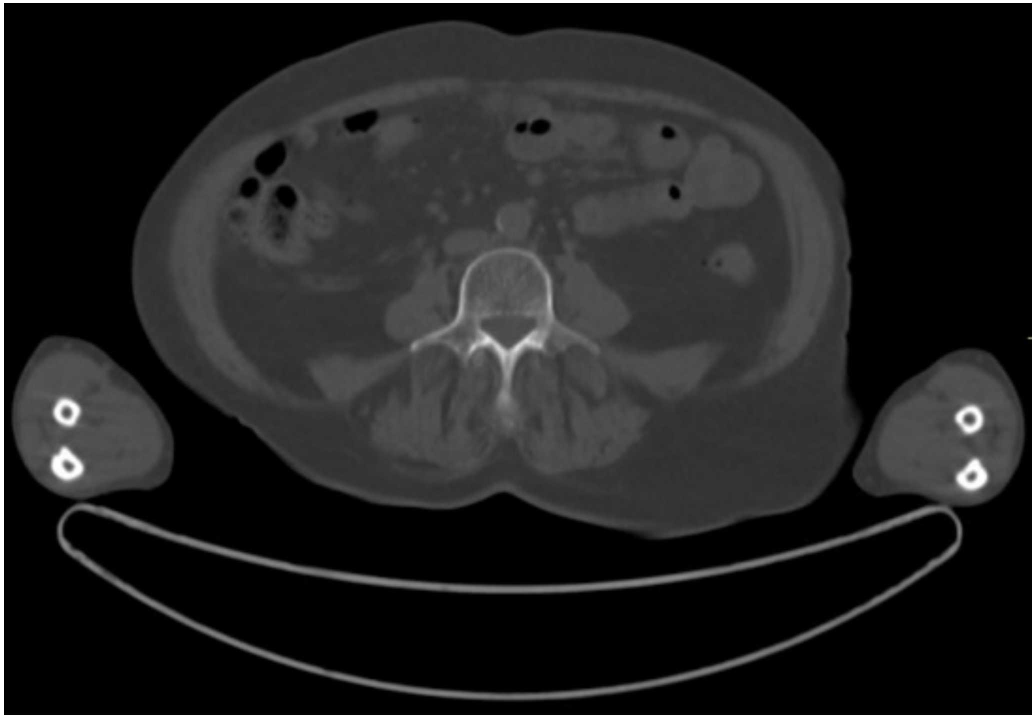


Wassenaar curve

$$C_{a,b,m}: \quad a(y+mx^2)^2 = 1 - bx^2$$

