Spectral analysis of matrices arising in isogeometric methods with GB-splines

Carla Manni[a] **Fabio Roman**[b] Hendrik Speleers[a]

[a] Università degli Studi di Roma Tor Vergata
[b] Università degli Studi di Torino

Due Giorni di Algebra Lineare Numerica - Como, February 16, 2017

Foreword

Motivations and goals

- Linear PDE $\mathcal{L}u = g$: implies a numerical method $A_n \mathbf{u}_n = \mathbf{g}_n$.
- $N_n := \dim(A_n) \to \infty$ as $n \to \infty$, $\{A_n\}_n$ sequence of matrices;
- $\{A_n\}$ has an asymptotic spectral distribution described by a *symbol* f.
- f could be useful for:
  - analysis of the spectral properties of A_n for large n;
  - design of fast (iterative) solvers;
  - theoretical tool to analyze whether the numerical method is appropriate to approximate the spectrum of $\mathcal{L}$
- Here we focus on IgA discretizations by means of GB-splines.

Summary

Preliminaries on GB-splines

Why defining GB-splines (generalized B-splines)

- Polynomial B-splines, and their rational form NURBS: standard in CAGD for decades, and also in IgA.
- Drawbacks:
  - B-splines do not allow the exact description of conic sections
  - NURBS are difficult to integrate/differentiate
- Popular GB-splines include trigonometric and hyperbolic functions:
  - trigonometric GBs: circles, ellipses, helices...
  - hyperbolic GBs: hyperbolae, catenaries...

Preliminaries on GB-splines

GB-splines

[Manni, Pelosi, Sampoli; 2011]

Smooth functions belonging piecewise in

$$\mathbb{P}_p^{U,V} := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle$$

Noteworthy cases:

- $(U(t), V(t)) = (t^{p-1}, t^p)$ (polynomial)
- $(U(t), V(t)) = (\cos(\omega t), \sin(\omega t))$ (trigonometric)
- $(U(t), V(t)) = (\cosh(\alpha t), \sinh(\alpha t))$ (hyperbolic)

Preliminaries on GB-splines

GB-splines

$$\mathbb{P}_p^{U,V} := \langle 1, t, \ldots, t^{p-2}, U(t), V(t) \rangle$$

Knot sequence $\{\xi_1, \ldots, \xi_{2p+n+1}\}$ to define GB-splines of degree p denoted by $N_{1,p}^{U,V}(t), \ldots, N_{n,p}^{U,V}(t)$ and defined recursively.

- Properties:
  - positivity
  - partition of unity (for $p \geq 2$)
  - compact minimum support
  - order of smoothness
  - easy derivative expression
  - local linear independence
  - convergence to the polynomial ones when parameter $\to 0$ (trig/hyp)

Preliminaries on GB-splines

Cardinal GB-splines

Knots (degree p): $0, 1, \ldots, p+1$. $\phi_p^{\mathbb{Q}_\alpha}$, $\mathbb{Q} = \mathbb{H}, \mathbb{T}$

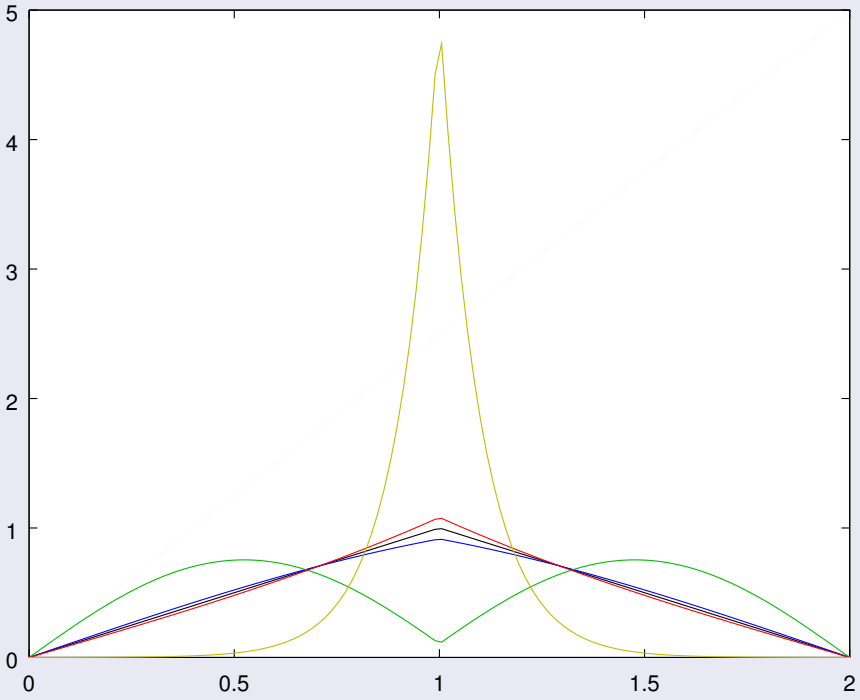

Figure : Degree 1.

