Spectral analysis of matrices arising in isogeometric methods with GB-splines

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Motivations and goals

- Linear PDE $\mathcal{L} u = g$: implies a numerical method $A_n u_n = g_n$.
- $N_n := \dim(A_n) \to \infty$ as $n \to \infty$, $\{A_n\}_n$ sequence of matrices;
- $\{A_n\}$ has an asymptotic spectral distribution described by a symbol $f$.
- $f$ could be useful for:
  - analysis of the spectral properties of $A_n$ for large $n$;
  - design of fast (iterative) solvers;
  - theoretical tool to analyze whether the numerical method is appropriate to approximate the spectrum of $\mathcal{L}$
- Here we focus on IgA discretizations by means of GB-splines.
Summary

1. Preliminaries on GB-splines
2. Preliminaries on spectral analysis
3. Setting for spectral analysis
4. Spectral analysis of GB-Galerkin matrices
5. Spectral analysis of GB-collocation matrices
Preliminaries on GB-splines

Why defining GB-splines (generalized B-splines)

- Polynomial B-splines, and their rational form NURBS: standard in CAGD for decades, and also in IgA.

- Drawbacks:
  - B-splines do not allow the exact description of conic sections
  - NURBS are difficult to integrate/differentiate

- Popular GB-splines include trigonometric and hyperbolic functions:
  - trigonometric GBs: circles, ellipses, helices...
  - hyperbolic GBs: hyperbolae, catenaries...
Preliminaries on GB-splines

GB-splines

NOTES ON GB-SPLINES

Smooth functions belonging piecewise in
\[ \mathbb{P}^U_V := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle \]

Noteworthy cases:

1. \((U(t), V(t)) = (t^{p-1}, t^p)\) (polynomial)
2. \((U(t), V(t)) = (\cos(\omega t), \sin(\omega t))\) (trigonometric)
3. \((U(t), V(t)) = (\cosh(\alpha t), \sinh(\alpha t))\) (hyperbolic)
Preliminaries on GB-splines

GB-splines

\[ \mathbb{P}_p^{U,V} := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle \]

Knot sequence \( \{\xi_1, \ldots, \xi_{2p+n+1}\} \) to define GB-splines of degree \( p \) denoted by \( N_{1,p}^{U,V}(t), \ldots, N_{n,p}^{U,V}(t) \) and defined recursively.

- Properties:
  - positivity
  - partition of unity (for \( p \geq 2 \))
  - compact minimum support
  - order of smoothness
  - easy derivative expression
  - local linear independence
  - convergence to the polynomial ones when parameter \( \rightarrow 0 \) (trig/hyp)
Preliminaries on GB-splines

Cardinal GB-splines

Knots (degree $p$): $0, 1, \ldots, p + 1$. $\phi_p^Q$, $Q = \mathbb{H}, \mathbb{T}$

Figure: Degree 1.

Figure: Degree 3.

Figure: Cardinal splines: B-spline (black), trig GB-spline $\omega = 1$ (blue) and $\omega = 3$ (green), hyp GB-spline $\alpha = 1$ (red) and $\alpha = 10$ (yellow).
Preliminaries on GB-splines

Fourier transform expressions

\[ \widehat{\phi_p}(\theta) = \left( \frac{1 - e^{-i\theta}}{i\theta} \right)^{p+1} \]

\[ \widehat{\phi_p^{T\omega}}(\theta) = \frac{\omega^2}{1 - \cos(\omega)} \cdot \frac{e^{-i\theta} \cdot (\cos(\theta) - \cos(\omega))}{(\omega^2 - \theta^2)} \cdot \left( \frac{1 - e^{-i\theta}}{i\theta} \right)^{p-1} \]

\[ \widehat{\phi_p^{H\alpha}}(\theta) = \frac{\alpha^2}{\cosh(\alpha) - 1} \cdot \frac{e^{-i\theta} \cdot (\cosh(\alpha) - \cos(\theta))}{(\alpha^2 + \theta^2)} \cdot \left( \frac{1 - e^{-i\theta}}{i\theta} \right)^{p-1} \]

