Matrix structures and image restoration: boundary conditions, and re-blurring

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Outline

1. Image restoration and boundary conditions
2. Antireflective Boundary Conditions
   - The antireflective algebras
   - Re-blurring
3. Numerical results
4. Conclusions

Outline

The model problem

The restored image $\mathbf{f}$ is obtained by "solving":

$$A\mathbf{f} = \mathbf{g}$$

- $\mathbf{g} = \mathrm{vec}(G)$ where G is the

  observed image = blurred image + noise

- A = two-level matrix associated to the point spread function (PSF).
- The PSF is the observation of one point (e.g., a star in astronomy) and we assume that it is space invariant.

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Requirements

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- Possibility to resort to fast transforms like FFT

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How to satisfy the requirements

1. Modelistic in the problem formalization
2. Computational in the definition of the regularization method (multigrid-type algorithms, regularizing preconditioners, etc.)

Classic boundary conditions

Classic Boundary Conditions (BCs) on the image F of size $n \times n$:

zero-Dirichlet	Periodic (P-BCs)	Reflective (R-BCs)
$\begin{matrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{matrix}$	$\begin{matrix} F & F & F \\ F & F & F \\ F & F & F \end{matrix}$	$\begin{matrix} F_{rc} & F_r & F_{rc} \\ F_c & F & F_c \\ F_{rc} & F_r & F_{rc} \end{matrix}$ F_c, F_r, F_{rc} "flip" of rows and/or columns of F
BTTB	BCCB	Block $T + H$ with $T + H$ blocks (DCT-III two-level)

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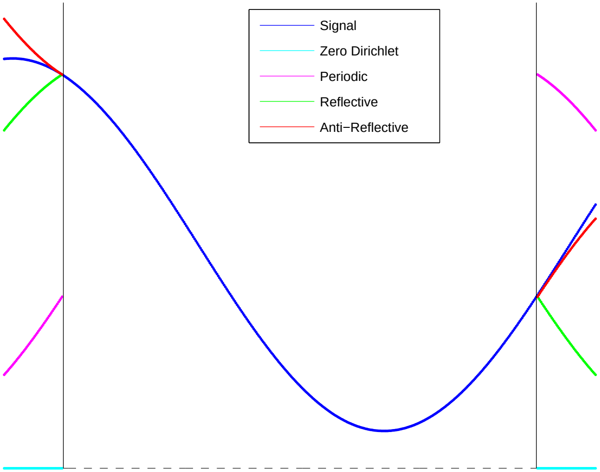
- The matrix vector product requires $O(n^2 \log(n))$ ops,
- The inversion requires $O(n^2 \log(n))$ ops only for P-BCs and DCT-III.

Outline

The 1D case


Signal
Zero Dirichlet
Periodic
Reflective
Anti−Reflective

Definition of antireflective BCs

- In 1D the antireflection is obtained by

$$\begin{aligned} f_{1-j} &= 2f_1 - f_{j+1} \\ f_{n+j} &= 2f_n - f_{n-j} \end{aligned}$$

- In the multidimensional case we perform an antireflection with respect to every edge.

$$\Downarrow$$

In the 2D case at the corner we antireflect first with respect to the x and then with respect to the y.

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The reflective BCs (R-BCs) assure the continuity at the edge, while the antireflective BCs (AR-BCs) assure also the continuity of the normal derivative.

Structural properties (1D case)

Generic PSF

- $A =$ Toeplitz $+$ Hankel $+$ rank 2.
- Matrix vector product in $O(n \log(n))$ ops.

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Strongly symmetric PSF

- For $S \in \mathbb{R}^{(n-2)\times(n-2)}$ diagonalizable using DST-I

$$A = \begin{bmatrix} 1 & & \\ * & & * \\ \vdots & S & \vdots \\ * & & * \\ & & 1 \end{bmatrix}$$

The $\mathcal{S}$ algebra

$$\mathcal{S} = \left\{ M \,\middle|\, M = \begin{bmatrix} \alpha & \mathbf{0}^T & 0 \\ \mathbf{v} & \hat{M} & \mathbf{w} \\ 0 & \mathbf{0}^T & \beta \end{bmatrix},\ \alpha, \beta \in \mathbb{R},\ \mathbf{v}, \mathbf{w}, \mathbf{0} \in \mathbb{R}^{n-2},\ \hat{M} \in \tau_{n-2} \right\}$$

Theorem

Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$, then for $M \in \mathcal{S}$:

- *(i) every linear system $M\mathbf{f} = \mathbf{g}$ can be solved in $O(n\log(n))$ ops if M is invertible;*
- *(ii) every matrix vector product $\mathbf{g} := M\mathbf{f}$ costs $O(n\log(n))$ ops;*
- *(iii) $\mathcal{S}$ is an algebra, i.e., it is closed under linear combinations, product and inversion (dimension $3n-4$).*

