Comparison of first-order methods for phase estimation in differential-interference-contrast microscopy

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Main references:
Differential-interference-contrast (DIC) microscopy

DIC phase estimation problem

Optimization methods

Numerical experience

Conclusions and future work
DIC microscopy

Differential-Interference-Contrast (DIC) microscopy is used to image transparent specimens not visible with ordinary transmitted-light microscopy without staining.

DIC images have a distinctive shadow-cast appearance.
DIC microscopy is a dual-beam polarized interference technique in which the polarized beams are first mutually orthogonal, then laterally separated and finally shifted in phase when passing through the specimen.

http://www.olympusmicro.com/primer/techniques/dic/dicintro.html
DIC imaging model

The acquisition of a grayscale DIC image is described by the following model [1]

\[ i(\mathbf{x}) = \int_{-\infty}^{+\infty} \alpha(\mathbf{\xi}) \left| \int_{-\infty}^{\infty} e^{-i\phi(\mathbf{x}_0)} h(\mathbf{x} - \mathbf{x}_0) h_c(\mathbf{\xi}; \mathbf{x}_0) d\mathbf{x}_0 \right|^2 d\mathbf{\xi} \]

where:

- \( \mathbf{\xi} = (\xi, \eta), \mathbf{x}_0 = (x_0, y_0) \) and \( \mathbf{x} = (x, y) \) are points in the front-focal plane of the condenser lens, in the object plane and in the image plane, respectively
- \( i(\mathbf{x}) \) is the DIC image intensity at pixel \( \mathbf{x} \)
- \( h(\mathbf{x}) \) is the complex-valued DIC point-spread function (PSF) at pixel \( \mathbf{x} \)
- \( \phi(\mathbf{x}_0) \) is the specimen’s phase function, which gives the phase difference between a pair of waves at point \( \mathbf{x}_0 \)
- \( \alpha(\mathbf{\xi}) \) is the illumination intensity
- \( h_c(\mathbf{\xi}; \mathbf{x}_0) \) specifies the complex amplitude of the illuminating plane waves

The image \( i(\mathbf{x}) \) is a nonlinear function of the phase differences between waves.

The DIC PSF and its Fourier Transform are expressed as

\[ h(x, y) = \frac{1}{2} e^{-i\Delta \theta} k(x - \Delta x, y) - \frac{1}{2} e^{i\Delta \theta} k(x + \Delta x, y) \]

\[ \mathcal{F} H(f, g) = -i \sin(2\pi f \Delta x + \Delta \theta) K(f, g) \]

where:

- \( k(x, y) \) is the coherent PSF of the microscope objective lens (FT of the pupil function)
- \( 2\Delta \theta \) is the DIC bias-retardation (expressed in rad)
- \( 2\Delta x \) is the shear distance (expressed in \( \mu m \))
- \( K(f, g) \) is the pupil function with radius \( f_c = \frac{NA}{\lambda} \), being NA the numerical aperture of the objective lens and \( \lambda \) the wavelength of the illumination light
Assuming coherent illumination, i.e. $\alpha(\xi) = \delta(\xi)$, the DIC model reduces to

$$i(x) = \left| \int_{-\infty}^{+\infty} e^{-i\phi(x_0)} h(x - x_0) dx_0 \right|^2$$

This is the so-called point-aperture model. If we also let $h(x, y)$ being an ideal PSF, namely

$$h(x, y) = \frac{1}{2} e^{-i\Delta \theta} \delta(x - \Delta x, y) - \frac{1}{2} e^{i\Delta \theta} \delta(x + \Delta x, y)$$

and $\Delta x \approx 0$, then the model further reduces to

$$i(x, y) \approx \sin^2 \left( \Delta x \frac{\partial \phi(x, y)}{\partial x} + \Delta \theta \right).$$

Thus the intensity of the DIC image is related to the first derivative of the specimen’s phase function along the direction of the shear or, equivalently, to the first derivative of the specimen’s optical path difference (OPD), if we recall that

$$\phi(x, y) = 2\pi \left( \frac{\text{OPL}_1 - \text{OPL}_2}{\lambda} \right)$$

where $\text{OPL} = nt$ is the product between the refractive index $n$ and thickness $t$. 
Disadvantages of DIC images

\[ i(x, y) \approx \sin^2 \left( \Delta x \frac{\partial \phi(x, y)}{\partial x} + \Delta \theta \right). \]

1st disadvantage: DIC imaging is direction-sensitive.

