Image deblurring with high order boundary conditions

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Outline

1. The model problem (signal deconvolution)
2. High order Boundary Conditions (BC)
   - Antireflective BC
   - The spectral decomposition
   - Algebraic high order BC
3. Regularization by filtering
   - The algorithm
   - Convergence
4. Numerical results

Outline

The model problem

- Problem: to approximate $f : \mathbb{R} \to \mathbb{R}$ from a blurred $g : \mathcal{I} \to \mathbb{R}$

$$g(x) = \int_{\mathcal{I}} k(x-y) f(y) \mathrm{d}y, \qquad x \in \mathcal{I} \subset \mathbb{R},$$

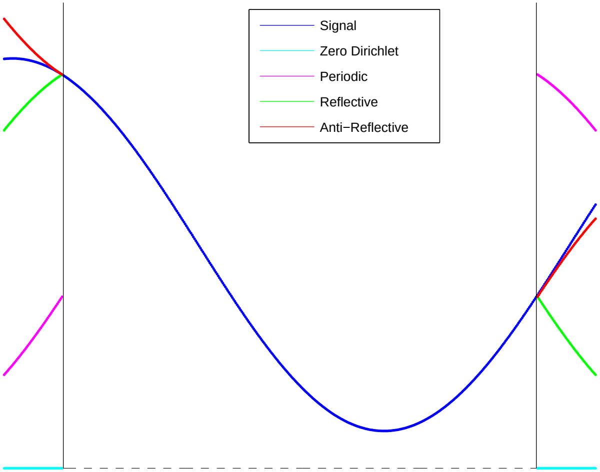
  the point spread function (PSF) k has compact support.

- Discretizing the integral by a rectangular quadrature rule and imposing boundary conditions:

$$A\mathbf{f} = \mathbf{g} + \text{noise}.$$

- The structure of A depends on k and the imposed boundary conditions.

Boundary conditions



Structure of the coefficient matrix A

Type	Generic PSF	Symmetric PSF
Zero Dirichlet	Toeplitz	Toeplitz
Periodic	Circulant	Circulant
Reflective	Toeplitz + Hankel	Cosine
Antireflective	Toeplitz + Hankel + rank 2	Sine + . . . =

Requests:

1. fast transforms
2. precise model

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Antireflective BC

- The 1D antireflection is obtained by

$$\begin{aligned} f_{1-j} &= 2f_1 - f_{j+1} \\ f_{n+j} &= 2f_n - f_{n-j} \end{aligned}$$

  [Serra-Capizzano, SISC. '03]

- In the multidimensional case we perform an antireflection with respect to every edge $\implies$ Tensor structure in the multidimensional case.

Approximation property

The reflective BC assure the continuity at the boundary,
while the antireflective BC assure also the
continuity of the first derivative.

Structural properties

- $A =$ Toeplitz $+$ Hankel $+$ rank 2.
- Matrix vector product in $O(n\log(n))$ ops.

Symmetric PSF

- $S \in \mathbb{R}^{(n-2)\times(n-2)}$ diagonalizable by discrete sine transforms (DST)

$$A = \begin{bmatrix} 1 & & \\ * & & * \\ \vdots & S & \vdots \\ * & & * \\ & & 1 \end{bmatrix}$$

AR Spectral decomposition

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$A = AR_n(h) = T_n \mathrm{diag}(h(\hat{\mathbf{x}}))\, T_n^{-1},$$

where $\hat{\mathbf{x}} = [0,\, \tilde{\mathbf{x}}^T,\, 0]^T$, $\tilde{\mathbf{x}} = [\frac{j\pi}{n-1}]_{j=1}^{n-2}$ and

$$T_n = \left[1 - \frac{\mathbf{x}}{\pi},\, \sin(\mathbf{x}),\, \ldots,\, \sin((n-2)\mathbf{x}),\, \frac{\mathbf{x}}{\pi}\right],$$

with $\mathbf{x} = [0,\, \tilde{\mathbf{x}}^T,\, \pi]^T$.

[Aricò, D., Nagy, and Serra-Capizzano, 2008]

Algebraic interpretation

Eigenvalues:

- $h(0)$ with multiplicity 2
- DST of the first column of $\tau_{n-2}(h)$

Eigenvectors:

A uniform sampling in $[0, \pi]$ of some special functions

1. fast transform: $\sin(k\mathbf{x})$, for $k = 1, \ldots, n-2$ (DST)
2. model precision: $\pi - \mathbf{x}$ and $\mathbf{x}$ are associated to $h(0) = 1 \Longrightarrow$ preserve linear functions.

