The Antireflective algebra with applications to image deblurring

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Outline

Signal/Image deblurring

Antireflective Boundary Conditions

- The $\mathcal{AR}$ algebra
- The spectral decomposition

Regularization by filtering

Numerical results

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The model problem

The restored signal (image) $\mathbf{f}$ is obtained by "solving":

$$A\mathbf{f} = \mathbf{g}$$

- $\mathbf{g}$ = observed signal = blurred signal + noise
- A = matrix associated to the point spread function (PSF).
- The PSF is the observation of one source point,
  we assume that it is space invariant.

Targets of the restoration

Requirements

- Restored signal of good quality
- Possibility to resort to fast transforms like FFT

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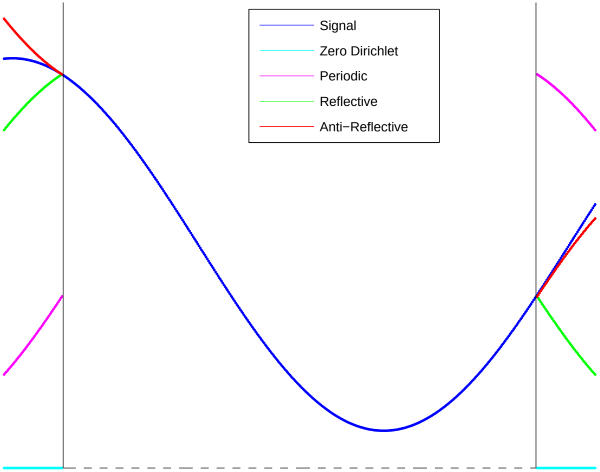
How to satisfy the requirements

- Problem formalization for the model
- Computational in the definition of the regularization method (multilevel algorithms, regularizing preconditioners, etc.)

Boundary conditions



The coefficient matrix A

The matrix-vector product can be done in $O(n \log n)$ in every case.

Type	Matrix structure	Matrix inversion
Zero Dirichlet	Toeplitz	$O(n^2)$
Periodic	Circulant	$O(n \log n)$ (FFT)
Reflective	Toeplitz + Hankel	generic PSF: $O(n^2)$ symm.: $O(n \log n)$ (DCT-III)
Anti-reflective	Toeplitz + Hankel + rank 2	generic PSF: $O(n^2)$ symm.: $O(n \log n)$ (DST-I + ...)

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Definition of antireflective BCs

- In 1D, the antireflection is obtained by

$$\begin{aligned} f_{1-j} &= 2f_1 - f_{j+1} \\ f_{n+j} &= 2f_n - f_{n-j} \end{aligned}$$

[S. Serra-Capizzano, SIAM J. Sci. Comput. 2003]

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The reflective BCs assure the continuity at the boundary, while the antireflective BCs assure also the continuity of the first (normal) derivative.

Structural properties

Generic PSF

- $A =$ Toeplitz $+$ Hankel $+$ rank 2.
- Matrix vector product in $O(n\log(n))$ ops.

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Symmetric PSF

- For $S \in \mathbb{R}^{(n-2)\times(n-2)}$ diagonalizable using DST-I

$$A = \begin{bmatrix} 1 & & \\ * & & * \\ \vdots & S & \vdots \\ * & & * \\ & & 1 \end{bmatrix}$$

The $\mathcal{AR}$ algebra

With h cosine real-valued polynomial of degree at most n-3

$$AR_n(h) = \begin{bmatrix} h(0) & & \\ \mathbf{v}_{n-2}(h) & \tau_{n-2}(h) & J\mathbf{v}_{n-2}(h) \\ & & h(0) \end{bmatrix},$$

where

- J is the flip matrix,
- $\tau_{n-2}(h) = Q\mathrm{diag}_{j=1,\ldots,n-2}(h(\frac{j\pi}{n-1}))Q$, with Q the DST-I,
- $\mathbf{v}_{n-2}(h) = \tau_{n-2}(\phi(h))\mathbf{e}_1$ with $[\phi(h)](x) = \frac{h(x)-h(0)}{2\cos(x)-2}$.

$$\mathcal{AR}_n = \{A \in \mathbb{R}^{n\times n} \mid A = AR_n(h)\}$$

Properties of the $\mathcal{AR}_n$ algebra

Computational properties:

- $\alpha AR_n(h_1) + \beta AR_n(h_2) = AR_n(\alpha h_1 + \beta h_2)$,
- $AR_n(h_1)AR_n(h_2) = AR_n(h_1 h_2)$,

Diagonalization

- $\mathcal{AR}_n$ is commutative, since $h = h_1 h_2 \equiv h_2 h_1$,
- the elements of $\mathcal{AR}_n$ are diagonalizable and have a common set of eigenvectors.
- not all matrices in $\mathcal{AR}_n$ are normal.

