

Regularization by multigrid-type algorithms

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1 Restoration of blurred and noisy images

The model problem

Properties of the PSF

Iterative regularization methods

2 Multigrid regularization

Multigrid methods

Iterative Multigrid regularization

Computational Cost

3 Numerical experiments

4 Conclusions



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Image restoration with Boundary Conditions

Using **Boundary Conditions (BCs)**, the restored image **f** is obtained solving: (in some way ...)

$$A\mathbf{f} = \mathbf{g} + \boldsymbol{\xi}$$

- **g** = blurred image,
- **ξ** = noise (random vector),
- **A** = two-level matrix depending on the point spread function (PSF) and the BCs.



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The **PSF** is the observation of a single point (e.g., a star in astronomy) that we assume shift invariant.



Coefficient matrix structure

The matrix-vector product computed in $O(n^2 \log(n))$ ops for $n \times n$ images while the inversion costs $O(n^2 \log(n))$ ops only in the periodic case.

BCs	A
Dirichlet periodic	Toeplitz circulant
Neumann (reflective) anti-reflective	Toeplitz + Hankel Toeplitz + Hankel



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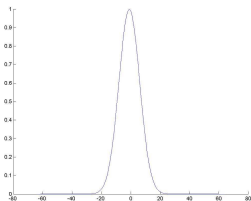
If the PSF is symmetric with respect to each direction:

BCs	A
Neumann (reflective) anti-reflective	DCT III DST I + low-rank

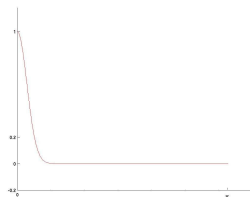


Generating function of PSF

- The eigenvalues of $A(z)$ are about a uniform sampling of z .



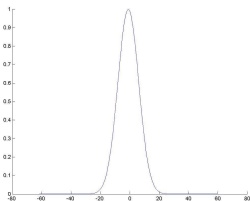
PSF



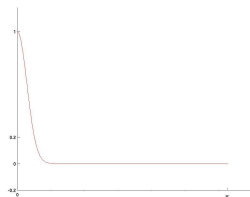
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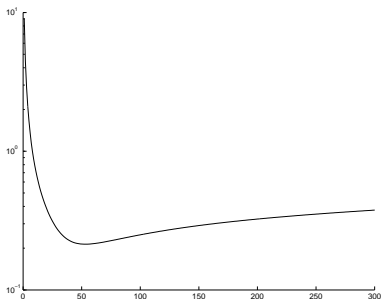
Generating function $z(x)$

- The ill-conditioned subspace is mainly constituted by the **middle/high frequencies**.

Semi-convergence of iterative regularization methods

Some iterative methods (CGNE, ...) have regularization properties:

- They firstly reduce the algebraic error in the low frequencies (well-conditioned subspace).
- When they arrive to reduce the **algebraic error** in the high frequencies then the **restoration error** increases because of the noise.



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Multigrid

① We use the **Galerkin approach**

- $P_i = R_i^T$
- $A_{i+1} = RA_i R_i^T$

where R_i and P_i are the restriction and the prolongation operators at the level i , respectively.



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- ③ Coarser grid of size 8×8 independent of the size of the finer grid.



Image deblurring and Multigrid

- In the **image deblurring** the **ill-conditioned subspace** is related to **high frequencies**, while the well-conditioned subspace is generated by low frequencies.



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- In order to obtain a fast convergence the **algebraic multigrid** projects in the high frequencies where the noise “lives” \implies noise explosion already at the first iteration (it requires **Tikhonov regularization** [Donatelli, NLAA, 12 (2005), pp. 715–729]).



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- In order to obtain a fast convergence the algebraic multigrid projects in the high frequencies where the noise “lives” \implies noise explosion already at the first iteration (it requires Tikhonov regularization [Donatelli, NLAA, 12 (2005), pp. 715–729]).
- In this case the **low-pass filter** projects in the well-conditioned subspace (low frequencies) \implies it is slowly convergent but it can be a good **iterative regularizer**.



If we have an *iterative regularization method* we can improve its regularizing properties and/or accelerate its convergence using it as *smoother* in a Multigrid algorithm.



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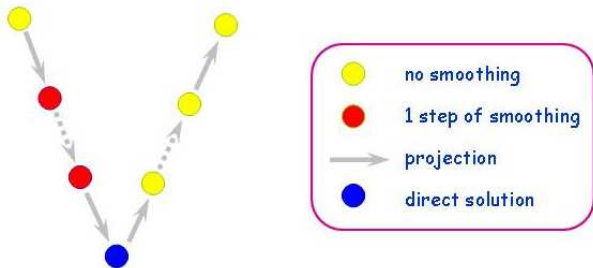
Two Level algorithm

- 1 No smoothing at the finer level
- 2 At the coarser level to apply one step of the smoother instead of to solve directly the linear system



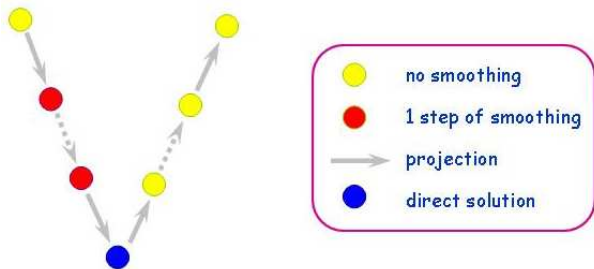
Multigrid regularization (applying recursively the TL)

V-cycle



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Using a **larger number of recursive calls** (e.g. *W*-cycle), the algorithm “works” more in the well-conditioned subspace, but it is more difficult to define an early stopping criterium.



