Approximated nonstationary iterated Tikhonov with application to image deblurring

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Main Issues

1. Nonstationary Iterated Tikhonov
2. Approximated algorithms
3. Numerical results

Large discrete ill-posed problems

$$\mathbf{y} = T\mathbf{x} + \mathbf{e}$$

- $T \in \mathbb{R}^{n \times n}$ large and severely ill-conditioned
- $\mathbf{y} \in \mathbb{R}^n$ known, measured data
- $\mathbf{e} \in \mathbb{R}^n$ noise, s.t. $\|\mathbf{e}\| = \delta$

Goal: compute approximation of the noise free solution $\mathbf{x}$

Nonstationary Iterated Tikhonov

Nonstationary iterated Tikhonov is given by

$$x_n = x_{n-1} + T^*(TT^* + \alpha_n I)^{-1}(y - Tx_{n-1}), \qquad (1)$$

which is equivalent to

$$x_n = x_{n-1} + (T^*T + \alpha_n I)^{-1}T^*(y - Tx_{n-1}). \qquad (2)$$

The iteration (2) can be interpreted as an iterative refinement. Let $e_{n-1} = x - x_{n-1}$ be the error at the $n-1$-th step, where x is the true solution. Solving the error equation $Te_{n-1} = r_{n-1}$ by Tikhonov, where $r_{n-1} = y - Tx_{n-1}$ is the residual, and using the solution to refine the previous approximation x_{n-1}, we obtain the equation (2).

Convergence

In the noise free case ($\delta = 0$)

Theorem

The method (1) *converges to* $x^\dagger = T^\dagger y$ *if and only if*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j^{-1} = \infty.$$

- In the stationary case, $\alpha_n = \alpha$, $\forall n \in \mathbb{N}$, assuming that $x^\dagger = (T^*T)^\nu w$ for some $\nu > 0$ with some $w \in \mathcal{D}((T^*T)^\nu)$, we have
$$\|x_n - x^\dagger\| = \mathcal{O}(n^{-(\nu+1)}).$$

The geometric sequence

- A classical choice for α_n is the successive geometric value

$$\alpha_n = \alpha q^{n-1}, \qquad 0 < q < 1. \tag{3}$$

- Under standard regularity assumptions on $x^\dagger$

$$\|x_n - x^\dagger\| = \mathcal{O}(q^{\nu n}).$$

[M. Hanke and C. W. Groetsh, J. Optim. Theory Appl., 1998].

The noise case

- Using the discrepancy principle, i.e., the iterative method is stopped at the first value of $n = n(\delta) \geq 1$ for which

$$\|y^\delta - Tx^\delta_{n(\delta)}\| \leq \tau\delta, \tag{4}$$

  with $\tau > 1$ fixed.

- For the geometric sequence

$$n(\delta) \leq \mathcal{O}(|\log\delta|).$$

- For the stationary sequence

$$n(\delta) = \mathcal{O}(\delta^{-\frac{2}{2\nu+1}}).$$

Geometric vs stationary sequence

- Under standard regularity assumptions on $x^\dagger$, iterated Tikhonov with the decreasing geometric sequence converges faster than the stationary method both for perturbed and nonperturbed data.
- On the other hand, the main drawback of the decreasing geometric sequence is a steep error curve. Therefore, we need a fair stopping criteria.
- Other nonstationary sequences could be considered if satisfy

$$\lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j^{-1} = \infty.$$

Approximated iterated Tikhonov

- In some applications $TT^* + \alpha_n I$ and $T^*T + \alpha_n I$ are computationally too expensive to invert.
- It is available a good approximation easy to invert.
- Image deblurring with space invariant point spread function: T is Toeplitz + something (Hankel, low rank, ...). A good approximation of T can be obtained by P in a matrix algebra diagonalizzable by unitary transforms.

