Antireflective boundary conditions for deblurring problems

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Outline

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The model problem

- Problem: to approximate $f : \mathbb{R} \rightarrow \mathbb{R}$ from a blurred $g : \mathcal{I} \rightarrow \mathbb{R}$

$$g(x) = \int_{\mathcal{I}} k(x-y)f(y)\mathrm{d}y, \qquad x \in \mathcal{I} \subset \mathbb{R},$$

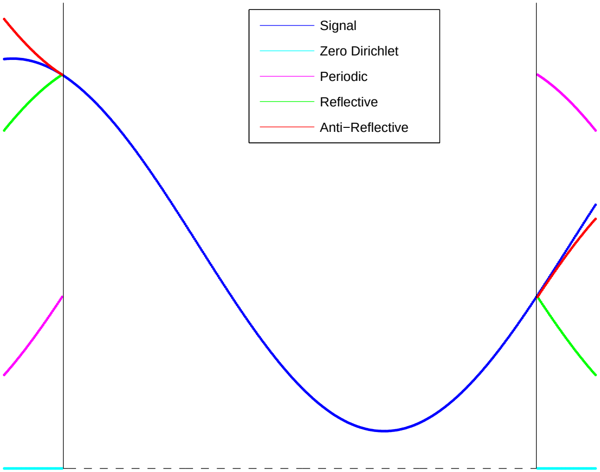
  the point spread function (PSF) k has compact support.

- Discretizing the integral by a rectangular quadrature rule and imposing boundary conditions:

$$A\mathbf{f} = \mathbf{g} + \text{noise}.$$

- The structure of A depends on k and the imposed boundary conditions.

Boundary conditions



Structure of the coefficient matrix A

Type	Generic PSF	Symmetric PSF
Zero Dirichlet	Toeplitz	Toeplitz
Periodic	Circulant	Circulant
Reflective	Toeplitz + Hankel	Cosine
Antireflective	Toeplitz + Hankel + rank 2	Sine + ... =

Requests:

1. fast transforms
2. precise model

Outline

Definition of antireflective BCs

- The 1D antireflection is obtained by

$$\begin{aligned} f_{1-j} &= 2f_1 - f_{j+1} \\ f_{n+j} &= 2f_n - f_{n-j} \end{aligned}$$

  [Serra-Capizzano, 2003]

- In the multidimensional case we perform an antireflection with respect to every edge $\Longrightarrow$ Tensor structure in the multidimensional case.

Approximation property

The reflective BCs assure the continuity at the boundary, while the antireflective BCs assure also the continuity of the first derivative.

Structural properties

- A = Toeplitz + Hankel + rank 2.
- Matrix vector product in $O(n\log(n))$ ops.

Symmetric PSF

- $S \in \mathbb{R}^{(n-2)\times(n-2)}$ diagonalizable by discrete sine transforms (DST)

$$A = \begin{bmatrix} 1 & & \\ * & & * \\ \vdots & S & \vdots \\ * & & * \\ & & 1 \end{bmatrix}$$

Spectral decomposition

Theorem

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$A = AR_n(h) = T_n \mathrm{diag}(h(\hat{\mathbf{x}}))\, T_n^{-1},$$

where $\hat{\mathbf{x}} = [0,\, \mathbf{x}^T,\, 0]^T$, $\mathbf{x} = [\frac{j\pi}{n-1}]_{j=1}^{n-2}$ *and*

$$T_n = \left(1 - \frac{\tilde{\mathbf{x}}}{\pi},\, \sin(\tilde{\mathbf{x}}),\, \ldots,\, \sin((n-2)\tilde{\mathbf{x}}),\, \frac{\tilde{\mathbf{x}}}{\pi}\right),$$

with $\tilde{\mathbf{x}} = [0,\, \mathbf{x}^T,\, \pi]^T$.

[Aricò, D., Nagy, and Serra-Capizzano, 2008]

Algebraic interpretation

Eigenvalues:

- $h(0)$ with multiplicity 2
- DST of the first column of $\tau_{n-2}(h)$

Eigenvectors:

A uniform sampling in $[0, \pi]$ of some special functions

1. fast transform: $\sin(k\mathbf{x})$, for $k = 1, \ldots, n-2$ (DST)
2. model precision: $\pi - \mathbf{x}$ and $\mathbf{x}$ are associated to $h(0) = 1 \Longrightarrow$ preserve linear functions.