$AR_n(\cdot)$ Jordan Canonical Form

Theorem

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$AR_n(h) = T_n \mathrm{diag}_{y \in G \cup \{0\}}(h(y))\, T_n^{-1},$$

where $G = \{\frac{j\pi}{n-1} | j = 0, \ldots, n-2\}$ *and*

$$T_n = \left(1 - \frac{y}{\pi}, \sin(y), \ldots, \sin((n-2)\pi), \frac{y}{\pi}\right)\Big|_{y \in G \cup \{\pi\}}.$$

Computational issues

- Inverse antireflective transform T_n^{-1} has a structure analogous to T_n.
- The matrix vector product with T_n and T_n^{-1} can be computed in $O(n\log(n))$, but they are not unitary.
- The eigenvalues are mainly obtained by DST-I.
  - $h(0)$ with multiplicity 2
  - DST-I of the first column of $\tau_{n-2}(h)$

Outline

Antireflective BCs and $\mathcal{AR}$ algebra

If the PSF is symmetric, imposing antireflective BCs
the matrix A belongs to $\mathcal{AR}$.

A possible problem

The $\mathcal{AR}$ algebra is not closed with respect to transposition.

Spectral properties

- Large eigenvalues are associated to lower frequencies.
- $h(0)$ is the largest eigenvalue and the corresponding eigenvector is the sampling of a linear function.
- Hanke et al. in [SISC '08] firstly compute the components of the solution related to the two linear eigenvectors and then regularize the inner part that is diagonalized by DST-I.

Regularization by filtering

- $A = AR_n(h) = T_n D_n T_n^{-1}$

$$T_n = \left[\begin{array}{ccc} \mathbf{t}_1 & \cdots & \mathbf{t}_n \end{array}\right], \quad D_n = \mathrm{diag}_{i=1,\dots,n}(d_i), \quad T_n^{-1} = \left[\begin{array}{c} \tilde{\mathbf{t}}_1^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{array}\right]$$

with $d_1 = d_n = h(0)$ and $d_i = h\left(\frac{(i-1)\pi}{n-1}\right)$, for $i = 2, \dots, n-2$,

- A spectral filter solution is given by

$$\mathbf{f}_{\mathrm{reg}} = \sum_{i=1}^{n} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i} \mathbf{t}_i \,, \qquad (1)$$

where $\mathbf{g}$ is the observed image and ϕ_i are the filter factors.

Filter factors

- Truncated spectral value decomposition (TSVD)

$$\phi_i^{\text{tsvd}} = \begin{cases} 1 & \text{if } d_i \geq \delta \\ 0 & \text{if } d_i < \delta \end{cases}$$

- Tikhonov regularization

$$\phi_i^{\text{tik}} = \frac{d_i^2}{d_i^2 + \lambda}, \qquad \lambda > 0,$$

- Imposing $\phi_1 = \phi_n = 1$, the solution $\mathbf{f}_{\text{reg}}$ is exactly that obtained by the homogeneous antireflective BCs in [Hanke et al. SISC '08].

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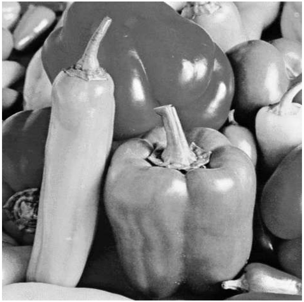
Tikhonov regularization

- Gaussian blur
- 1% of white Gaussian noise



True image

Observed image

Restored images.

Reflective

Antireflective

Best restoration errors

Relative restoration error defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$,
where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.

noise	Reflective	Anti-Reflective
10%	0.1284	0.1261
1%	0.1188	0.1034
0.1%	0.1186	0.0989

Conclusions

- The AR-BCs have the same computationally properties of the R-BCs but lead to better restorations.
- The importance of to have good BCs increases when the PSF has a large support and the noise is not huge.
- The spectral decomposition of AR matrices could be useful for other regularization methods?
- At my home-page:

  http://scienze-como.uninsubria.it/mdonatelli/

  Preprints, software, slides, . . .