We note that \( \phi_p^{H\alpha} = \phi_p^{T_{i\alpha}} \), so by computing in \( \mathbb{C} \), \( \phi_p^{T\omega} \) and \( \phi_p^{H\alpha} \) are the same case. Hyperbolic is trigonometric with \( \alpha := i\omega \).
Distribution in the sense of the eigenvalues

\{X_n\} sequence of matrices, \(\dim(X_n) = d_n < d_{n+1}\), \(f : D \subset \mathbb{R}^d \to \mathbb{C}\).

\(\{X_n\}\) is distributed like \(f\), \(\{X_n\} \sim \lambda f\), if, \(\forall F \in C_c(\mathbb{C}, \mathbb{C})\)

\[
\lim_{n \to \infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(f(x_1, \ldots, x_d)) \, dx_1 \cdots dx_d.
\]

The function \(f\) is referred to as the symbol of the sequence \(\{X_n\}\).
Informally: for high \(n\), spectrum of \(X_n \approx\) uniform sampling of \(f\) over \(D\).

*Small* (in a sense) perturbations of \(\{X_n\} \sim \lambda f\) are still \(\sim \lambda f\).
Preliminaries on spectral analysis

Association function/sequence of Toeplitz matrices

\( f : [-\pi, \pi] \rightarrow \mathbb{R} \in L_1([-\pi, \pi]) \) is associated with \( \{ T_m(f) \}_m \):

\[
T_m(f) := \begin{bmatrix}
    f_0 & f_{-1} & \cdots & \cdots & f_{-(m-1)} \\
    f_1 & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    \vdots & & \ddots & \ddots & \vdots \\
    f_{m-1} & \cdots & \cdots & f_1 & f_0
\end{bmatrix} \in \mathbb{C}^{m \times m}
\]

\( f_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \exp(-ik\theta) d\theta, \quad k \in \mathbb{Z} \)

\( f_k \) : Fourier coefficients of \( f \)

\[ \{ T_m(f) \}_m \sim \lambda f \]
Spectral analysis of GB-Galerkin matrices

A second order linear elliptic PDE in dimension 1

\[
\begin{aligned}
- u'' + \beta u' + \gamma u &= f, \quad 0 < x < 1, \\
u(0) &= 0, \quad u(1) = 0
\end{aligned}
\]

\(f \in L_2(0, 1), \beta \in \mathbb{R}, \gamma \geq 0.\)

Weak form: find \(u \in V := H_0^1(\Omega)\) such that

\[
a(u, v) = F(v), \forall v \in V
\]

\[
a(u, v) := \int_\Omega (\nabla u \cdot \nabla v + \beta \cdot \nabla uv + \gamma uv) d\Omega, \quad F(v) := \int_\Omega fvd\Omega.
\]

\[
\langle \varphi_1, \ldots, \varphi_N \rangle = W \subset V, \text{ seek } u_W \in W \text{ s.t. } a(u_W, v) = F(v), \forall v \in W
\]

\[
u = [u_1 \cdots u_N]^T, \quad A = [a(\varphi_j, \varphi_i)]_{i,j=1}^N, \quad f = [F(\varphi_i)]_{i=1}^N; \quad u_W = \sum_{j=1}^N u_j \varphi_j.
\]

\[
Au = f
\]

\[
\varphi_i := N_{i+1,p}^{U,V} \quad \text{GB-splines}
\]
Spectral analysis of GB-Galerkin matrices

Decompositions

Sequence: \( \frac{1}{n} A_{n,p}^Q = K_{n,p}^Q + \frac{\beta}{n} H_{n,p}^Q + \frac{\gamma}{n^2} M_{n,p}^Q, \quad A_{n,p}^Q = A \)

- \( K_{n,p}^Q = \left[ \frac{1}{n} \int_0^1 (N_{j+1,p}^Q)'(x)(N_{i+1,p}^Q)'(x) \, dx \right]_{i,j=1}^{n+p-2} \)