Inverse antireflective transform T_n^{-1} has a structure analogous to T_n.

Spectral properties

- Large eigenvalues are associated to lower frequencies.
- $h(0)$ is the largest eigenvalue and the corresponding eigenvector is the sampling of a linear function.
- Hanke and Christiansen (2008) firstly compute the components of the solution related to the two linear eigenvectors and then regularize the inner part that is diagonalized by DST.

Outline

Regularization by filtering

- $A = T_n D_n T_n^{-1}$ where $\mathbf{d} = h(\hat{\mathbf{x}})$ and

$$T_n = \begin{bmatrix} \mathbf{t}_1 & \cdots & \mathbf{t}_n \end{bmatrix}, \quad D_n = \mathrm{diag}(\mathbf{d}), \quad T_n^{-1} = \begin{bmatrix} \tilde{\mathbf{t}}_1^T \\ \vdots \\ \tilde{\mathbf{t}}_n^T \end{bmatrix}$$

- A spectral filter solution is given by

$$\mathbf{f}_{\mathrm{reg}} = \sum_{i=1}^{n} \phi_i \frac{\tilde{\mathbf{t}}_i^T \mathbf{g}}{d_i} \mathbf{t}_i \,, \qquad (1)$$

where $\mathbf{g}$ is the observed image and ϕ_i are the filter factors.

Filter factors

- Truncated spectral value decomposition (TSVD)

$$\phi_i^{\text{tsvd}} = \begin{cases} 1 & \text{if } d_i \geq \delta \\ 0 & \text{if } d_i < \delta \end{cases}$$

- Tikhonov regularization

$$\phi_i^{\text{tik}} = \frac{d_i^2}{d_i^2 + \alpha}, \qquad \alpha > 0,$$

- Imposing $\phi_1 = \phi_n = 1$, the solution $\mathbf{f}_{\text{reg}}$ is exactly that obtained by homogeneous antireflective BCs in [Hanke and Christiansen 2008].

Reblurring

Filtering with the Tikhonov filter ϕ_i^{tik} is equivalent to solve

$$(A^2 + \alpha I)\,\mathbf{f}_{\text{reg}} = A\mathbf{g}$$

- This is the reblurring approach where for a symmetric PSF A^T is replaced by A itself [D. and Serra-Capizzano, 2005].
- In the general case (nonsymmetric PSF), the reblurring replace the transposition with the correlation.
- Reblurring is equivalent to regularize the continuous problem and then to discretize imposing the boundary conditions.

Convergence

From the SVD of the matrix T_n [D. and Hanke, 2010]

Theorem

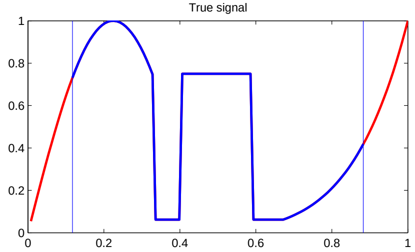
$\|T_n\|_2\|T_n^{-1}\|_2 \doteq \sqrt{2n}$.

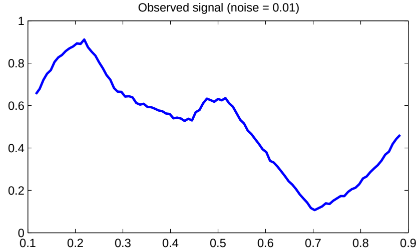
Theorem

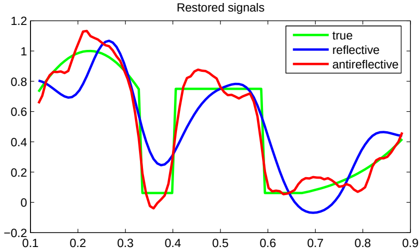
The reblurring with antireflective boundary conditions and a symmetric PSF is a regularization method.