The $\mathcal{AR}$ subalgebra

Let h be real-valued cosine polynomial

$$AR_n(h) = \begin{bmatrix} h(0) & & \\ \mathbf{v}_{n-2}(h) & \tau_{n-2}(h) & J\mathbf{v}_{n-2}(h) \\ & & h(0) \end{bmatrix},$$

where

- J is the flip matrix,
- $\tau_{n-2}(h) = Q\mathrm{diag}_{j=1,\dots,n-2}(h(\frac{j\pi}{n-1}))Q$, with Q the DST-I,
- $\mathbf{v}_{n-2}(h) = \tau_{n-2}(\phi(h))\mathbf{e}_1$ with $[\phi(h)](x) = \frac{h(x)-h(0)}{2\cos(x)-2}$.

Properties of the $\mathcal{AR}$ subalgebra

- $\mathcal{AR} \subset \mathcal{S}$,
- $\dim(\mathcal{AR}) = n - 2$,

Computational properties:

- $\alpha AR_n(h_1) + \beta AR_n(h_2) = AR_n(\alpha h_1 + \beta h_2)$,
- $AR_n(h_1) AR_n(h_2) = AR_n(h_1 h_2)$,

Diagonalization

- $\mathcal{AR}$ is commutative, since $h = h_1 h_2 \equiv h_2 h_1$,
- the elements of $\mathcal{AR}$ are diagonable and have a common set of eigenvectors.

$AR_n(\cdot)$ Jordan Canonical Form

Theorem

Let h be a cosine real-valued polynomial of degree at most n-3, then

$$AR_n(h) = T_n \mathrm{diag}_{y \in G}(h(y)) T_n^{-1},$$

where $G = \{\frac{j\pi}{n-1} | j = 0, \ldots, n-1\}$ *and*

$$T_n = \left(1 - \frac{y}{\pi}, \sin(y), \ldots, \sin((n-2)\pi), \frac{y}{\pi}\right)\Big|_{y \in G}.$$

Remark

The matrix vector product with T_n and T_n^{-1} can be computed in $O(n \log(n))$ but they can not be unitary.

The multidimensional case

It is trivial to extend the previous results to the multidimensional case thanks to the tensor structure.

Tensor structure in the BCs

The tensor structure arise naturally imposing the antireflection with respect to every variable separately.

Preliminary results

- In the noise free case the AR-BCs lead to a restoration qualitatively better than the other BCs.
- In the noise case regularization is necessary. For instance Tikhonov or CG for normal equations.
- In such cases the AR-BCs can be worse than the other BCs (at least with respect to R-BCs).
- The AR-BCs lose the algebra structure also for strongly symmetric PSFs.

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First Motivation

Let the PSF be strongly symmetric. For the classic BCs $A^T = A$, while with AR-BCs $A^T \neq A$ and it is no longer a blurring operator (low-pass filter).

Re-blurring

Proposal: To replace A^T with A' obtained imposing the BCs to the PSF rotated of 180°.

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Remark:

The re-blurring replaces transposition with correlation

The origin of re-blurring

The continuous model

$$g(x) = (Kf)(x) = \int k(x-y)f(y)\mathrm{d}y$$

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Normal equations

1. Discretization and imposition of BCs
2. Look at a least squares solution

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Re-blurring ($1 \leftrightarrow 2$)

1. Look at a least squares solution
2. Discretization and imposition of BCs

Algebraic properties

- In the 1D case and generic PSF the re-blurring matrix is

$$A' = JAJ$$

  where J is the flip matrix.
- For T Toeplitz $T^T = JTJ \Rightarrow$ for D-BCs and P-BCs the re-blurring and normal equations are the same.
- For H Hankel usually $H^T \neq JHJ \Rightarrow$ if the PSF is not at least centro-symmetric for the R-BCs, the re-blurring is different from normal equations (for AR-BCs always!).

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Remark: For AR-BCs and strongly symmetric PSF we overcome the computational problems, since we are working in the $\mathcal{S}_d$ algebra.

