Rigorous computation of invariant measures with a two-grid approach

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Rigorous numerical methods based on interval arithmetic have become popular—as a tool to assist in theorem-proving—among the mathematicians working on dynamical systems, after their success in solving a famous open problem (existence of the Lorenz attractor [2]). By 'rigorous' we mean that the method provides a result with an explicit error bound that is guaranteed to hold in exact arithmetic, even if the computation is performed on a computer in floating point.

Here we consider the problem of finding a rigorous method to compute the so-called *absolutely continuous invariant measure* of the dynamical system obtained by the repeated application of a map $T : [0, 1] \to [0, 1]$.

Although this is not a differential problem, its numerical solution can be approached using the Galerkin method; it is a sort of (non-selfadjoint) finite-element eigenvalue problem. The simplest discretization strategy (relying on piecewise constant functions) is called *Ulam's method* [3], and has an elementary interpretation: one computes the invariant measure of a Markov chain with transition matrix $P \in \mathbb{R}^{n\times n}$, where P_{ij} is given by the probability that T maps a randomly chosen point in $[(j-1)/n,\, j/n]$ into $[(i-1)/n,\, i/n]$.

Galatolo and Nisoli [1] have provided a first rigorous implementation of this method with explicit error bounds, using interval arithmetic and explicit forward error tracking. The main computational burden in their method is obtaining a sufficiently tight upper bound for the norms of a certain matrix power $\|P^k\|_1$ (restricted to the subspace of zero-sum vectors). This computation is equivalent to estimating rigorously the convergence speed of the power method to the Perron vector of P (mixing time), a problem that has appeared independently in many other settings.

We show that this costly computation can be performed instead on a coarser discretization of the problem, by using ideas similar to those that lead to the approximation of the infinite problem with a finite one. We combine this idea with tighter inequalities that make use of all the intermediate values $\|P^j\|_1$, $j = 1, 2, \ldots, k$, instead of only the final one, and with inexact eigenvalue computation with an Arnoldi method. Some care is required to make sure that all the bounds obtained are rigorous irrespective of computational errors.

The new estimation strategy reduces the total computational cost of the original algorithm from $O(n^2)$ to $O(n)$, resulting in a much faster method.

References

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