2nd disadvantage: The (false) three dimensional appearance of DIC images strongly depends on the value of the bias retardation \(2\Delta \theta\).
\[(o_k, \lambda_\ell)_j = \left|(h_{k, \lambda_\ell} \otimes e^{-i\phi/\lambda_\ell})_j\right|^2 + (\eta_{k, \lambda_\ell})_j, \quad k = 1, \ldots, K, \quad \ell = 1, 2, 3, \quad j \in \chi\]

- \(k\) is the index of the rotation of the specimen w.r.t. the horizontal axis, \(\ell\) is the index denoting one of the RGB channels and \(j = (j_1, j_2)\) is a 2D–index varying in the set \(\chi = \{1, \ldots, M\} \times \{1, \ldots, P\}, \quad N = MP\)
- \(\lambda_\ell\) is the \(\ell\)–th illumination wavelength
- \(o_{k, \lambda_\ell} \in \mathbb{R}^N\) is the \(\ell\)–th color component of the \(k\)–th observed image
- \(o_k = (o_{k, \lambda_1}, o_{k, \lambda_2}, o_{k, \lambda_3}) \in \mathbb{R}^{N \times 3}\)
- \(\phi \in \mathbb{R}^N\) is the unknown phase vector and \(e^{-i\phi/\lambda_\ell} \in \mathbb{C}^N\) is defined by
- \((e^{-i\phi/\lambda_\ell})_j = e^{-i\phi_j/\lambda_\ell}\)
- \(h_{k, \lambda_\ell} \in \mathbb{C}^N\) is the discretization of the continuous DIC Point Spread Function at the \(k\)-th direction and wavelength \(\lambda_\ell\)
- \(\eta_{k, \lambda_\ell} \in \mathbb{R}^N\) is the noise corrupting the data, \(\eta_{k, \lambda_\ell} \sim \mathcal{N}(0, \sigma^2 I_N)\).

Optimization problem

Problem:

given $K$ images $o_1, \ldots, o_K$, retrieve the phase function $\phi$ by solving

$$\min_{\phi \in \mathbb{R}^N} J(\phi) \equiv J_0(\phi) + J_{TV}(\phi),$$

(1)

- $J_0$ is the nonlinear least-squares distance

$$J_0(\phi) = \sum_{\ell=1}^{3} \sum_{k=1}^{K} \sum_{j \in \chi} \left[(o_k, x_{\ell})_j - \left| \left(h_k, x_{\ell} \otimes e^{-i\phi/\lambda_{\ell}}\right)_j \right|^2 \right]^2.$$

- $J_{TV}$ is the smooth total variation functional (also known as hypersurface potential - HS) defined as

$$J_{TV}(\phi) = \mu \sum_{j \in \chi} \sqrt{((D\phi)_j)_1^2 + ((D\phi)_j)_2^2 + \delta^2},$$

where $\mu > 0$ is a regularization parameter and $\delta > 0$ plays the role of a threshold for the gradient of the phase.

Remark:

- $J$ admits at least one minimum point (even if it is not coercive).
- Since $J(\phi + c1) = J(\phi)$ for all $c \in \mathbb{R}$, $J$ admits infinitely many minimum points.
Proposed method #1

Case $\delta > 0$: the DIC optimization problem is differentiable and nonconvex. We use a steepest descent method with variable steplengths:

$$\phi^{(n+1)} = \phi^{(n)} - \alpha_n \nabla J(\phi^{(n)}), \quad n = 0, 1, \ldots$$


Regard the matrix $B(\alpha_n) = (\alpha_n I)^{-1}$ as an approximation of the Hessian $\nabla^2 J(\phi^{(n)})$, by imposing a quasi-Newton property on $B(\alpha_n)$

$$\alpha_{n}^{BB1} = \arg \min_{\alpha \in \mathbb{R}} \| B(\alpha) s^{(n-1)} - z^{(n-1)} \|$$

where $s^{(n-1)} = \phi^{(n)} - \phi^{(n-1)}$ and $z^{(n-1)} = \nabla J(\phi^{(n)}) - \nabla J(\phi^{(n-1)})$.

