

An algebraic generalization of the local Fourier analysis

Marco Donatelli

University of Insubria
Department of Physics and Mathematics

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Outline

- 1 Classic convergence analysis for geometric multigrid
- 2 Multigrid methods for Toeplitz matrices
- 3 Equivalence of the two approaches
- 4 Consequences
- 5 Numerical results



The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- d -dimensional PDE with **constant coefficients**

$$(-1)^q \sum_{i=1}^d \frac{d^{2q}}{dx_i^{2q}} u(x) = g(x), \quad x \in \Omega = (0, 1)^d, \quad q \geq 1.$$

- **Periodic boundary conditions** on $\partial\Omega$ or an infinite domain.
- Discretization by centered finite difference of minimal precision on a uniform grid.



Local Fourier Analysis

The **Fourier transform** of the discrete differential operator is

$$\hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j e^{i \langle jh | \omega \rangle},$$

where $\omega \in [-\pi/h, \pi/h]^d$ denotes the frequencies for the current discretization step h and

$$l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h, \pi/h]^d} \hat{L}(\omega) e^{-i \langle jh | \omega \rangle} d\omega.$$



The convergence result

Theorem (Convergence)

Given a constant-coefficient PDE of order m , a necessary condition for nonincreasing high frequencies arising from a coarse grid correction is

$$\gamma_r + \gamma_p \geq m, \quad (1)$$

where γ_p and γ_r are the order of the prolongation and of the restriction respectively.

Definition

A prolongation (restriction) has **order** γ_p if it (its transpose) leaves unchanged all polynomials of degree $\gamma_p - 1$.



More general orders

Definition

The set of all **corners** of x is

$$\Omega(x) = \{y \mid y_j \in \{x_j, \pi + x_j\}, j = 1, \dots, d\}$$

and the set of the **“mirror” points** of x is $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$.



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Definition (P. W. Hemker 1990)

For a grid transfer operator $B \in \{R, P\}$ (B is multiplied by 2^d when $B = P$), for $x = \omega h$, $|x| \rightarrow 0$, the largest $s \geq 0$ such that

$$\hat{B}(x) = 1 + O(|x|^s),$$

$$\hat{B}(y) = O(|x|^s), \quad \forall y \in \mathcal{M}(x),$$

is the **Low Frequency (LF) order**

is the **High Frequency (HF) order**



New convergence conditions

Theorem

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If a prolongation leaves all polynomials of degree $k - 1$ invariant, then both the LF and HF orders are at least k .



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Lemma (P. W. Hemker 1990)

If a prolongation leaves all polynomials of degree $k - 1$ invariant, then both the LF and HF orders are at least k .

Remark (I. Yavneh 1998)

*The same analysis for the Galerkin approach requires **LF** > 0 .*



Grid transfer operators

Interpolation operators

- Linear interpolation: $\frac{1}{2}[1 \ 2 \ 1]$
LF order = HF order = 2
- cubic interpolation: $\frac{1}{16}[-1 \ 0 \ 9 \ 16 \ 9 \ 0 \ -1]$
LF order = HF order = 4



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Noninterpolation operator

- Refinement coefficients of the B-spline of order 4:
 $\frac{1}{16}[1 \ 4 \ 6 \ 4 \ 1] \implies$ LF order = 2, HF order = 4.



Toeplitz matrices and $\hat{L}(\omega)$

- The d -level **Toeplitz** matrix $T_n(f)$ is such that

$$[T_n(f)]_{r,s} = a_{s-r} = a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x) e^{-i\langle j|x \rangle} dx, \quad r, s, j \in \mathbb{Z}^d.$$

- $f \geq 0 \iff T_n(f)$ is positive definite.



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Example

1D Laplacian: $\hat{L}(\omega) = \frac{1}{h^2}(2 - 2 \cos(\omega h))$. The Toeplitz approach moves the factor $\frac{1}{h^2}$ to the rhs, thus $A_n = T_n(f)$, where $f(x) = 2 - 2 \cos(x)$, which vanishes at the origin with order 2.



MGM convergence for Toeplitz matrices

- **Galerkin approach**: $A_k = P^T A_n P$ (A_n positive definite).
- Convergence analysis for the algebra case like τ or **circulant algebra**.



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Theorem (G. Fiorentino and S. Serra-Capizzano 1991, 1996, (2002))

Let $A_n = C_n(f)$ be circulant with f having a unique zero at x^0 . Defining $P = C_n(p)K_n^T$, where K_n is the down-sampling and p is a trigonometric polynomial non identically zero and such that for each $x \in [-\pi, \pi]^d$

$$\limsup_{x \rightarrow x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \quad \forall y \in \mathcal{M}(x), \quad (2a)$$

$$\sum_{y \in \Omega(x)} p(y)^2 > 0, \quad (2b)$$

then the TGM converges in a number of iterations independent of n .



Equivalence of the two approaches

Theorem

In the case of

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- *periodic boundary conditions,*
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Remark

The (2b) is equivalent to require $LF > 0$.



Consequences of such equivalence

- 1 For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance some integral problems.



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- ② Allow to define a MGM for Toeplitz matrices with $R \neq P$.



Consequences of such equivalence

- 1 For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance some integral problems.
- 2 Allow to define a MGM for Toeplitz matrices with $R \neq P$.
- 3 Allow to compare the grid transfer operators used in the two approaches.



Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^d (1 + \cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.



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- In the PDE setting $x^{(0)} = 0$ and $p(x)$ can be generalized as

$$\varphi_m(x) = 2^{-dm} \prod_{j=1}^d (1 + e^{-ix_j})^m.$$

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- The grid transfer operators with HF= m can be obtained by $\varphi_m(x)\psi_m(x)$ such that $\psi_m(y) \neq 0$ for all $y \in \mathcal{M}(0)$ and $\psi_m(0) = 1$.