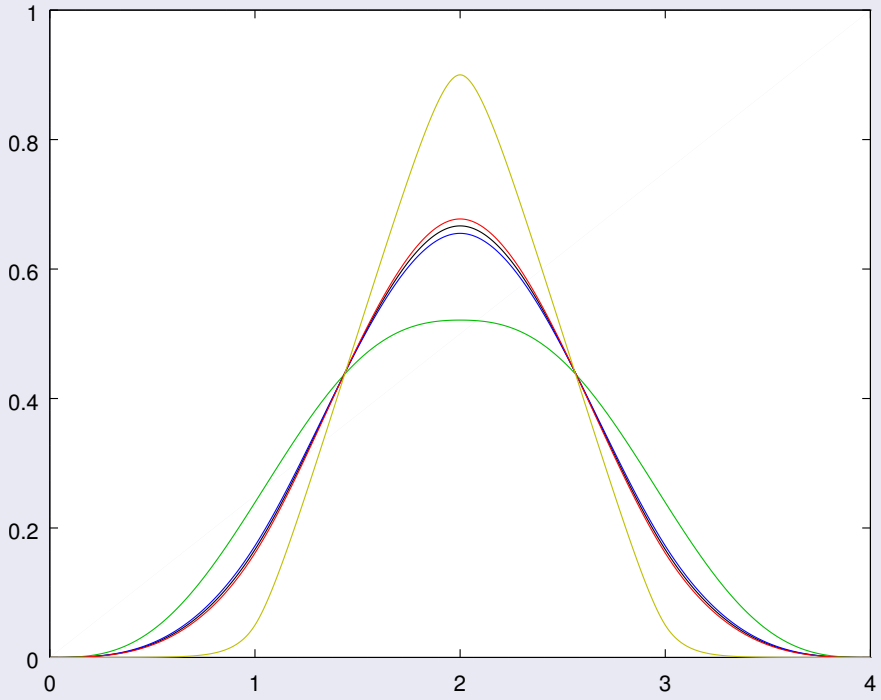

Figure : Degree 3.

Figure : Cardinal splines: B-spline (black), trig GB-spline $\omega = 1$ (blue) and $\omega = 3$ (green), hyp GB-spline $\alpha = 1$ (red) and $\alpha = 10$ (yellow).

Preliminaries on GB-splines

Fourier transform expressions

$$\widehat{\phi_p}(\theta) = \left(\frac{1 - e^{-i\theta}}{i\theta}\right)^{p+1}$$

$$\widehat{\phi_p^{\mathbb{T}_\omega}}(\theta) = \frac{\omega^2}{1 - \cos(\omega)} \cdot \frac{e^{-i\theta} \cdot (\cos(\theta) - \cos(\omega))}{(\omega^2 - \theta^2)} \cdot \left(\frac{1 - e^{-i\theta}}{i\theta}\right)^{p-1}$$

$$\widehat{\phi_p^{\mathbb{H}_\alpha}}(\theta) = \frac{\alpha^2}{\cosh(\alpha) - 1} \cdot \frac{e^{-i\theta} \cdot (\cosh(\alpha) - \cos(\theta))}{(\alpha^2 + \theta^2)} \cdot \left(\frac{1 - e^{-i\theta}}{i\theta}\right)^{p-1}$$

We note that $\phi_p^{\mathbb{H}_\alpha} = \phi_p^{\mathbb{T}_{i\alpha}}$, so by computing in $\mathbb{C}$, $\phi_p^{\mathbb{T}_\omega}$ and $\phi_p^{\mathbb{H}_\alpha}$ are the same case. Hyperbolic is trigonometric with $\alpha := i\omega$.