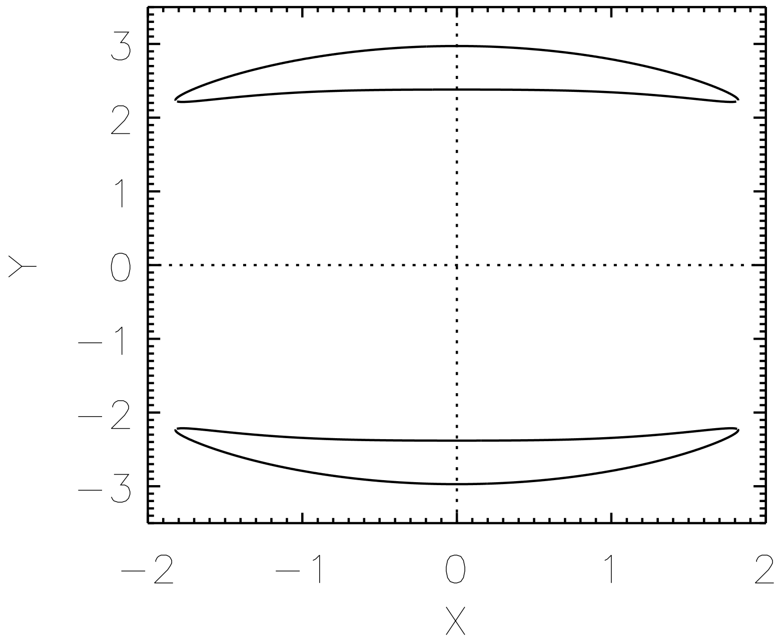


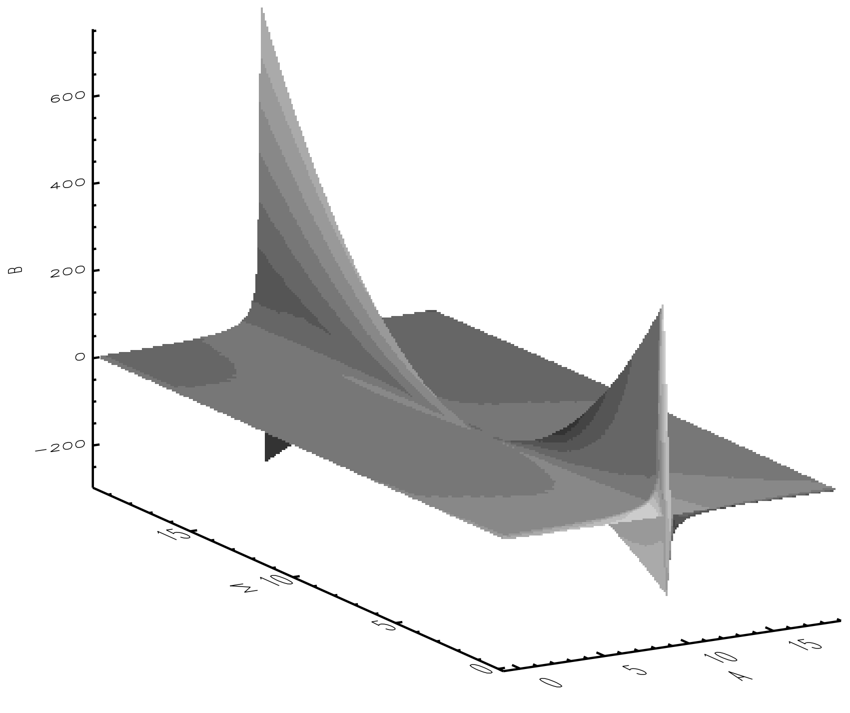
….. Not only bones …..