Computational Cost

Assumptions: $n \times n$ images and $m \times m$ PSFs with $m \ll n$.

- Let $S(n)$ be the computational cost of one smoother iteration.
- The computational cost of one iteration of our multigrid regularization method with γ recursive calls is

$$C(\gamma, n) \approx \begin{cases} \frac{1}{3}S(n), & \gamma = 1 \\ S(n), & \gamma = 2 \\ 3S(n), & \gamma = 3 \end{cases}$$

- if $m \approx n$ then $S(n) = O(n^2 \log(n))$.



Remarks on the TL regularization

Theoretical analysis

- **Intuitively:** the regularization properties of the smoother are preserved since it is combined with a low-pass filter.



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- From the **filter factor analysis** of the Two Level (TL) method we have:
 - a condition to choose p s.t. it is a regularizer
 - in such case it improves the regularization properties of the iterative method used as smoother.



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Theoretical analysis

- Intuitively: the regularization properties of the smoother are preserved since it is combined with a low-pass filter.
- From the filter factor analysis of the Two Level (TL) method we have:
 - a condition to choose p s.t. it is a regularizer
 - in such case it improves the regularization properties of the iterative method used as smoother.
- In the **noise free case** the TL method does not compute the exact solution.



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An airplane

- Periodic BCs
- Gaussian PSF (A spd)
- noise = 1%



Original
Image

An airplane

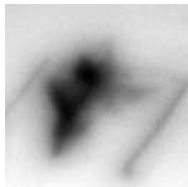
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Original
Image



Inner part 128×128



Observed image



Restored with MGM

Restoration error: noise = 1%

$e_j = \|\mathbf{f} - \mathbf{f}^{(j)}\|_2 / \|\mathbf{f}\|_2$ restoration error at the j -th iteration.



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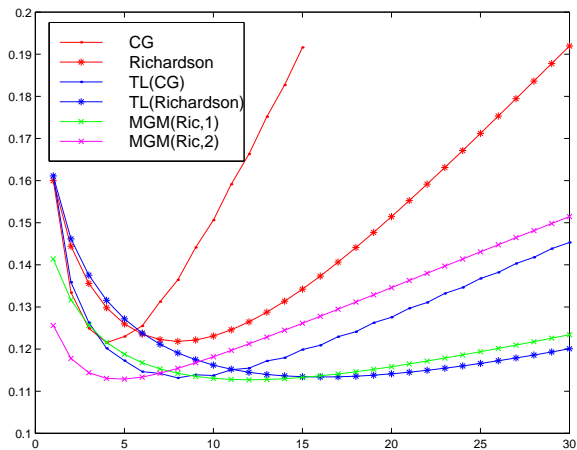
Minimum restoration error

Method	$\min_{j=1,\dots} (e_j)$	$\arg \min_{j=1,\dots} (e_j)$
CG	0.1215	4
Richardson	0.1218	8
TL(CG)	0.1132	8
TL(Rich)	0.1134	16
MGM(Rich, 1)	0.1127	12
MGM(Rich, 2)	0.1129	5
CGNE	0.1135	178
RichNE	0.1135	352



Restoration error: noise = 1%

Relative error vs. number of iterations



Noise = 10%

For CG and Richardson it is better to resort to normal equations.



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Minimum restoration error

Method	$\min_{j=1,\dots} (e_j)$	$\arg \min_{j=1,\dots} (e_j)$
CGNE	0.1625	30
RichNE	0.1630	59
TL(CGNE)	0.1611	48
TL(RichNE)	0.1613	97
MGM(RichNE,1)	0.1618	69
MGM(RichNE,2)	0.1621	26
MGM(Rich,1)	0.1648	3
MGM(Rich,2)	0.1630	1



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Possible generalizations

- Include nonnegativity constraints.
- Improve the projector:

$$p(x, y) = (1 + \cos(x))^\alpha (1 + \cos(y))^\alpha, \quad \alpha \in \mathbb{N}^+.$$

- The γ regularization:

*varying γ , the proposed multigrid is a direct (one step) regularization method with **regularization parameter γ** .*

The computational cost increases with γ but not so much (e.g. $\gamma = 8 \Rightarrow O(N^{1.5})$ where $N = n^2$).



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- It can be combined with other techniques and it can lead to several generalizations (e.g., nonnegativity constraints).



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Reference

M. Donatelli and S. Serra Capizzano, *On the regularizing power of multigrid-type algorithms*, SIAM J. Sci. Comput., 27–6 (2006) pp. 2053–2076.