Preliminaries on spectral analysis

Distribution in the sense of the eigenvalues

$\{X_n\}$ sequence of matrices, $\dim(X_n) = d_n < d_{n+1}$, $f : D \subset \mathbb{R}^d \to \mathbb{C}$.
$\{X_n\}$ is distributed like f, $\{X_n\} \sim_\lambda f$, if, $\forall F \in C_c(\mathbb{C}, \mathbb{C})$

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(f(x_1, \ldots, x_d))\, \mathrm{d}x_1 \cdots \mathrm{d}x_d.$$

The function f is referred to as the symbol of the sequence $\{X_n\}$.
Informally: for high n, spectrum of $X_n \approx$ uniform sampling of f over D.

- *Small* (in a sense) perturbations of $\{X_n\} \sim_\lambda f$ are still $\sim_\lambda f$.

Preliminaries on spectral analysis

Association function/sequence of Toeplitz matrices

$f : [-\pi, \pi] \to \mathbb{R} \in L_1([-\pi, \pi])$ is associated with $\{T_m(f)\}_m$:

$$T_m(f) := \begin{bmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-(m-1)} \\ f_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & f_{-1} \\ f_{m-1} & \cdots & \cdots & f_1 & f_0 \end{bmatrix} \in \mathbb{C}^{m \times m}$$

$$f_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \exp(-ik\theta) d\theta, \qquad k \in \mathbb{Z}$$

f_k : Fourier coefficients of f

- $\{T_m(f)\}_m \sim_\lambda f$

Spectral analysis of GB-Galerkin matrices

A second order linear elliptic PDE in dimension 1

$$\begin{cases} -u'' + \beta u' + \gamma u = f, & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$

$f \in L_2(0,1), \beta \in \mathbb{R}, \gamma \geq 0.$

Weak form: find $u \in \mathcal{V} := H_0^1(\Omega)$ such that $\quad a(u,v) = F(v), \forall v \in \mathcal{V}$

$$a(u,v) := \int_\Omega (\nabla u \cdot \nabla v + \beta \cdot \nabla u v + \gamma u v) d\Omega, \qquad F(v) := \int_\Omega f v d\Omega.$$

$\langle \varphi_1, \ldots, \varphi_N \rangle = \mathcal{W} \subset \mathcal{V}$, seek $u_\mathcal{W} \in \mathcal{W}$ s.t. $\quad a(u_\mathcal{W}, v) = F(v), \forall v \in \mathcal{W}$
$\mathbf{u} = [u_1 \cdots u_N]^T$, $A = [a(\varphi_j, \varphi_i)]_{i,j=1}^N$, $\mathbf{f} = [F(\varphi_i)]_{i=1}^N$; $u_\mathcal{W} = \sum_{j=1}^N u_j \varphi_j$.

$$A\mathbf{u} = \mathbf{f}$$

$$\varphi_i := N_{i+1,p}^{U,V} \quad \text{GB-splines}$$

Spectral analysis of GB-Galerkin matrices

Decompositions

Sequence: $\frac{1}{n}A_{n,p}^{\mathbb{Q}_\mu} = K_{n,p}^{\mathbb{Q}_\mu} + \frac{\beta}{n}H_{n,p}^{\mathbb{Q}_\mu} + \frac{\gamma}{n^2}M_{n,p}^{\mathbb{Q}_\mu}, \quad A_{n,p}^{\mathbb{Q}_\mu} = A$

- $K_{n,p}^{\mathbb{Q}_\mu} = \left[\frac{1}{n}\int_0^1 (N_{j+1,p}^{\mathbb{Q}_\mu})'(x)(N_{i+1,p}^{\mathbb{Q}_\mu})'(x)dx\right]_{i,j=1}^{n+p-2}$
  - $K_{n,p}^{\mathbb{Q}_\mu} = T_{n+p-2}(f_p^{\mathbb{Q}_\mu/n}) +$ small rank matrix
- $M_{n,p}^{\mathbb{Q}_\mu} = \left[n\int_0^1 N_{j+1,p}^{\mathbb{Q}_\mu}(x)N_{i+1,p}^{\mathbb{Q}_\mu}(x)dx\right]_{i,j=1}^{n+p-2}$
  - $M_{n,p}^{\mathbb{Q}_\mu} = T_{n+p-2}(h_p^{\mathbb{Q}_\mu/n}) +$ small rank matrix
- $H_{n,p}^{\mathbb{Q}_\mu} = \left[\int_0^1 (N_{j+1,p}^{\mathbb{Q}_\mu})'(x)N_{i+1,p}^{\mathbb{Q}_\mu}(x)dx\right]_{i,j=1}^{n+p-2}$ has imaginary spectrum

- *Small* (in a sense) perturbations of $\{X_n\} \sim_\lambda f$ are still $\sim_\lambda f$.