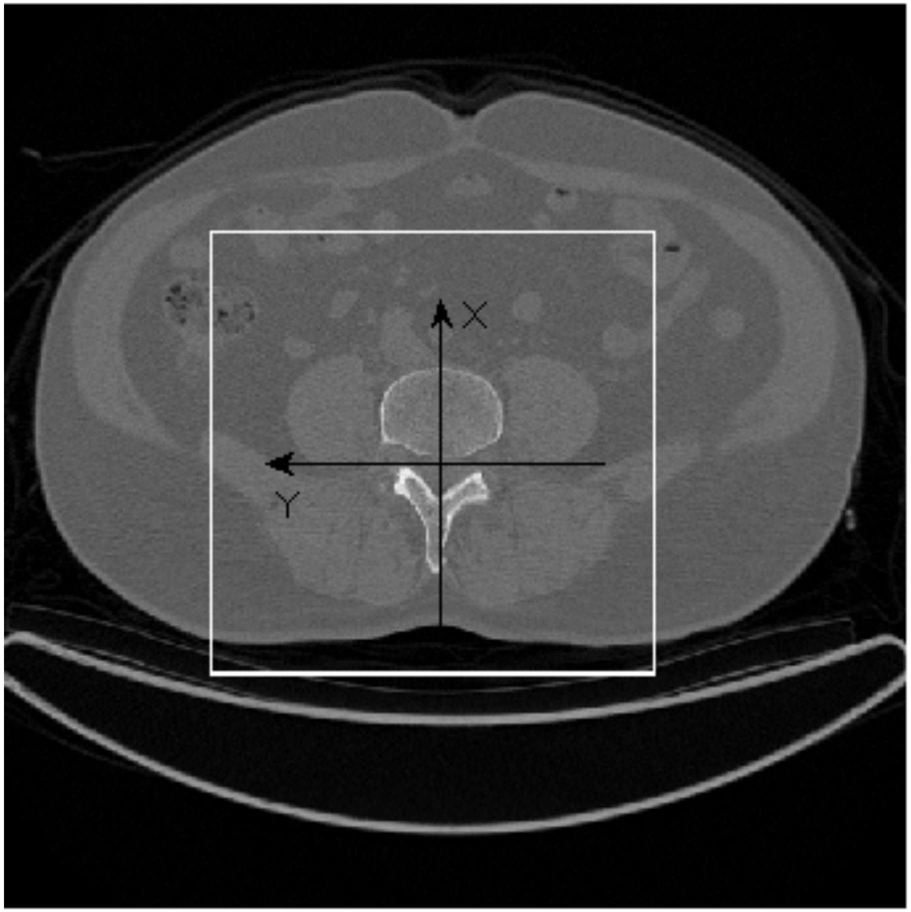
Watt curve

$$C_{a,b,m}: \quad (x^2+y^2)(x^2+y^2-m)^2+4ay^2(x^2+y^2-b)=0$$

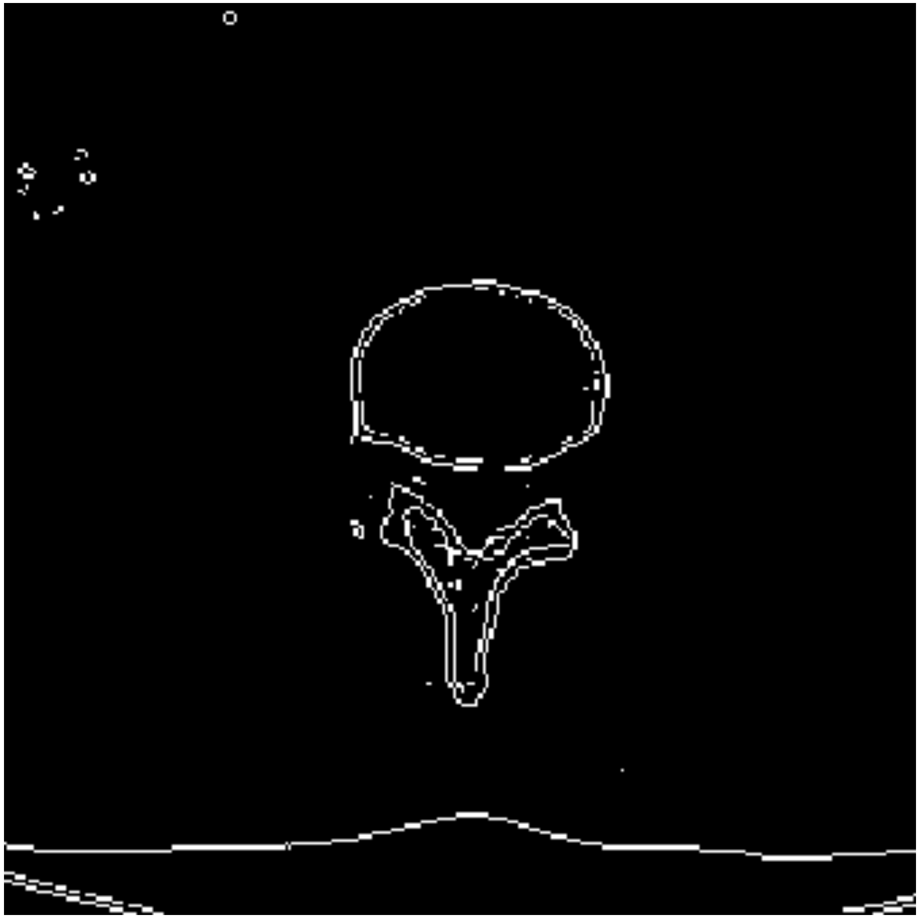




Detection of the external vertebral profile: conchoid of Slüse

