The Antireflective algebra and applications

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Outline

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1. The model problem (signal deconvolution)
2. Antireflective Boundary Conditions
   - The $\mathcal{AR}$ algebra
   - The spectral decomposition
3. Regularization by filtering
4. Numerical results

The model problem

- Problem: to approximate $f : \mathbb{R} \to \mathbb{R}$ from a blurred $g : \mathcal{I} \to \mathbb{R}$

$$g(x) = \int_{\mathcal{I}} k(x-y) f(y) \mathrm{d}y, \qquad x \in \mathcal{I} \subset \mathbb{R},$$

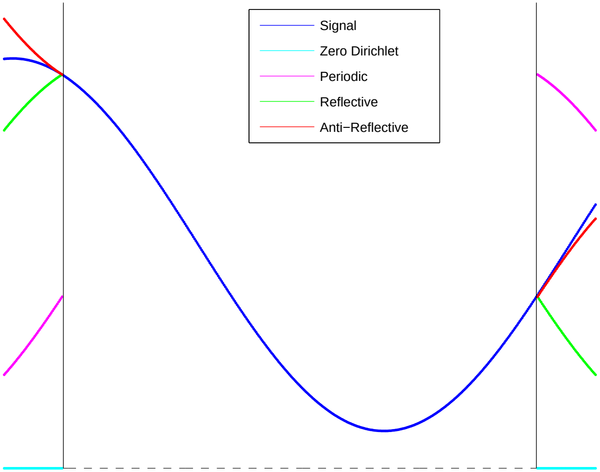
  the point spread function (PSF) k has compact support.

- Discretizing the integral by a rectangular quadrature rule and imposing boundary conditions:

$$A\mathbf{f} = \mathbf{g} + \text{noise}.$$

- The structure of A depends on k and the imposed boundary conditions.

Boundary conditions



Structure of the coefficient matrix A

Type	Generic PSF	Symmetric PSF
Zero Dirichlet	Toeplitz	Toeplitz
Periodic	Circulant	Circulant
Reflective	Toeplitz + Hankel	Cosine
Antireflective	Toeplitz + Hankel + rank 2	Sine + ... = Antireflective

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Definition of antireflective BCs

- The 1D antireflection is obtained by

$$\begin{aligned} f_{1-j} &= 2f_1 - f_{j+1} \\ f_{n+j} &= 2f_n - f_{n-j} \end{aligned}$$

[Serra-Capizzano, SISC. '03]

- In the multidimensional case we perform an antireflection with respect to every edge $\implies$ Tensor structure in the multidimensional case.

Approximation property

The reflective BCs assure the continuity at the boundary,
while the antireflective BCs assure also the
continuity of the first derivative.

Structural properties

- A = Toeplitz + Hankel + rank 2.
- Matrix vector product in $O(n\log(n))$ ops.

Symmetric PSF

- $S \in \mathbb{R}^{(n-2)\times(n-2)}$ diagonalizable by discrete sine transforms (DST)

$$A = \begin{bmatrix} 1 & & \\ * & & * \\ \vdots & S & \vdots \\ * & & * \\ & & 1 \end{bmatrix}$$

The $\mathcal{AR}$ algebra

With h cosine real-valued polynomial of degree at most n-3

$$AR_n(h) = \left[\begin{array}{ccc} h(0) & & \\ \mathbf{v}_{n-2}(h) & \tau_{n-2}(h) & J\mathbf{v}_{n-2}(h) \\ & & h(0) \end{array} \right],$$

where J is the flip matrix and

- $\tau_{n-2}(h) = Q\mathrm{diag}(h(\mathbf{x}))Q$, with Q being the DST and $\mathbf{x} = [\frac{j\pi}{n-1}]_{j=1}^{n-2}$
- $\mathbf{v}_{n-2}(h) = \tau_{n-2}(\phi(h))\mathbf{e}_1$, with $[\phi(h)](x) = \frac{h(x)-h(0)}{2\cos(x)-2}$.

$$\mathcal{AR}_n = \{A \in \mathbb{R}^{n\times n} \mid A = AR_n(h)\}$$

Properties of the $\mathcal{AR}_n$ algebra

Computational properties:

- $\alpha AR_n(h_1) + \beta AR_n(h_2) = AR_n(\alpha h_1 + \beta h_2)$,
- $AR_n(h_1)AR_n(h_2) = AR_n(h_1 h_2)$,

Diagonalization

- $\mathcal{AR}_n$ is commutative, since $h = h_1 h_2 \equiv h_2 h_1$,
- the elements of $\mathcal{AR}_n$ are diagonalizable and have a common set of eigenvectors.
- not all matrices in $\mathcal{AR}_n$ are normal.

$AR_n(\cdot)$ Jordan Canonical Form

Theorem

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$AR_n(h) = T_n \mathrm{diag}(h(\hat{\mathbf{x}}))\, T_n^{-1},$$

where $\hat{\mathbf{x}} = [0, \mathbf{x}^T, 0]^T$, $\mathbf{x} = [\frac{j\pi}{n-1}]_{j=1}^{n-2}$ *and*

$$T_n = \left(1 - \frac{\tilde{\mathbf{x}}}{\pi}, \sin(\tilde{\mathbf{x}}), \ldots, \sin((n-2)\tilde{\mathbf{x}}), \frac{\tilde{\mathbf{x}}}{\pi}\right),$$

with $\tilde{\mathbf{x}} = [0, \mathbf{x}^T, \pi]^T$.