Spectral analysis of GB-Galerkin matrices

Lemma on $h_p^{\mathbb{Q}_\mu}$

Let $p \geq 2$, and let $h_p^{\mathbb{Q}_\mu} : [-\pi, \pi] \to \mathbb{R}$,

$$h_p^{\mathbb{Q}_\mu}(\theta) := \sum_{k=-p}^{p} \int_{\mathbb{R}} \phi_p^{\mathbb{Q}_\mu}(t)\phi_p^{\mathbb{Q}_\mu}(t-k)dt \cdot \cos(k\theta)$$

Then, $\forall \theta \in [-\pi, \pi]$:

1. $h_p^{\mathbb{Q}_\mu}(\theta) = \sum_{k\in\mathbb{Z}} \left|\widehat{\phi_p^{\mathbb{Q}_\mu}}(\theta + 2k\pi)\right|^2$.
2. $h_p^{\mathbb{Q}_\mu}(\theta) \leq 1$, $h_p^{\mathbb{Q}_\mu}(\theta) \geq c_{\alpha,p} > 0$ $(\mathbb{Q} = \mathbb{H}, \mathbb{T})$.

h_p: [Garoni, Manni, Pelosi, Serra-Capizzano, Speleers; 2014]

$h_p^{\mathbb{Q}_\mu}$: [Roman, Manni, Speleers; 2016]

Spectral analysis of GB-Galerkin matrices

Lemma on $f_p^{\mathbb{Q}_\mu}$

Let $p \geq 3$, and let $f_p^{\mathbb{Q}_\mu} : [-\pi, \pi] \to \mathbb{R}$,

$$f_p^{\mathbb{Q}_\mu}(\theta) := \sum_{k=-p}^{p} \int_{\mathbb{R}} \dot{\phi}_p^{\mathbb{Q}_\mu}(t)\dot{\phi}_p^{\mathbb{Q}_\mu}(t-k)dt \cdot \cos(k\theta)$$

and $M_{f_p^{\mathbb{Q}_\mu}} := \max_{\theta \in [-\pi,\pi]} f_p^{\mathbb{Q}_\mu}(\theta)$. Then:

1. $\forall \theta \in [-\pi, \pi], \qquad f_p^{\mathbb{Q}_\mu}(\theta) = (2 - 2\cos\theta) h_{p-1}^{\mathbb{Q}_\mu}(\theta)$;
2a. $\min_{\theta \in [-\pi,\pi]} f_p^{\mathbb{Q}_\mu}(\theta) = f_p^{\mathbb{Q}_\mu}(0) = 0$;
2b. $\theta = 0$ unique zero of $f_p^{\mathbb{Q}_\mu}$ over $[-\pi, \pi]$;
2c. $M_{f_p^{\mathbb{Q}_\mu}} \to 0$ as $p \to \infty$.