Algorithm 1

- Replacing $(TT^* + \alpha_n I)^{-1}$ with $(PP^* + \alpha_n I)^{-1}$ in (1) we obtain

$$x_n = x_{n-1} + T^*(PP^* + \alpha_n I)^{-1}(y - Tx_{n-1}). \qquad (5)$$

- Defining the function

$$\Phi(x) = \|Tx - y\|^2_{(PP^*+\alpha I)^{-1}}$$

in the stationary case $(\alpha_n = \alpha)$ the iteration (5) can be rewritten as

$$x_n = x_{n-1} - \nabla\Phi(x_{n-1}).$$

Algorithm 2

- Replacing $(TT^* + \alpha_n I)^{-1}$ with $(PP^* + \alpha_n I)^{-1}$ in (2) we obtain

$$x_n = x_{n-1} + (P^*P + \alpha_n I)^{-1} T^*(y - Tx_{n-1})$$

- In the stationary case, the previous method is the

  Preconditioned Landweber

  method previously investigated in [P. Brianzi, F. Di Benedetto, and C. Estatico, SIAM J. Sci. Comput., 2008]

- In the stationary case, it is a quasi-Newton method for the minimum problem

$$\min_x \|Tx - y\|^2,$$

  where the Hessian $(T^*T)^{-1}$ is replaced with $(P^*P + \alpha I)^{-1}$.

Algorithm 3

- Using the iterative refinement interpretation, an approximation of (2) can be derived solving the error equation

$$Pe_n = r_n$$

instead of $Te_n = r_n$, where the residual r_n is defined by the "true" model as $r_n = y - Tx_n$. This produces the iteration

$$x_n = x_{n-1} + (P^*P + \alpha_n I)^{-1} P^*(y - Tx_{n-1}).$$

- For the error equation accurate boundary conditions are not necessary since the error is about uniformly distributed on the domain if the model represented by T is accurate. Hence periodic boundary conditions, such that P is diagonalized by FFT, are an accurate model for the error equation even if they are not good for the original linear system.

Computational cost

- Algorithms 1 and 2 have the same computational cost for each iteration.
- Algorithms 1 and 2 require at each iteration a product with T and T^*, while the Algorithm 3 needs only one product with T.
- Let P be a circulant matrix, then the matrix-vector product with $(P^*P + \alpha_n I)^{-1}P^*$ requires 2 FFTs like $(P^*P + \alpha_n I)^{-1}$.

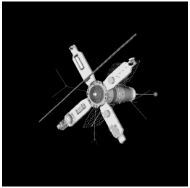
Numerical Experiments

- The matrix P is defined by imposing periodic BCs to the PSF such that P is diagonalized by FFT.
- We do not consider the stopping criteria.
- The maximum number of iterations is fixed to 300.
- For the nonstationary case, we use the geometric decreasing sequence with $\alpha = 1$.
- The relative restoration error (RRE) is

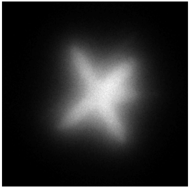
$$\mathrm{RRE} = \frac{\|\tilde{x} - x\|}{\|x\|},$$

where $\tilde{x}$ is the computed solution.