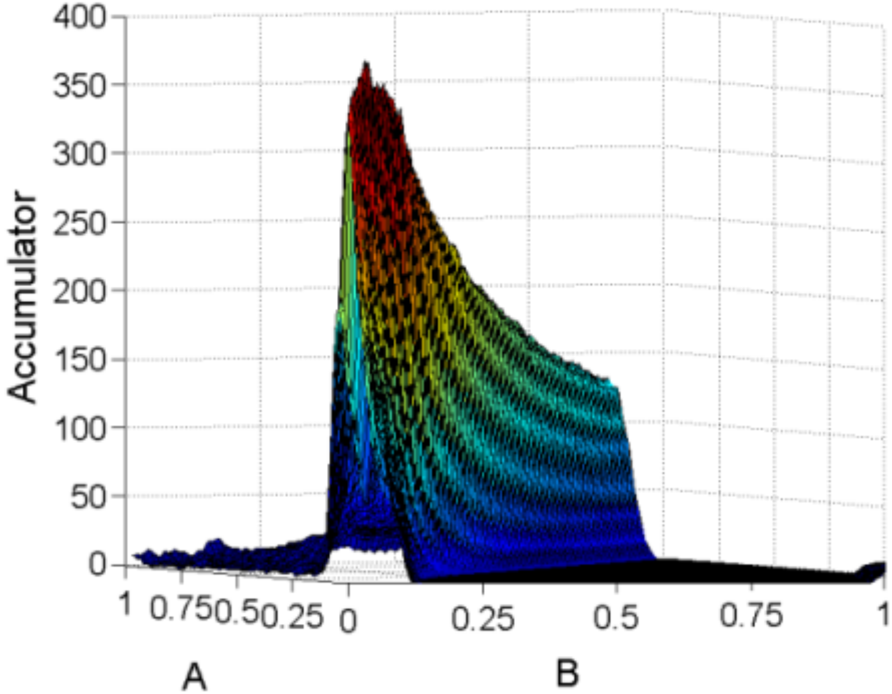


a) CT image

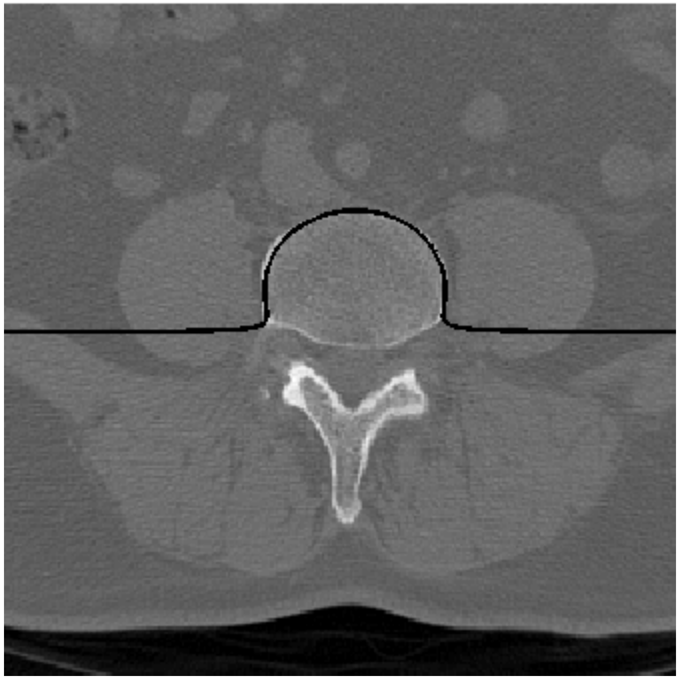


b) Edge detection

Detection of the external vertebral profile: conchoid of Slüse