Essentially based on the information from the last two iterations

$$\phi^{(n)}, \phi^{(n-1)}, \nabla J(\phi^{(n)}), \nabla J(\phi^{(n-1)})$$


Recently proposed for limited memory steepest descent methods. The gradients of the last $m$ it. are exploited ($m$ small, $m = 3, 4, 5$)

$$\nabla J(\phi^{(n)}), \ldots, \nabla J(\phi^{(n-m+1)})$$
Proposed method #1

Unconstrained quadratic problem:

\[
\min_{\phi \in \mathbb{R}^N} J(\phi), \quad J(\phi) = \frac{1}{2} \phi^T A \phi - \phi^T b, \quad g(\phi) = \nabla J(\phi)
\]

Basic properties

Consider the Krylov sequence \( \{g^{(n-m)}, Ag^{(n-m)}, \ldots, A^{m-1}g^{(n-m)}\} \)

- Lanczos iterative process generates orthonormal basis vectors for the Krylov sequence:
  \[
  Q = [q_1, \ldots, q_m]
  \]

- The eigenvalues (Ritz values) of the tridiagonal matrix
  \[
  T = Q^T A Q
  \]

are estimates of the eigenvalues \( \lambda_i \) of \( A \).
Proposed method #1

Main steps of the rule

- Define $T$ starting from

$$G = [g^{(n-m)}, \ldots, g^{(n-1)}] \quad (m \text{ small}; \ m=3,4,5)$$

without explicit use of $Q$ and $A$

- Compute the Ritz values $\theta_i, \ i = 1, \ldots, m$

- Exploit $\alpha_{n-1+i} = \frac{1}{\theta_i}, \ i = 1, \ldots, m$ for $m$ iterations of the gradient methods

$$\phi^{(n+i)} = \phi^{(n-1+i)} - \alpha_{n-1+i} g^{(n-1+i)}, \ i = 1, \ldots, m$$

For a general objective function $J(\phi)$, $T$ is upper Hessenberg and the Ritz-like values are obtained by computing the eigenvalues of a symmetric and tridiagonal approximation $\tilde{T}$ of $T$ defined as

$$\tilde{T} = \text{diag}(T) + \text{tril}(T, -1) + \text{tril}(T, -1)^T$$
Compute the tridiagonal matrix $T$

$$G = \begin{bmatrix} g^{(n-m)}, \ldots, g^{(n-1)} \end{bmatrix};$$

- $g^{(n-i)} \in \text{span}\{g^{(n-m)}, Ag^{(n-m)}, \ldots, A^{m-1}g^{(n-m)}\}, \quad i = 1, \ldots, m$

- There exists $R_{m \times m}$ upper triangular such that $G = QR$; then

$$T = QT AQ = R^{-T} G^T AG R^{-1} \quad \quad G^T G = R^T R$$

- We have $AG = \begin{bmatrix} G & g^{(n)} \end{bmatrix} \Gamma$ \hspace{1cm} $\Gamma = \begin{pmatrix} \alpha^{-1}_{n-m} \\ -\alpha^{-1}_{n-m} \\ \ddots \\ \ddots \\ \alpha^{-1}_{n-1} \\ -\alpha^{-1}_{n-1} \end{pmatrix}$

$$T = [R \ r] \Gamma R^{-1} \quad \text{where} \quad R^T r = G^T g^{(n)}$$

\[
\phi^{(n+1)} = \phi^{(n)} - \alpha_n g^{(n)} \quad \Rightarrow \quad A\phi^{(n+1)} = A\phi^{(n)} - \alpha_n Ag^{(n)} \\
\Rightarrow \quad g^{(n+1)} = g^{(n)} - \alpha_n Ag^{(n)}
\]
Proposed method #1: LMSD

**Algorithm 1** Limited memory steepest descent (LMSD) method

Set $\rho, \omega \in (0, 1)$, $m > 0$, $\alpha_0^{(0)}, \ldots, \alpha_{m-1}^{(0)} > 0$, $\phi^{(0)} \in \mathbb{R}^N$, $G = [\ ]$, $\Theta = [\ ]$, $n = 0$.

