Numerical approximation of coupled PDEs on embedded manifolds with high dimensionality gap

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The coupling of three-dimensional (3D) continua with embedded (1D) networks is not well investigated yet from the standpoint of mathematical analysis and numerical approximation, although it arises in applications of paramount importance such as microcirculation, flow through perforated media and the study of reinforced materials, just to make a few examples.

We address this mathematical problem within a unified framework, designed to formulate and approximate coupled partial differential equations (PDEs) on manifolds with heterogeneous dimensionality, arising from topological model reduction. The main difficulty consists in the ill-posedness of restriction operators (such as the trace operator) applied on manifolds with co-dimension larger than one. Partial results about the analysis and the approximation of this type of problems have appeared only recently [1, 2, 3, 4, 5].

We will overcome the challenges of defining and approximating PDEs on manifolds with high dimensionality gap by means of nonlocal restriction operators that combine standard traces with mean values of the solution on low dimensional manifolds. This new approach has the fundamental advantage to enable the approximation of the problem using Galerkin projections on Hilbert spaces, which can not be otherwise applied because of regularity issues. Furthermore, combining the numerical error analysis with the model reduction approach, the concurrent modeling and discretization errors in the approximation of the original fully dimensional problem can be quantified and balanced.

Our ultimate objective is to exploit topological model reduction to perform large scale simulations of microcirculation, see for example [6] and of perforated reservoirs, which represent problems with significant impact on medicine and geophysics.

References

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