Computational issues

- Inverse antireflective transform T_n^{-1} has a structure analogous to T_n.
- The matrix vector product with T_n and T_n^{-1} can be computed in $O(n\log(n))$, but they are not unitary.
- The eigenvalues are mainly obtained by DST.
  - $h(0)$ with multiplicity 2
  - DST of the first column of $\tau_{n-2}(h)$

Outline

Antireflective BCs and $\mathcal{AR}$ algebra

If the PSF is symmetric, imposing antireflective BCs
the matrix A belongs to $\mathcal{AR}$.

A possible problem

The $\mathcal{AR}$ algebra is not closed with respect to transposition.

Spectral properties

- Large eigenvalues are associated to lower frequencies.
- $h(0)$ is the largest eigenvalue and the corresponding eigenvector is the sampling of a linear function.
- Hanke et al. in [SISC '08] firstly compute the components of the solution related to the two linear eigenvectors and then regularize the inner part that is diagonalized by DST.

Regularization by filtering

- $A = T_n D_n T_n^{-1}$ where $\mathbf{d} = h(\hat{\mathbf{x}})$ and

$$T_n = \begin{bmatrix} \mathbf{t}_1 & \cdots & \mathbf{t}_n \end{bmatrix}, \quad D_n = \mathrm{diag}(\mathbf{d}), \quad T_n^{-1} = \begin{bmatrix} \tilde{\mathbf{t}}_1^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{bmatrix}$$

- A spectral filter solution is given by

$$\mathbf{f}_{\mathrm{reg}} = \sum_{i=1}^{n} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i} \mathbf{t}_i \,, \tag{1}$$

where $\mathbf{g}$ is the observed image and ϕ_i are the filter factors.

Filter factors

- Truncated spectral value decomposition (TSVD)

$$\phi_i^{\text{tsvd}} = \begin{cases} 1 & \text{if } d_i \geq \delta \\ 0 & \text{if } d_i < \delta \end{cases}$$

- Tikhonov regularization

$$\phi_i^{\text{tik}} = \frac{d_i^2}{d_i^2 + \alpha}, \qquad \alpha > 0,$$

- Imposing $\phi_1 = \phi_n = 1$, the solution $\mathbf{f}_{\text{reg}}$ is exactly that obtained by the homogeneous antireflective BCs in [Hanke et al. SISC '08].

Reblurring

Filtering with the Tikhonov filter ϕ_i^{tik} is equivalent to solve

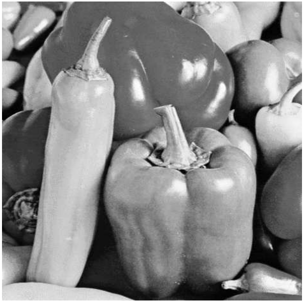
$$(A^2 + \alpha I)\,\mathbf{f}_{\text{reg}} = A\mathbf{g}$$

- This is the reblurring approach where for a symmetric PSF A^T is replace by A itself [D. and Serra-Capizzano, IP '05].
- In the general case (nonsymmetric PSF), the reblurring replace the transposition with the correlation.
- Reblurring is equivalent to regularize the continuous problem and then to discretize imposing the boundary conditions.

Outline

Tikhonov regularization

- Gaussian blur
- 1% of white Gaussian noise



True image

Observed image

Restored images.

Reflective

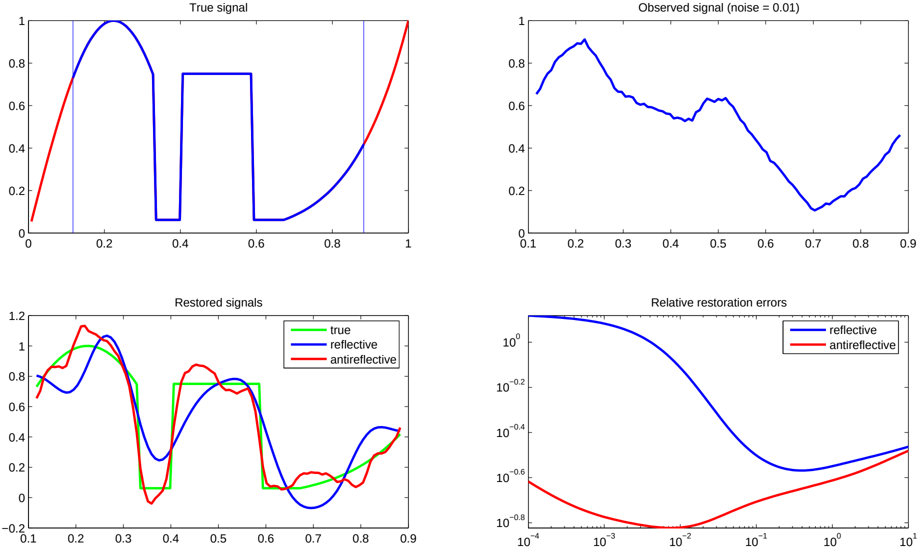
Antireflective

Best restoration errors

Relative restoration error defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$,
where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.

noise	Reflective	Antireflective
10%	0.1284	0.1261
1%	0.1188	0.1034
0.1%	0.1186	0.0989

1D Example (Tikhonov with Laplacian)



Conclusions

Summarizing

- The antireflective have the same computationally properties of the reflective boundary conditions but usually lead to better restorations.
- The importance of to have good boundary conditions increases when the PSF has a large support and the noise is not huge.

Work in progress ...

- Other applications (other regularization methods, filtering for trend estimation of time series, . . .).
- Theoretical analysis of the reblurring strategy.

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