WHILE True

FOR $l = 1, \ldots, m$

1. Compute the smallest non-negative integer $i_n$ such that $\alpha_n = \alpha_n^{(0)} \rho^{i_n}$ satisfies
   \[ J(\phi^{(n)} - \alpha_n \nabla J(\phi^{(n)})) \leq J(\phi^{(n)}) - \omega \alpha_n \| \nabla J(\phi^{(n)}) \|^2. \]

2. Compute $\phi^{(n+1)} = \phi^{(n)} - \alpha_n \nabla J(\phi^{(n)})$.
3. Update $G = [G \ \nabla J(\phi^{(n)})]$ and $\Theta = [\Theta \ \alpha_n^{-1}]$.

4. Set $n = n + 1$.

END

5. Define the $(m + 1) \times m$ matrix $\Gamma = \begin{bmatrix} \text{diag}(\Theta) & 0 \\ 0 & \text{diag}(\Theta) \end{bmatrix}$.

6. Compute the Cholesky factorization $R^T R$ of the $m \times m$ matrix $G^T G$.

7. Solve the linear system $R^T r = G^T \nabla J(\phi^{(n)})$.

8. Define the $m \times m$ matrix $T = [R, r] \Gamma R^{-1}$ and its approximation $\tilde{T}$.

9. Compute eigenvalues $\theta_1, \ldots, \theta_m$ of $\tilde{T}$ and define $\alpha_n^{(0)} = 1/\theta_i$, $i = 1, \ldots, m$.

END
Theorem

Let \( J \) be the DIC functional, \( \{\phi^{(n)}\}_{n \in \mathbb{N}} \) the sequence generated by Algorithm 1 and assume that \( \alpha_n^{(0)} \leq \alpha_{\text{max}} \), where \( \alpha_{\text{max}} > 0 \). If \( \phi^* \) is a limit point of \( \{\phi^{(n)}\}_{n \in \mathbb{N}} \), then \( \phi^* \) is a stationary point of \( J \) and \( \phi^{(n)} \) converges to \( \phi^* \).

The proof relies onto the fact that the objective function satisfies a specific analytical property (the Kurdyka-Łojasiewicz property) [1,2,3].

Case $\delta = 0$: the DIC optimization problem is non differentiable and nonconvex. We use a proximal-gradient method with variable steplengths:

**Algorithm 2 Inexact Line–search based Algorithm (ILA) [1,2]**

Set $\rho, \omega \in (0, 1)$, $0 < \alpha_{\text{min}} \leq \alpha_{\text{max}}$, $\tau > 0$, $\phi(0) \in \mathbb{R}^N$, $n = 0$. FOR $n = 0, 1, 2, \ldots$

1. Choose $\alpha_n = \max \left\{ \min \left\{ \alpha_n^{(0)}, \alpha_{\text{max}} \right\}, \alpha_{\text{min}} \right\}$, where $\alpha_n^{(0)}$ is as in Algorithm 1.

2. Let $\psi(n) = \text{prox}_{\alpha_n J_{TV}} \left( \phi(n) - \alpha_n \nabla J_0(\phi(n)) \right) = \arg\min_{\phi \in \mathbb{R}^N} h(n)(\phi)$.

   Compute $\tilde{\psi}(n) \in \mathbb{R}^N$ such that
   $$h(n)(\tilde{\psi}(n)) - h(n)(\psi(n)) \leq \epsilon_n$$
   $$0 \leq \epsilon_n \leq -\tau h(n)(\tilde{\psi}(n)).$$

3. Set $d(n) = \tilde{\psi}(n) - \phi(n)$.

4. Compute the smallest non-negative integer $i_n$ such that $\lambda_n = \rho^{i_n}$ satisfies
   $$J(\phi(n) + \lambda_n d(n)) \leq J(\phi(n)) + \omega \lambda_n h(n)(\tilde{\psi}(n)).$$

5. Compute the new point as $\phi(n+1) = \phi(n) + \lambda_n d(n)$.

END


Let $J$ be the DIC functional and \{\phi^{(n)}\}_{n \in \mathbb{N}} sequence generated by Algorithm 2. Any limit point $\phi^*$ of $\{\phi^{(n)}\}_{n \in \mathbb{N}}$ is a stationary point of $J$.