- \( K_{n,p}^Q = T_{n+p-2}(f_{p}^Q/n) + \text{small rank matrix} \)

- \( M_{n,p}^Q = \left[ n \int_0^1 N_{j+1,p}^Q(x)N_{i+1,p}^Q(x) \, dx \right]_{i,j=1}^{n+p-2} \)

- \( M_{n,p}^Q = T_{n+p-2}(h_{p}^Q/n) + \text{small rank matrix} \)

- \( H_{n,p}^Q = \left[ \int_0^1 (N_{j+1,p}^Q)'(x)N_{i+1,p}^Q(x) \, dx \right]_{i,j=1}^{n+p-2} \) has imaginary spectrum

Small (in a sense) perturbations of \( \{X_n\} \sim f \) are still \( \sim f \).
Lemma on $h_p^Q\mu$

Let $p \geq 2$, and let $h_p^Q\mu : [-\pi, \pi] \to \mathbb{R}$,

$$h_p^Q\mu(\theta) := \sum_{k=-p}^{p} \int_{\mathbb{R}} \phi_p^Q\mu(t) \phi_p^Q\mu(t - k) dt \cdot \cos(k\theta)$$

Then, $\forall \theta \in [-\pi, \pi]$:

1. $h_p^Q\mu(\theta) = \sum_{k \in \mathbb{Z}} \left| \widehat{\phi_p^Q\mu}(\theta + 2k\pi) \right|^2$.
2. $h_p^Q\mu(\theta) \leq 1$, $h_p^Q\mu(\theta) \geq c_{\alpha, p} > 0$ ($Q = \mathbb{H}, \mathbb{T}$).

$h_p$: [Garoni, Manni, Pelosi, Serra-Capizzano, Speleers; 2014]

$h_p^Q\mu$: [Roman, Manni, Speleers; 2016]
Lemma on $f_p^{Q,\mu}$

Let $p \geq 3$, and let $f_p^{Q,\mu} : [-\pi, \pi] \rightarrow \mathbb{R}$,

$$f_p^{Q,\mu}(\theta) := \sum_{k=-p}^{p} \int_{\mathbb{R}} \phi_p^{Q,\mu}(t) \phi_p^{Q,\mu}(t-k) dt \cdot \cos(k\theta)$$

and $M_{f_p^{Q,\mu}} := \max_{\theta \in [-\pi, \pi]} f_p^{Q,\mu}(\theta)$. Then:

1. $\forall \theta \in [-\pi, \pi], \quad f_p^{Q,\mu}(\theta) = (2 - 2\cos\theta) h_p^{Q,\mu}(\theta)$;

2a. $\min_{\theta \in [-\pi, \pi]} f_p^{Q,\mu}(\theta) = f_p^{Q,\mu}(0) = 0$;

2b. $\theta = 0$ unique zero of $f_p^{Q,\mu}$ over $[-\pi, \pi]$;

2c. $M_{f_p^{Q,\mu}} \rightarrow 0$ as $p \rightarrow \infty$. 
Spectral analysis of GB-Galerkin matrices

Figure: [top] $h_{p}$ (left), $h_{p}^{T1}$ and $h_{p}^{T3}$ (center), $h_{p}^{H1}$ and $h_{p}^{H10}$ (right).

[bottom] $f_{p}$ (left), $f_{p}^{T1}$ and $f_{p}^{T3}$ (center), $f_{p}^{H1}$ and $f_{p}^{H10}$ (right).