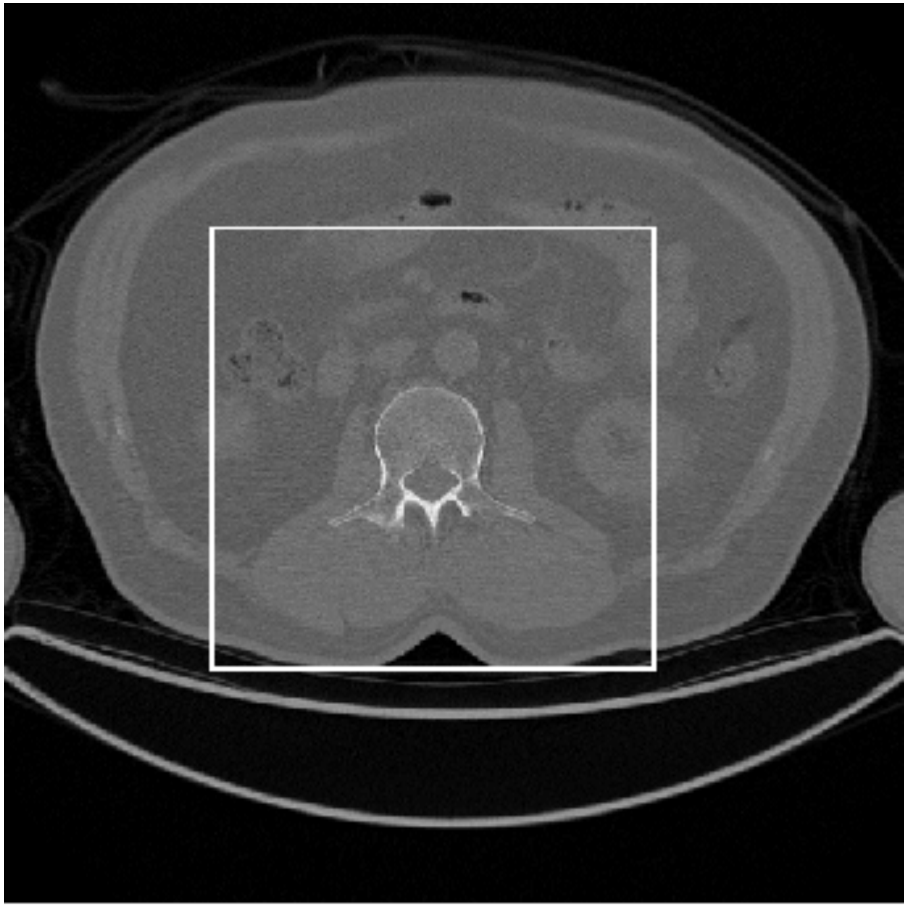


c) Accumulator matrix

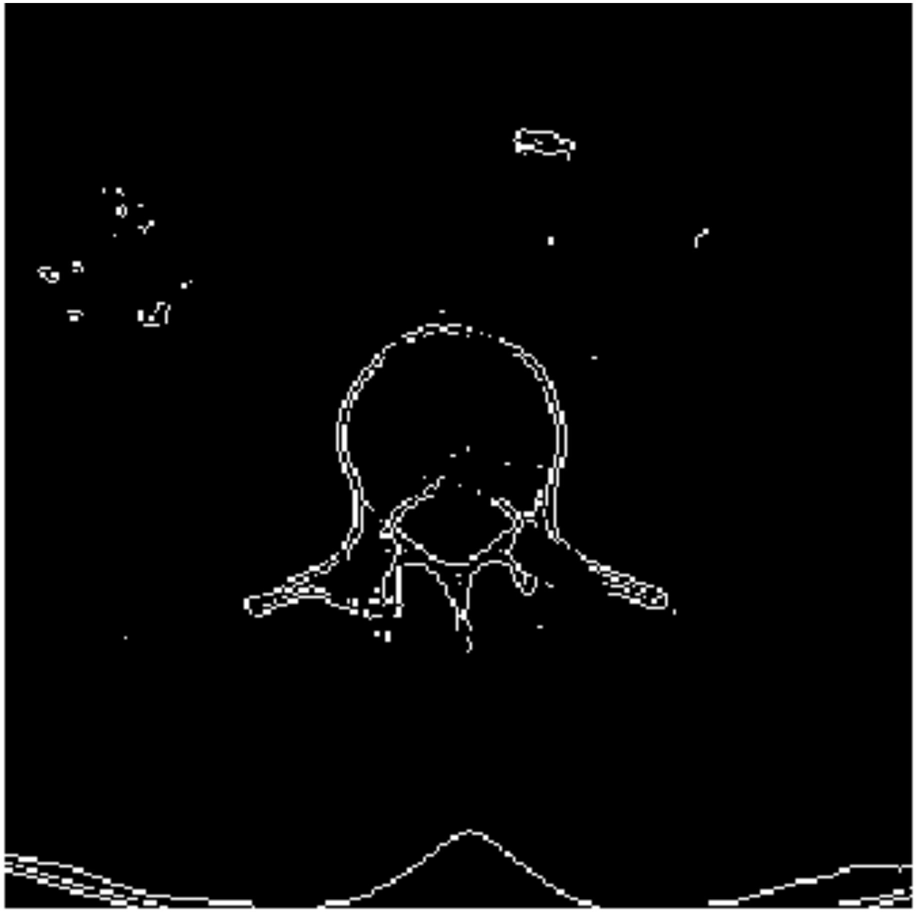


d) Recognized curve superimposed on the image

Detection of the external vertebral profile: elliptic curve