Convergence under the Kurdyka-Łojasiewicz property can be proved when the proximal point is computed exactly [1].

Comparison #1: LMSD versus standard gradient method

Microscope parameters:
- Shear: $2\Delta x = 0.68 \, \mu m$
- Bias: $2\Delta \theta = \pi/2 \, \text{rad}$
- Numerical aperture: $\text{NA} = 0.3$
- $K = 2$ shear directions

Regularization parameters $\mu$ and $\delta$:
- Cone: $\mu = 10^{-2}, \delta = 10^{-2}$
- Cross: $\mu = 4 \cdot 10^{-2}, \delta = 10^{-3}$

Methods:
- **LMSD**: steepest descent equipped with Ritz values and monotone linesearch
- **BB1**: steepest descent equipped with BB1 rule and nonmonotone linesearch

Error:
$$\|\phi^{(n)} - \phi^* - \bar{c}1\| / \|\phi^*\|$$
where $\bar{c} = \sum_{x \in \chi} [\phi^{(n)}(x) - \phi^*(x)] / N$
Algorithm 3 Conjugate gradient (CG) method

Choose $\phi^{(0)} \in \mathbb{R}^N$ and set $n = 0$, $p^{(0)} = -\nabla J(\phi^{(0)})$, $\epsilon > 0$.
While $\|\nabla J(\phi^{(n)})\| > \epsilon$

1. Compute $\alpha_n$ and set $\phi^{(n+1)} = \phi^{(n)} + \alpha_n p^{(n)}$ ← line search parameter
2. Evaluate $\nabla J(\phi^{(n+1)})$
3. Choose the scalar parameter $\beta_{n+1}$ ← CG direction parameter
4. $p^{(n+1)} = -\nabla J(\phi^{(n+1)}) + \beta_{n+1} p^{(n)}$
5. $n = n + 1$

End

<table>
<thead>
<tr>
<th>CG algorithm</th>
<th>$\beta_{n+1}^{\text{FR}} = \frac{\nabla J(\phi^{(n+1)})^T \nabla J(\phi^{(n+1)})}{\nabla J(\phi^{(n)})^T \nabla J(\phi^{(n)})}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fletcher-Reeves (FR)</td>
<td>$\beta_{n+1}^{\text{PR}} = \frac{\nabla J(\phi^{(n+1)})^T (\nabla J(\phi^{(n+1)}) - \nabla J(\phi^{(n)}))}{\nabla J(\phi^{(n)})^T \nabla J(\phi^{(n)})}$</td>
</tr>
<tr>
<td>Polak-Ribière (PR)</td>
<td>$\beta_{n+1}^{\text{PR}+} = \max(\beta_{n+1}^{\text{PR}}, 0)$</td>
</tr>
</tbody>
</table>
| PR$^+$               | $\beta_{n+1}^{\text{FR–PR}} = \begin{cases} 
\beta_{n+1}^{\text{PR}} & \text{if } |\beta_{n+1}^{\text{PR}}| \leq \beta_{n+1}^{\text{FR}} \\
\beta_{n+1}^{\text{FR}} & \text{otherwise} \end{cases}$ |

Table: Choice of the parameter $\beta_{n+1}$ in CG methods.
Choice of the CG line search parameter #1

Require the steplength $\alpha_n$ to satisfy the strong Wolfe conditions [1]

\[
J(\phi^{(n)} + \alpha_n p^{(n)}) \leq J(\phi^{(n)}) + c_1 \alpha_n \nabla J(\phi^{(n)})^T p^{(n)} \\
|\nabla J(\phi^{(n)} + \alpha_n p^{(n)})^T p^{(n)}| \leq c_2 |\nabla J(\phi^{(n)})^T p^{(n)}|
\]

where $0 < c_1 < c_2 < \frac{1}{2}$ (e.g. $c_1 = 10^{-4}$, $c_2 = 0.1$).

✓ Let $\beta_n = \beta_n^{FR}$ or $\beta_n = \beta_n^{FR-PR}$. If the strong Wolfe conditions hold, each limit point of the sequence $\{\phi^{(n)}\}_{n \in \mathbb{N}}$ is stationary.