Spectral analysis of GB-Galerkin matrices

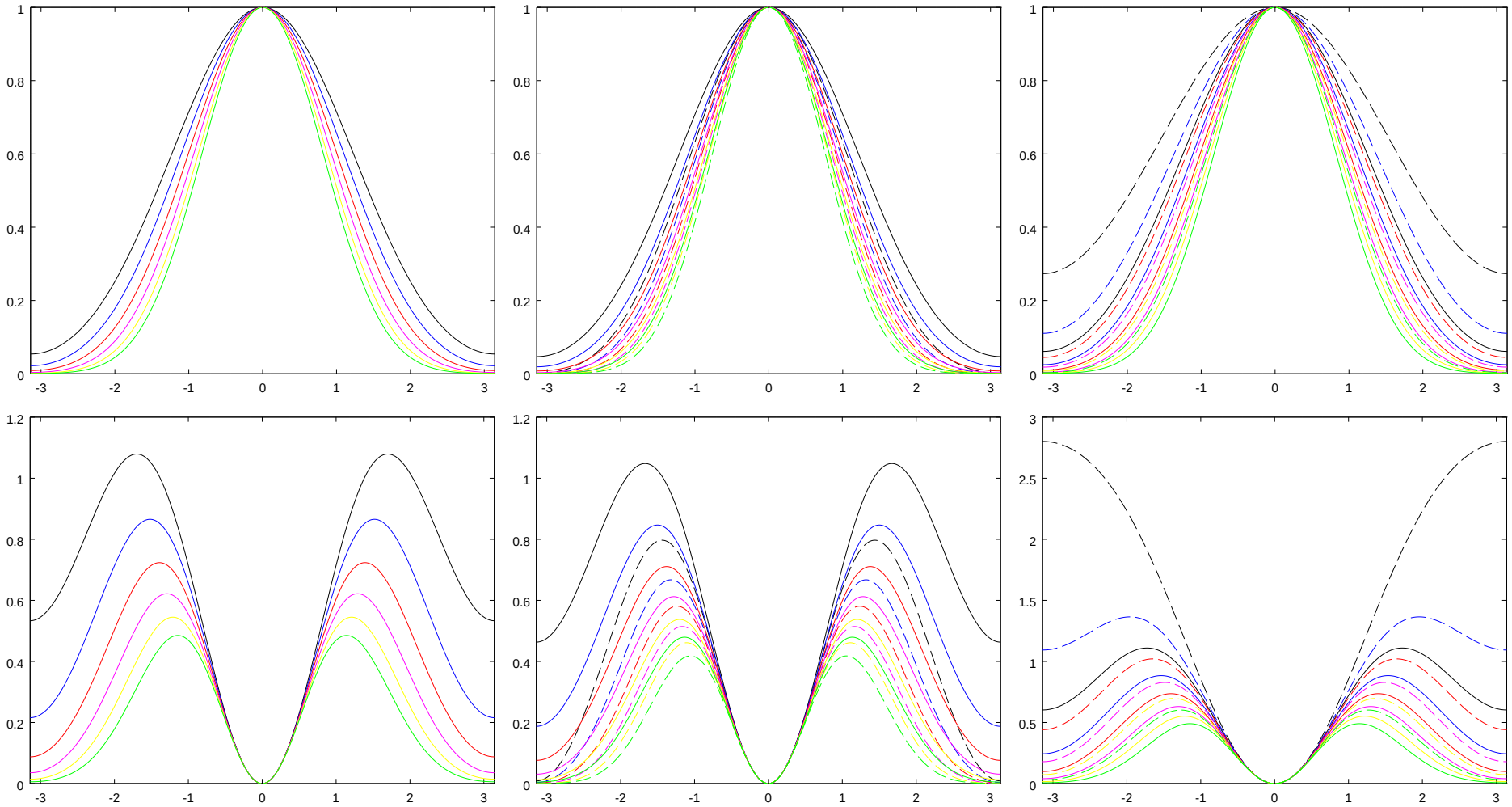

Figure : [top] h_p (left), $h_p^{\mathbb{T}_1}$ and $h_p^{\mathbb{T}_3}$ (center), $h_p^{\mathbb{H}_1}$ and $h_p^{\mathbb{H}_{10}}$ (right).
[bottom] f_p (left), $f_p^{\mathbb{T}_1}$ and $f_p^{\mathbb{T}_3}$ (center), $f_p^{\mathbb{H}_1}$ and $f_p^{\mathbb{H}_{10}}$ (right).

Dashed line: higher parameter; different colors: $p = 3, \ldots, 8$, downward.

Setting for spectral analysis

Nested case vs non-nested case

$$N_{i,p}^{U,V}(x) = \phi_p^{(U,V)(\cdot/n)}(nx - i + p + 1), \quad i = p+1, \dots, n$$

From this one, two associations: $N_{i,p}^{\mathbb{Q}_{n\alpha}}$ with $\phi_p^{\mathbb{Q}_\alpha}$, and $N_{i,p}^{\mathbb{Q}_\alpha}$ with $\phi_p^{\mathbb{Q}_{\alpha/n}}$

- Nested case: constant parameter in n for $N_{i,p}^{\mathbb{Q}_\bullet}$; spaces implied $\langle 1, t, \dots, t^{p-2}, \cos(\omega t), \sin(\omega t)\rangle$ and $\langle 1, t, \dots, t^{p-2}, \cosh(\alpha t), \sinh(\alpha t)\rangle$.
- Non-nested case: constant parameter in n for $\phi_p^{\mathbb{Q}_\bullet}$; spaces implied $\langle 1, t, \dots, t^{p-2}, \cos(n\omega t), \sin(n\omega t)\rangle$, $\langle 1, t, \dots, t^{p-2}, \cosh(n\alpha t), \sinh(n\alpha t)\rangle$.