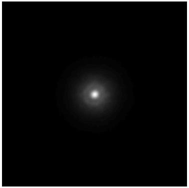
Satelitte test



True



Observed

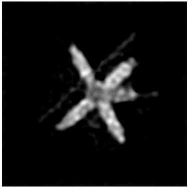


PSF

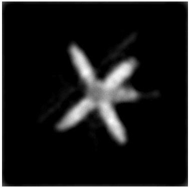
Stationary case

First row $\rightarrow \alpha = 0.06$, second row $\rightarrow \alpha = 0.01$

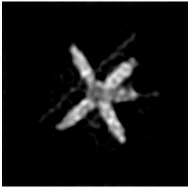
Algorithm 1 | Algorithm 2 | Algorithm 3



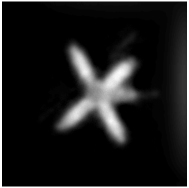
RRE $= 0.3306$, it. 300



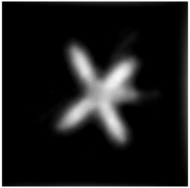
RRE $= 0.3966$, it. 51



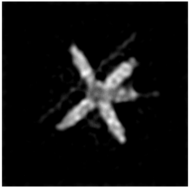
RRE $= 0.3308$, it. 300



RRE $= 0.4547$, it. 4



RRE $= 0.4942$, it. 2

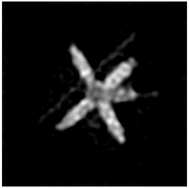


RRE $= 0.3329$, it. 51

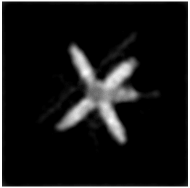
Nonstationary case

First row $\rightarrow q = 0.98$, second row $\rightarrow q = 0.8$

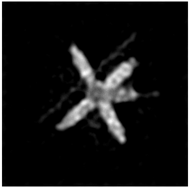
Algorithm 1	Algorithm 2	Algorithm 3



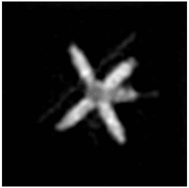
RRE $= 0.3321$, it. 227



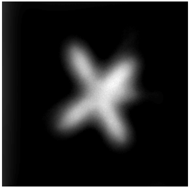
RRE $= 0.3671$, it. 161



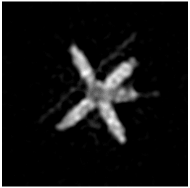
RRE $= 0.3323$, it. 227



RRE $= 0.3537$, it. 26



RRE $= 0.4108$, iter. 22



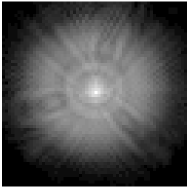
RRE $= 0.3392$, it. 30

Example 2

- Antireflective BCs
- 0.1% of white Gaussian noise

True

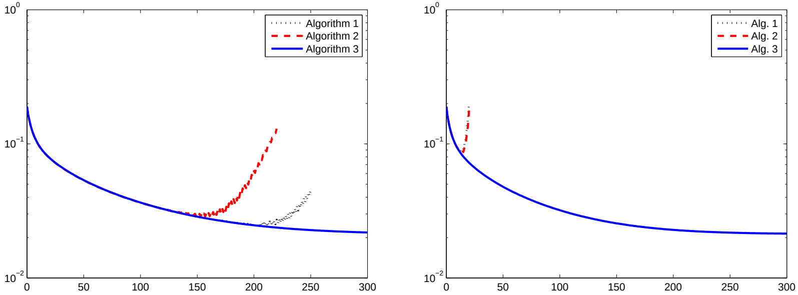
Observed



$\log(\mathrm{PSF})$

Stationary case

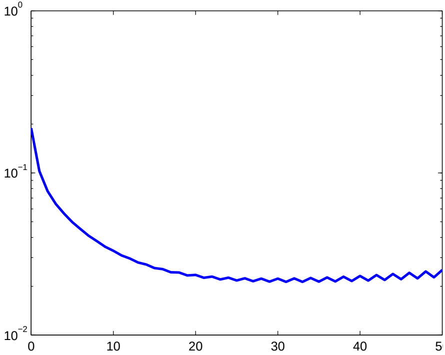
RRE varying the iteration number



$\alpha = 0.5$ $\qquad$ $\alpha = 0.4$

Algorithm 3

Algorithm 3 with $\alpha = 0.04$



RRE

RRE=0.0213, it. 33

Nonstationary case

First row $\rightarrow q = 0.98$, second row $\rightarrow q = 0.9$

Algorithm 1 | Algorithm 2 | Algorithm 3



RRE $= 0.0379$, it. 77 | RRE $= 0.0380$, it. 78 | RRE $= 0.0218$, it. 145



RRE $= 0.072$, it. 18 | RRE $= 0.072$, it. 18 | RRE $= 0.0214$, it. 43

Conclusions and work in progress

- For a quasi-symmetric PSF the Algorithm 3 computes the best restoration and it is robust varying the regularization parameter.
- The geometric decreasing sequence $\{\alpha_n\}$ avoids a fair estimation of the spectral thresholding parameter.
- For a strongly nonsymmetric PSF the Algorithm 3 is robust but it does not compute the best approximation.
- The convergence and the stability of algorithms 1–3 should be investigated.