B-spline refinement coefficients

- The coefficients of φ_m are the refinement coefficients of the **B-spline of order m** in the MRA.
- $\phi_m(x) = \varphi_m(x)e^{ix\lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.



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The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^m \phi_m$ in the 1D case.

| m | h_{-2} | h_{-1} | h_0 | h_1 | h_2 |
|-----|----------|----------|-------|-------|-------|
| 1 | | 1 | 1 | | |
| 2 | | 1 | 2 | 1 | |
| 3 | 1 | 3 | 3 | 1 | |
| 4 | 1 | 4 | 6 | 4 | 1 |



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| 4 | 1 | 4 | 6 | 4 | 1 |

- $m = 2q \Rightarrow$ vertex centered discretization.
- $m = 2q + 1 \Rightarrow$ cell centered discretization.



Numerical results: an integral equation

$$g(x) = \int_{\mathbb{R}} t(x - \theta) f(\theta) d\theta, \quad x, \theta \in \mathbb{R}.$$

A discretization on a equispaced grid and zero-Dirichlet boundary conditions leads to $A_n = T_n(z)$. We consider $z(x) = (2 + 2 \cos(x))^3$.

- Symbol of grid transfer operators: $(2 + 2 \cos(x - x_0))^{\frac{\delta_z}{2}}$, $z \in \{r, p\}$.
- Smoother = weighted Richardson.

TGM
iterations.

| n | $\delta_r = 2$ | $\delta_r = 2$ | $\delta_r = 4$ |
|-----|----------------|----------------|----------------|
| | $\delta_p = 2$ | $\delta_p = 4$ | $\delta_p = 4$ |
| 15 | 219 | 65 | 51 |
| 31 | 607 | 72 | 52 |
| 63 | 1501 | 76 | 51 |
| 127 | > 2000 | 77 | 50 |
| 255 | > 2000 | 78 | 49 |



Grid transfer operators comparison

We consider the following PDE

$$\left\{ \begin{array}{l} \frac{d^2}{dx^2} \left(a(x) \frac{d^2}{dx^2} u(x) \right) = g(x), \quad x \in (0, 1), \\ \text{homogeneous boundary conditions} \end{array} \right.$$

with nonconstant $a(x)$.

- It has order $m = 4$.
- Smoother: Gauss-Seidel
- Galerkin approach
- The condition for **V-cycle** is at least $\gamma_r + \gamma_p > m$.



Iteration numbers

V-cycle iteration numbers varying the problem size n and $a(x) = (x - 0.5)^2$.

| restriction prolongation | ϕ_2 ϕ_2 | ϕ_2 ϕ_4 | ϕ_2 g_c | ϕ_4 ϕ_4 | ϕ_4 g_c |
|-----------------------------|----------------------|----------------------|-------------------|----------------------|-------------------|
| n | # iterations | | | | |
| 15 | 15 | 10 | 10 | 9 | 9 |
| 31 | 33 | 13 | 17 | 10 | 11 |
| 63 | 61 | 17 | 24 | 13 | 11 |
| 127 | 101 | 26 | 27 | 17 | 13 |
| 255 | 155 | 35 | 29 | 20 | 16 |
| 511 | 221 | 44 | 36 | 24 | 19 |
| 1023 | 284 | 53 | 46 | 27 | 22 |

- g_c = cubic interpolation
- For the choices (ϕ_2, g_c) and (ϕ_4, ϕ_4) , the coarse matrices have the same bandwidth.



About V-cycle ...

V-cycle optimality with Ruge-Stüben theory and Perron-Frobenius theorem.

Theorem (A. Aricò, M. Donatelli, and S. Serra-Capizzano, 2004)

TGM \Rightarrow *V-cycle*

$$\limsup_{x \rightarrow x^0} \left| \frac{p(y)}{f(x)} \right| = c < +\infty, \quad \forall y \in \mathcal{M}(x),$$

[Multidimensional case A. Aricò and M. Donatelli, 2007]



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Remark

This is equivalent to $\gamma_p \geq m$ ($\gamma_p + \gamma_r \geq 2m$) instead of $2\gamma_p > m$ ($\gamma_p + \gamma_r > m$).

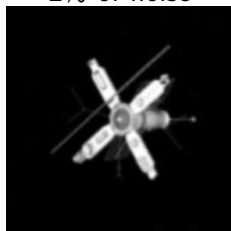


An image deblurring problem

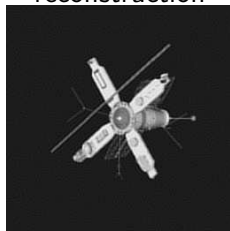
Riley regularization:

$$T_{n,n}(z + \theta)f = g$$

Observed image
2% of noise



MGM
reconstruction



| θ | 10^{-2} | 10^{-3} | 10^{-4} |
|----------------------|-----------|-----------|-----------|
| Linear interpolation | 45 | 286 | > 1000 |
| Toeplitz (*) | 49 | 73 | 73 |

$$(*) p(x, y) = (2 - 2 \cos(x))^2 (2 - 2 \cos(y))^2$$

Summary

- In the case of elliptic PDEs and Galerkin approach, the LFA and the theory for Toeplitz matrices give the same conditions.
- The LFA does not requires $R = P^T \implies R \neq P^T$ also for Toeplitz matrices.
- The convergence theory for Toeplitz matrices includes other applications (e.g. image deblurring).
- Introduce a class of grid transfer operators associated to B-spline.