Spectral analysis of GB-Galerkin matrices

Distribution theorem

Nested case:

The sequence of matrices $\{\frac{1}{n}A_{n,p}^{\mathbb{Q}_\alpha}\}$ is distributed like the function f_p in the sense of the eigenvalues.

Non-nested case:

The sequence of matrices $\{\frac{1}{n}A_{n,p}^{\mathbb{Q}_{n\alpha}}\}$ is distributed like the function $f_p^{\mathbb{Q}_\alpha}$ in the sense of the eigenvalues.

Spectral analysis of GB-collocation matrices

A second order linear elliptic PDE in dimension 1

$$\begin{cases} -u'' + \beta u' + \gamma u = \mathrm{f}, & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$

We look for a function $u_{\mathcal{W}} \in \langle \varphi_1, \dots, \varphi_N \rangle =: \mathcal{W}$ such that

$$-u''_{\mathcal{W}}(\tau_i) + \beta u'_{\mathcal{W}}(\tau_i) + \gamma u_{\mathcal{W}}(\tau_i) = \mathrm{f}(\tau_i), \quad \tau_i \text{ collocation points}$$

$$u_{\mathcal{W}} = \sum_{j=1}^{N} u_j \varphi_j, \quad \text{solve} \quad A\mathbf{u} = \mathbf{f}, \quad \mathbf{u} = [u_1 \cdots u_N]^T, \quad \mathbf{f} = [f(\tau_i)]_{i=1}^{N}$$

$$A := [-\varphi_j''(\tau_i) + \beta \varphi_j'(\tau_i) + \gamma \varphi_j(\tau_i)]_{i,j=1}^{N} \in \mathbb{R}^{N \times N}$$

$$\varphi_i := N_{i+1,p}^{U,V} \quad \text{GB-splines}, \qquad \tau_i \text{ Greville abscissae}$$

Spectral analysis of GB-collocation matrices

Distribution theorem

Nested case:

The sequence of matrices $\{\frac{1}{n^2}A^{\mathbb{Q}_\alpha}_{C,n,p}\}$ is distributed like a function $f_{C,p}$ in the sense of the eigenvalues.

Non-nested case:

The sequence of matrices $\{\frac{1}{n^2}A^{\mathbb{Q}_{n\alpha}}_{C,n,p}\}$ is distributed like a function $f^{\mathbb{Q}_\alpha}_{C,p}$ in the sense of the eigenvalues.

- $f_{C,2p+1} = f_p, \quad f^{\mathbb{Q}_\alpha}_{C,2p+1} \to f^{\mathbb{Q}_\alpha}_p$ if $\alpha \to 0$;
- $-\kappa(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) = \mathrm{f}(x)$ in (a, b) can be considered;
- $G : [0, 1] \to [a, b]$, s.t. $\exists G^{-1}, G(\partial([0, 1])) = \{a, b\}$, as a geometry map.

Concluding remarks

We show that...

- GB-splines:
  - enjoy the same properties of B-splines, and can be refined similarly;
  - plug-to-plug compatible with B-splines in IgA.
- GB-splines behave similarly to B-splines and NURBS in IgA, with improvements;
- IgA Galerkin/collocation methods based on GB-splines $\Rightarrow$ matrices with spectral properties similar to the polynomial case;
- This ensures that multi-iterative techniques for B-splines discretizations can be adapted to GB-spline setting.

Essential bibliography

- M. Donatelli, C. Garoni, C. Manni, S. Serra-Capizzano, H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric collocation methods*, Math. Comp., **85** (2016), 1639-1680.
- C. Garoni, C. Manni, F. Pelosi, S. Serra-Capizzano, H. Speleers, *On the spectrum of stiffness matrices arising from isogeometric analysis*, Numer. Math., **127 (4)** (2014), 751-799.
- C. Garoni, C. Manni, S. Serra-Capizzano, D. Sesana, H. Speleers, *Spectral analysis and spectral symbol of matrices in isogeometric Galerkin methods*, Math. Comp., To appear in 2017, DOI:10.1090/mcom/3143.
- F. Roman, C. Manni, H. Speleers, *Spectral analysis of matrices in Galerkin methods based on generalized B-splines with high smoothness*, Numer. Math., **135 (1)** (2017), 169-216.
- C. Manni, F. Pelosi, M.L. Sampoli, *Generalized B-splines as a tool in isogeometric analysis*, Comput. Methods Appl. Mech. Engrg., **200** (2011), 867-881.

Thanks!