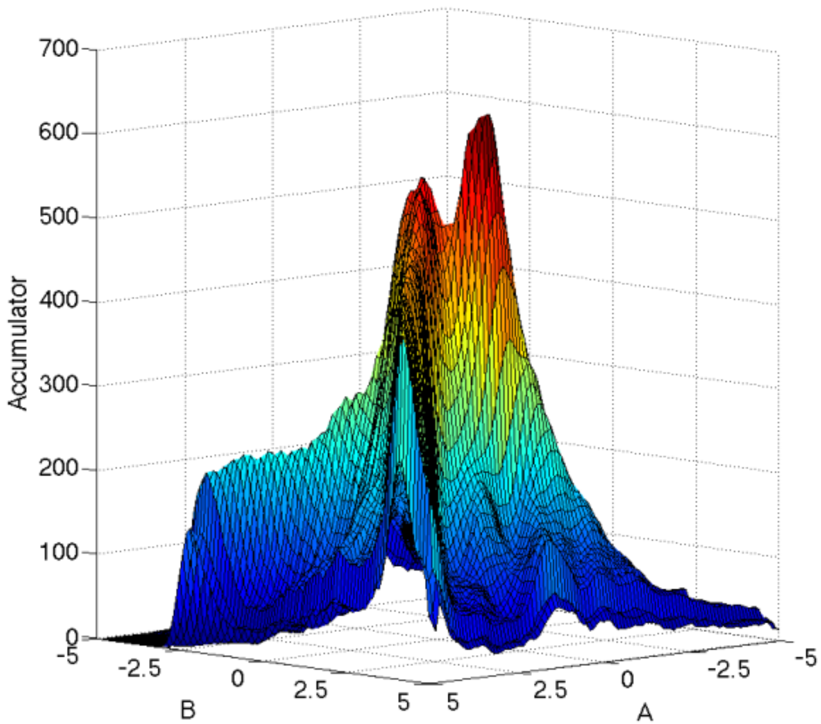


a) CT image



b) Edge detection

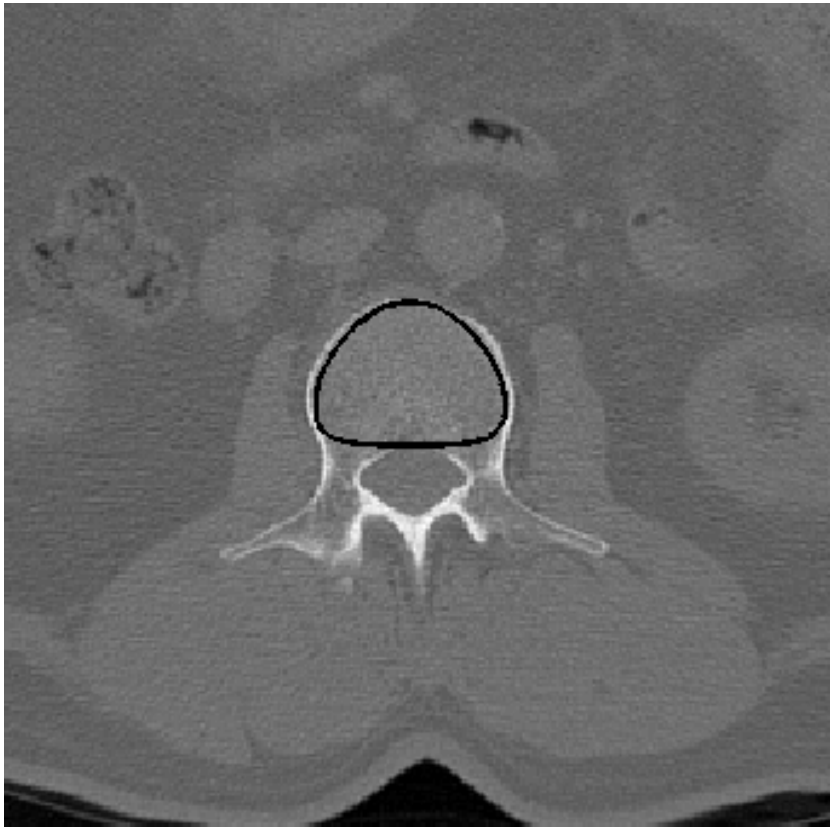
Detection of the external vertebral profile: elliptic curve