Inverse antireflective transform T_n^{-1} has a structure analogous to T_n.

Algebraic BC with cosine basis

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$A_C(h) = C_n \mathrm{diag}(h(\hat{\mathbf{x}})) C_n^{-1},$$

where $\hat{\mathbf{x}} = [0,\, \tilde{\mathbf{x}}^T,\, 0]^T$, $\tilde{\mathbf{x}} = \left[\frac{(j-1)\pi}{n-2}\right]_{j=1}^{n-2}$ and

$$C_n = \left[(\frac{(2n-3)\pi}{2n-4} - \mathbf{x})^2,\ \cos(0\mathbf{x}),\ \ldots,\ \cos((n-3)\mathbf{x}),\ \mathbf{x}^2 \right],$$

with $\mathbf{x} = \left[\frac{(2i-1)\pi}{2n-4}\right]_{j=0}^{n-1}$.

[D., 2009]

Notes on high order BC

1. The first and the last rows are not separable (otherwise we introduce a discontinuity).
2. When the PSF is nonsymmetric it is possible to use the exponential basis (DFT).
3. Some a-priori knowledge on the true object can be included as an eigenvector associated to the eigenvalue 1.

Outline

Regularization by filtering

- $A = T_n D_n T_n^{-1}$ where $\mathbf{d} = h(\hat{\mathbf{x}})$ and

$$T_n = \begin{bmatrix} \mathbf{t}_1 & \cdots & \mathbf{t}_n \end{bmatrix}, \quad D_n = \mathrm{diag}(\mathbf{d}), \quad T_n^{-1} = \begin{bmatrix} \tilde{\mathbf{t}}_1^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{bmatrix}$$

- A spectral filtered solution is given by

$$\mathbf{f}_{\mathrm{reg}} = \sum_{i=1}^{n} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i} \mathbf{t}_i \,, \qquad (1)$$

where $\mathbf{g}$ is the observed signal and ϕ_i are the filter factors.

Algorithm

1. Transform the signal to the frequency domain
2. Filtering of the components
3. Inverse transform to come back into the original space

Examples: TSVD and Tikhonov with the reblurring approach

$$\Downarrow$$

$$A_n(h)^* \text{ is replaced by } A_n(\bar{h})$$

.

Convergence

For antireflective BCs, from the SVD of T_n [D. and M. Hanke 2009]

Theorem

$\|T_n\|_2\|T_n^{-1}\|_2 \doteq \sqrt{2n}.$

Theorem

The reblurring with antireflective boundary conditions and a symmetric PSF is a regularization method.

Outline

Tikhonov regularization

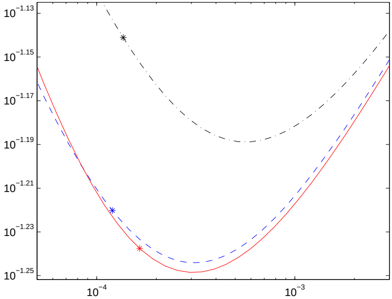
- Out of focus blur
- 0.1% of Gaussian noise

True image

Observed image

Restoration errors

Relative restoration error (RRE) defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$, where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.



RRE: — A_C, - - - antireflective, - · - reflective ($*$ is $\mu_{\rm GCV}$)

	$\mu_{\rm opt}$	$\mu_{\rm GCV}$
reflective	0.064	0.072
antirefl.	0.057	0.060
A_C	0.056	0.057

Restored images.

Reflective

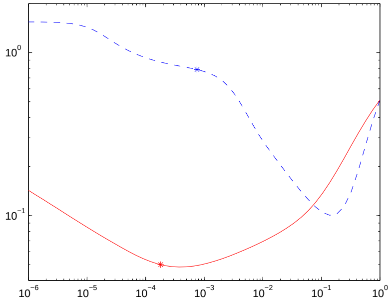
Antireflective

A_C

Nonsymmetric PSF.

Out of focus + moving blur

Observed image



RRE: — A_F, - - - periodic
($*$ is μ_{GCV}).

Restored images.

periodic with μ_{opt}

A_F with μ_{opt}

A_F with μ_{GCV}

Summarizing

The importance of good boundary conditions increases when the PSF has a large support and the noise is not huge.

High order BC:

- Filtering in $O(n \log(n))$ by few fast transforms.
- Work both for symmetric and nonsymmetric PSF.
- Can include some a-priori knowledge on the true object.