Outline

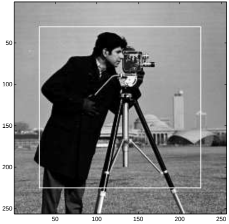
Tikhonov regularization

Classical Tikhonov equation for R-BCs:

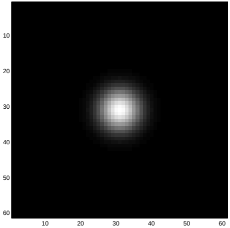
$$(A^T A + \mu I)\mathbf{f} = A^T \mathbf{g}$$

and its reblurring version for AR-BCs:

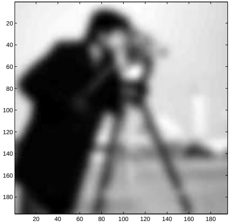
$$(A' A + \mu I)\mathbf{f} = A' \mathbf{g}.$$



True image



Gaussian PSF



Blurred image

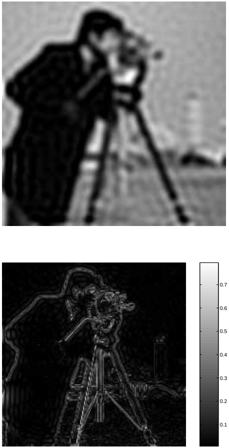
Best restoration errors for Tikhonov

Relative restoration error (RRE) defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$, where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.

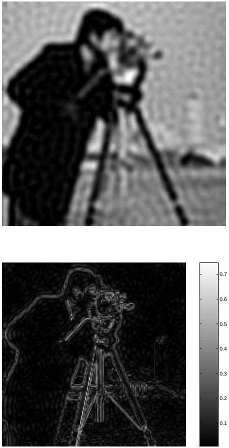
SNR	Reflective	Anti-Reflective
∞	0.1974	0.1815
50	0.1974	0.1839
40	0.1980	0.1900
30	0.2056	0.2031
20	0.2174	0.2175
10	0.2458	0.2500

$\text{SNR} = 20 \log_{10} \|\mathbf{g}_b\|_2 / ||\boldsymbol{\nu}||_2$, where $\mathbf{g}_b$ is the blurred image without noise and $\boldsymbol{\nu}$ is the noise vector

Restored images with SNR $= 40$.



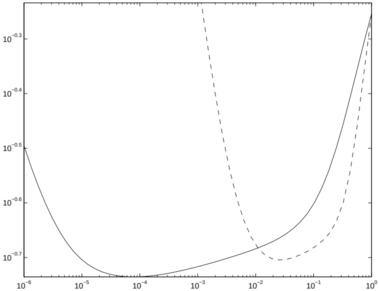
Reflective



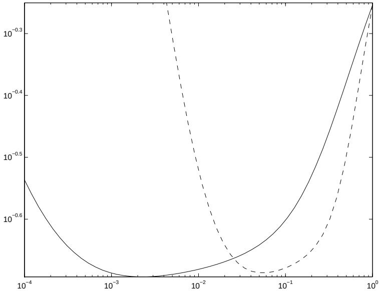
Anti-reflective

RRE vs μ

$-$ Antireflective and $--$ Reflective BCs



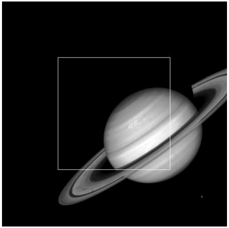
SNR$=$ 50



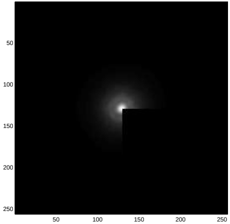
SNR$=$ 30

Conjugate Gradient regularization (CGLS)

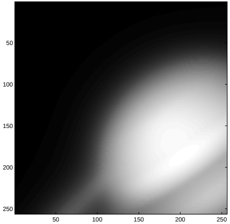
- We upgraded the Matlab Toolbox (RestoreTools) of J. Nagy et al. adding the AR-BCs for iterative methods.
- This toolbox implements the reblurring approach.
- Example



True image



Gaussian PSF



Blurred image

CGLS minimum RREs.

SNR	Periodic	Reflective	Anti-Reflective
∞	0.2353	0.1521	0.0816
50	0.2353	0.1521	0.0819
40	0.2356	0.1525	0.0837
30	0.2371	0.1552	0.1063
20	0.2529	0.1807	0.1553
10	0.3717	0.3336	0.3319

RRE vs number of iterations

$-$ Antireflective and $--$ Reflective BCs

SNR$=$ 50

SNR$=$ 30

Outline

Summarizing...

- The AR-BCs with reblurring strategy have the same computationally properties of the R-BCs but lead to better restorations.
- With the AR-BCs the matrix-vector product can be computed within $O(n^2 \log(n))$ operations also for nonsymmetric PSFs.
- The importance of to have good BCs increases when the PSF has a large support and the noise is not too huge.

Future work

- Theoretical analysis of the re-blurring strategy and an extensive comparison with the classic normal equations.
- In general (reflective or antireflective BCs in our tests) $A'A$ is not always s.p.d. but CG works fine.
- Applications of the AR-BCs spectral decomposition like filtering methods, etc.

Resources

At my home-page:

http://scienze-como.uninsubria.it/mdonatelli/

there are:

- Preprints
- Software
- Slides

THANKS!