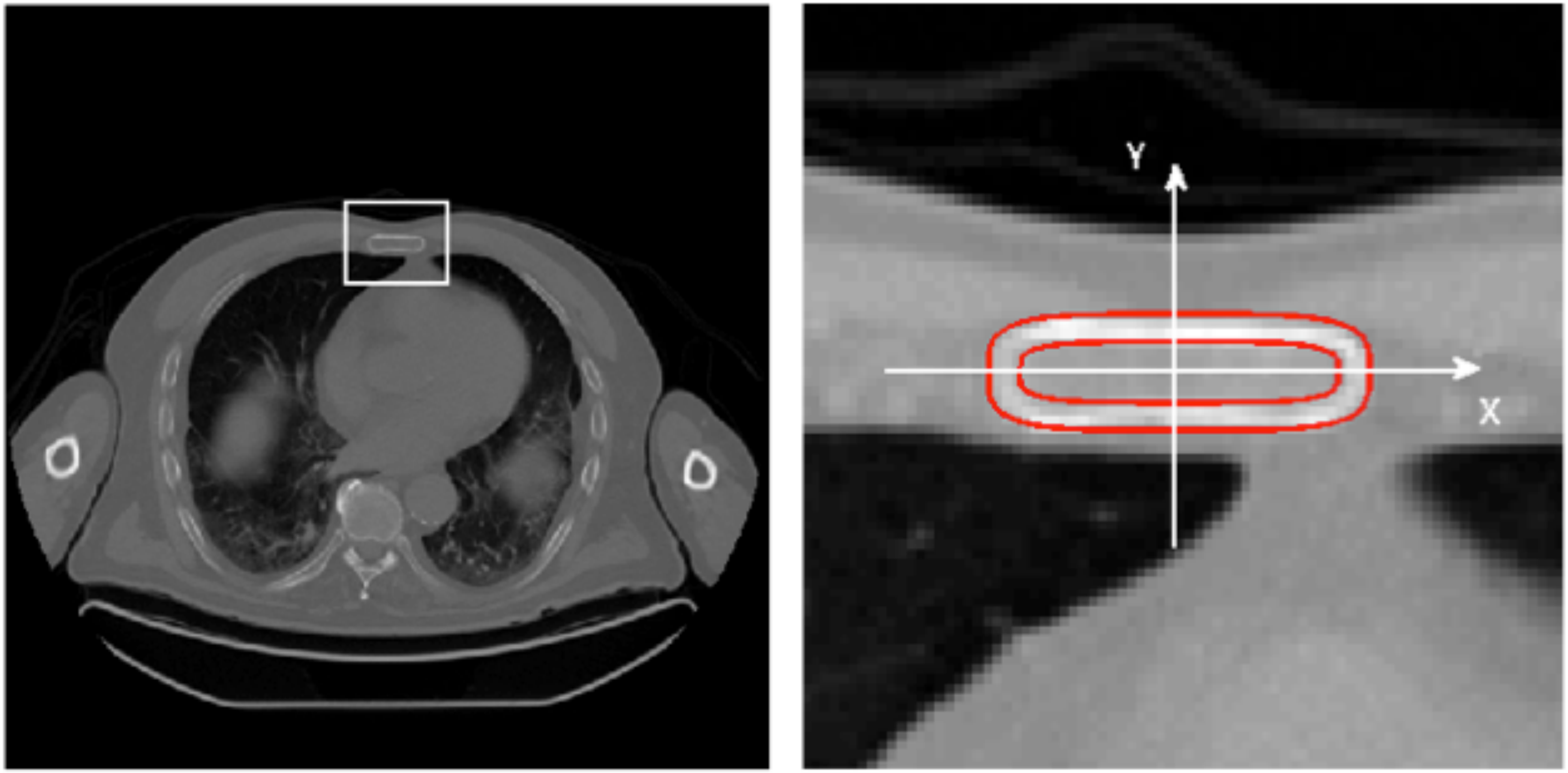
c) Accumulator matrix

d) Recognized curve superimposed on the image

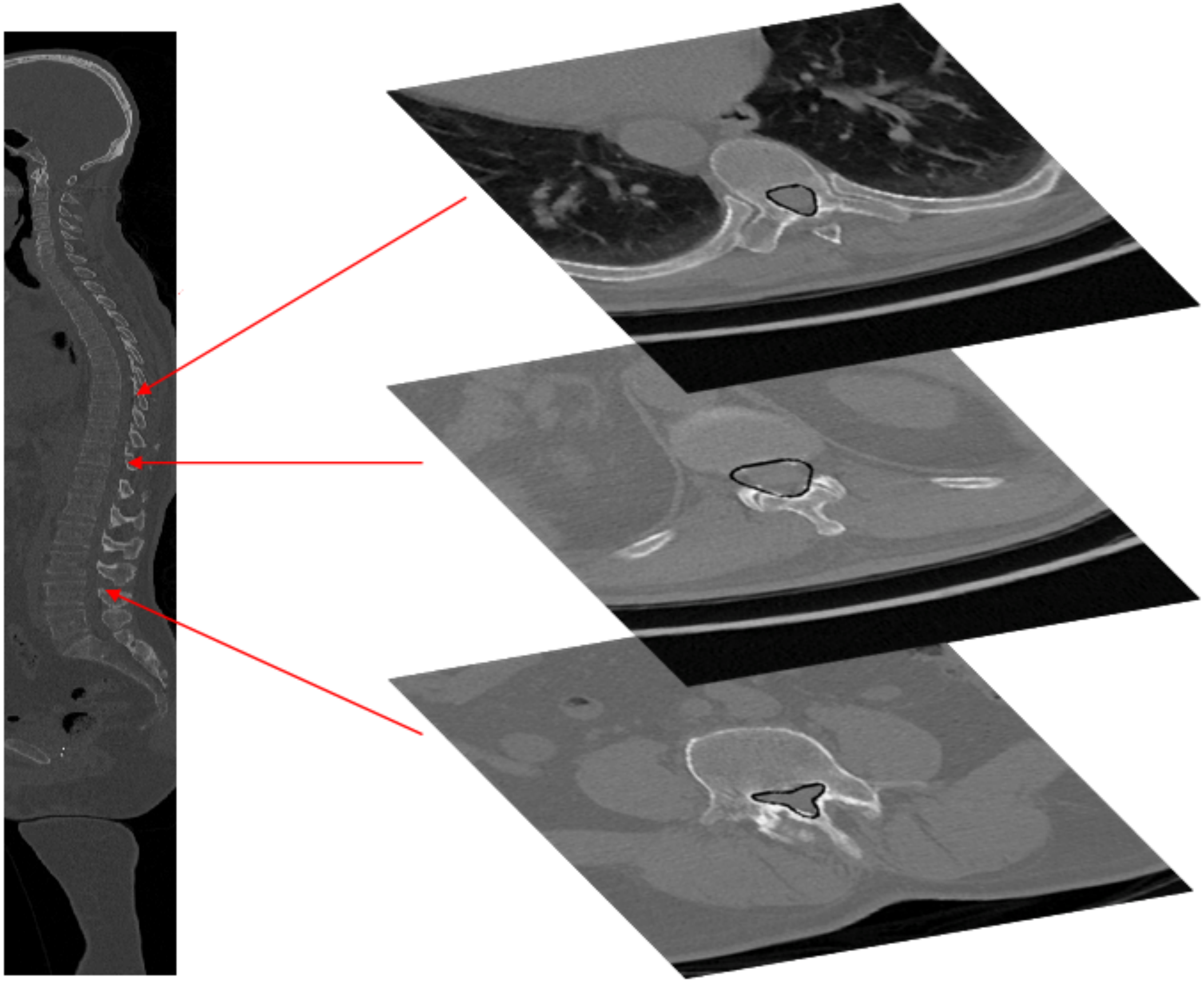
Detection of the internal vertebral profile: Wassenaar curve



Detection of the sternum: Lamet curve



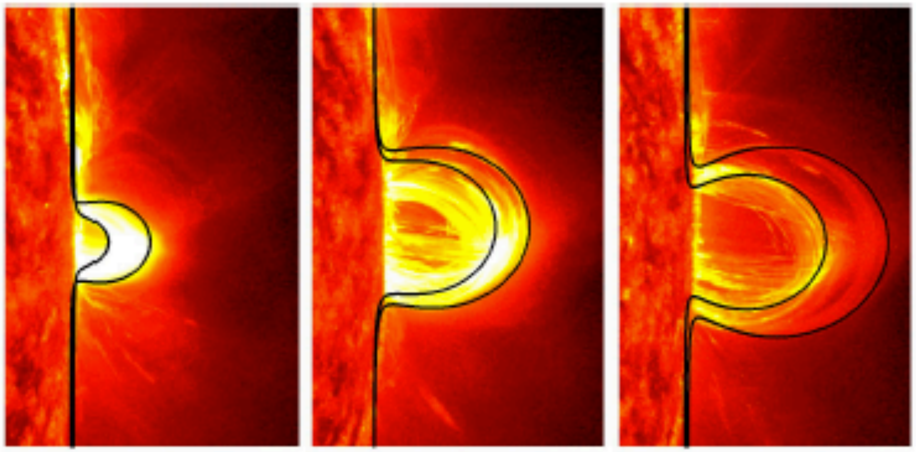
Detection of the spinal canal: curve with three convexities



Astronomical imaging: temporal evolution of a solar flare

(Massone, Perasso, Campi and Beltrametti, *Journal of Mathematical Imaging and Vision*, 2015)

The July 19^{th}, 2012 event:



Conchoid of Slüse:

$\mathcal{C}_{a,b} \ : \ a(X-a)(X^2+Y^2) = b^2X^2$

- 05:44:56 UT:
  $a = 0.12$, $b = 0.21$
  $a = 0.09$, $b = 0.28$
- 08:44:56 UT:
  $a = 0.17$, $b = 0.48$
  $a = 0.17$, $b = 0.55$
- 12:45:08 UT:
  $a = 0.09$, $b = 0.38$
  $a = 0.12$, $b = 0.53$