✓ If $\beta_n = \beta_n^{PR^+}$, the same convergence result hold by also assuming that the following descent condition is satisfied

\[
\nabla J(\phi^{(n)})^T p^{(n)} \leq -c_3 \|\nabla J(\phi^{(n)})\|^2, \quad \text{with } 0 < c_3 \leq 1.
\]

✗ In the DIC problem the evaluation of the gradient $\nabla J$ is computational demanding and its nonlinearity w.r.t $\alpha$ requires a new computation for each step of the backtracking loop

Algorithm 4 Line–search based on polynomial approximation [1]

Let $\psi(\alpha) := J(\phi^{(n)} + \alpha p^{(n)})$, set $t > 0$, $a = 0$, $b = t$ and compute $\psi(a)$ and $\psi(b)$.

1. Find a point $c \in [a, b]$ such that $\Psi(a) > \Psi(c) < \Psi(b)$ as follows
   
   If $\psi(b) < \psi(a)$
   
   Set $c = 2b$ and compute $\psi(c)$.
   
   While $\psi(c) \leq \psi(b)$
   
   Set $a = b$, $b = c$, $c = 2c$ and compute $\psi(c)$.
   
   End
   
   Else
   
   Set $c = \frac{b}{2}$ and compute $\psi(c)$.
   
   While $\psi(c) \geq \psi(a)$
   
   Set $b = c$, $c = \frac{c}{2}$ and compute $\psi(c)$.
   
   End
   
   End

2. Compute $\alpha_n$ as the minimum point of the parabola interpolating the points $(a, \psi(a))$, $(b, \psi(b))$, $(c, \psi(c))$.

✓ "Gradient-free" linesearch

✗ Quite sensitive to the choice of the parameter $t$ and no guarantee of convergence


Three-point approximation
Comparison #2: LMSD and ILA versus CG methods

![Error VS Time for the cone (left) and cross (right). Noise level SNR = 4.5 dB.](image)

<table>
<thead>
<tr>
<th>Object</th>
<th>Algorithm</th>
<th>Iterations</th>
<th># f</th>
<th># g</th>
<th>Time (s)</th>
<th>Obj fun</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>FR–PA</td>
<td>303</td>
<td>2164</td>
<td>303</td>
<td>11.74</td>
<td>8.57</td>
<td>3.63 %</td>
<td></td>
</tr>
<tr>
<td>PR–PA</td>
<td>98</td>
<td>997</td>
<td>98</td>
<td>4.97</td>
<td>8.57</td>
<td>3.63 %</td>
<td></td>
</tr>
<tr>
<td>FR–SW</td>
<td>299</td>
<td>1705</td>
<td>1705</td>
<td>24.28</td>
<td>8.57</td>
<td>3.63 %</td>
<td></td>
</tr>
<tr>
<td>Cross</td>
<td>FR-PR–SW</td>
<td>96</td>
<td>300</td>
<td>4.41</td>
<td>8.57</td>
<td>3.63 %</td>
<td></td>
</tr>
<tr>
<td>PR⁺–SW</td>
<td>98</td>
<td>326</td>
<td>326</td>
<td>4.74</td>
<td>8.57</td>
<td>3.63 %</td>
<td></td>
</tr>
<tr>
<td>LMSD</td>
<td>152</td>
<td>221</td>
<td>152</td>
<td>2.75</td>
<td>8.57</td>
<td>3.64 %</td>
<td></td>
</tr>
<tr>
<td>ILA</td>
<td>97</td>
<td>179</td>
<td>97</td>
<td>5.26</td>
<td>8.47</td>
<td>3.46 %</td>
<td></td>
</tr>
</tbody>
</table>

# f: total number of function evaluations. # g: total number of gradient evaluations.

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Conclusions:

- we have proposed a steepest descent method (LMSD) and a proximal–gradient method (ILA) to address the problem of phase estimation from DIC images
- both methods exploit a recently proposed rule to update the steplength parameter
- accurate reconstruction of the DIC phase are obtained with a reduced computational time with respect to widely used conjugate gradient methods

Future work:

- application to real images
- reparametrize the DIC problem by setting $u = e^{-i\phi}$ and therefore minimize

$$
\min_{u \in \mathbb{C}^N} \sum_{k=1}^{K} \sum_{j=1}^{N} \left[ o_{k,j} - \left| (h_k \otimes u)_j \right|^2 \right]^2.
$$