Dashed line: higher parameter; different colors: $p = 3, \ldots, 8$, downward.
Nested case vs non-nested case

\[ N_{i,p}^{U,V}(x) = \phi_p^{(U,V)(\cdot/n)}(nx - i + p + 1), \quad i = p + 1, \ldots, n \]

From this one, two associations: \( N_{i,p}^{Qn,\alpha} \) with \( \phi_p^{Q\alpha} \), and \( N_{i,p}^{Q\alpha} \) with \( \phi_p^{Q\alpha/n} \)

- **Nested case**: constant parameter in \( n \) for \( N_{i,p}^{Q\bullet} \); spaces implied
  \[ \langle 1, t, \ldots, t^{p-2}, \cos(\omega t), \sin(\omega t) \rangle \text{ and } \langle 1, t, \ldots, t^{p-2}, \cosh(\alpha t), \sinh(\alpha t) \rangle. \]

- **Non-nested case**: constant parameter in \( n \) for \( \phi_p^{Q\bullet} \); spaces implied
  \[ \langle 1, t, \ldots, t^{p-2}, \cos(n\omega t), \sin(n\omega t) \rangle, \langle 1, t, \ldots, t^{p-2}, \cosh(n\alpha t), \sinh(n\alpha t) \rangle. \]
Distribution theorem

Nested case:

The sequence of matrices \( \{ \frac{1}{n} A_{n,p}^{\alpha} \} \) is distributed like the function \( f_p \) in the sense of the eigenvalues.

Non-nested case:

The sequence of matrices \( \{ \frac{1}{n} A_{n,p}^{\alpha} \} \) is distributed like the function \( f_p^{\alpha} \) in the sense of the eigenvalues.
A second order linear elliptic PDE in dimension 1

\[
\begin{aligned}
- u'' + \beta u' + \gamma u &= f, & 0 < x < 1, \\
u(0) &= 0, & u(1) = 0
\end{aligned}
\]

We look for a function \( u_\mathcal{W} \in \langle \phi_1, \ldots, \phi_N \rangle =: \mathcal{W} \) such that

\[
-u''_{\mathcal{W}}(\tau_i) + \beta u'_{\mathcal{W}}(\tau_i) + \gamma u_{\mathcal{W}}(\tau_i) = f(\tau_i), \quad \tau_i \text{ collocation points}
\]

\[
u_{\mathcal{W}} = \sum_{j=1}^{N} u_j \phi_j, \quad \text{solve} \quad A u = f, \quad u = [u_1 \cdots u_N]^T, \quad f = [f(\tau_i)]_{i=1}^{N}
\]

\[A := [-\phi''_{j}(\tau_i) + \beta \phi'_{j}(\tau_i) + \gamma \phi_{j}(\tau_i)]_{i,j=1}^{N} \in \mathbb{R}^{N \times N}\]

\[\phi_i := N_{i+1,p}^{U,V} \quad \text{GB-splines}, \quad \tau_i \text{ Greville abscissae}\]
Distribution theorem

Nested case:
The sequence of matrices \( \left\{ \frac{1}{n^2} A_{C,n,p}^{Q,\alpha} \right\} \) is distributed like a function \( f_{C,p} \) in the sense of the eigenvalues.

Non-nested case:
The sequence of matrices \( \left\{ \frac{1}{n^2} A_{C,n,p}^{Q,\alpha} \right\} \) is distributed like a function \( f_{C,p}^{Q,\alpha} \) in the sense of the eigenvalues.

- \( f_{C,2p+1} = f_{p} \), \( f_{C,2p+1}^{Q,\alpha} \rightarrow f_{P,\alpha}^{Q,\alpha} \) if \( \alpha \rightarrow 0 \);
- \( -\kappa(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) = f(x) \) in \((a, b)\) can be considered;
- \( G : [0, 1] \rightarrow [a, b] \), s.t. \( \exists G^{-1}, G(\partial([0, 1])) = \{a, b\} \), as a geometry map.
Concluding remarks

We show that...

- GB-splines:
  - enjoy the same properties of B-splines, and can be refined similarly;
  - plug-to-plug compatible with B-splines in IgA.

- GB-splines behave similarly to B-splines and NURBS in IgA, with improvements;

- IgA Galerkin/collocation methods based on GB-splines ⇒ matrices with spectral properties similar to the polynomial case;

- This ensures that multi-iterative techniques for B-splines discretizations can be adapted to GB-spline setting.


Thanks!