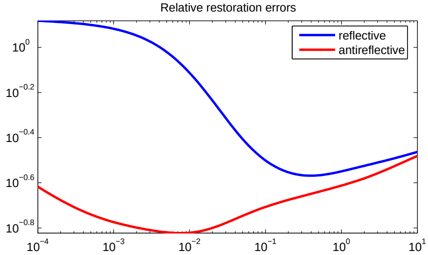
Outline

1D Example (Tikhonov with Laplacian)



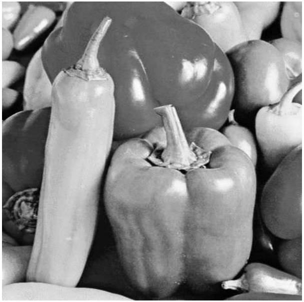






Tikhonov regularization

- Gaussian blur
- 1% of white Gaussian noise



True image

Observed image

Restored images.

Reflective

Antireflective

Best restoration errors

Relative restoration error defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$,
where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.

noise	Reflective	Antireflective
10%	0.1284	0.1261
1%	0.1188	0.1034
0.1%	0.1186	0.0989

Outline

1. The model problem (signal deconvolution)
2. Antireflective Boundary Conditions
   - The definition
   - The spectral decomposition
3. Regularization by filtering
   - The algorithm
   - The convergence
4. Numerical results
5. High order BCs

Algebraic BC with cosine basis

Let h be a cosine real-valued polynomial of degree at most n-3. Then

$$A_C(h) = C_n \mathrm{diag}(h(\hat{\mathbf{x}})) C_n^{-1},$$

where $\hat{\mathbf{x}} = [0, \tilde{\mathbf{x}}^T, 0]^T$, $\tilde{\mathbf{x}} = \left[\frac{(j-1)\pi}{n-2}\right]_{j=1}^{n-2}$ and

$$C_n = \left[\left(\frac{(2n-3)\pi}{2n-4} - \mathbf{x}\right)^2, \cos(0\mathbf{x}), \ldots, \cos((n-3)\mathbf{x}), \mathbf{x}^2\right],$$

with $\mathbf{x} = \left[\frac{(2i-1)\pi}{2n-4}\right]_{j=0}^{n-1}$.

[D., 2010]

Notes on high order BC

1. The first and the last rows are not separable (otherwise we introduce a discontinuity).
2. When the PSF is nonsymmetric it is possible to use the exponential basis (DFT).
3. Some a-priori knowledge on the true object can be included as an eigenvector associated to the eigenvalue 1.

Tikhonov regularization

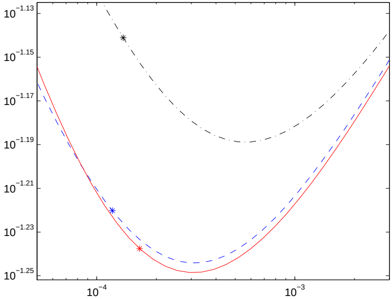
- Out of focus blur
- 0.1% of Gaussian noise

True image

Observed image

Restoration errors

Relative restoration error (RRE) defined as $\|\hat{\mathbf{f}} - \mathbf{f}\|_2 / \|\mathbf{f}\|_2$, where $\hat{\mathbf{f}}$ is the computed approximation of the true image $\mathbf{f}$.



RRE: — A_C, - - - antireflective,
- · - reflective ($*$ is μ_{GCV})

	μ_{opt}	μ_{GCV}
reflective	0.064	0.072
antirefl.	0.057	0.060
A_C	0.056	0.057

Restored images.

Reflective

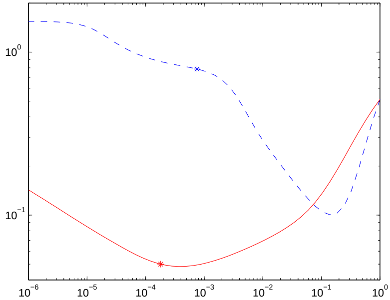
Antireflective

A_C

Nonsymmetric PSF.

Out of focus + motion blur

Observed image



RRE: — A_F, - - - periodic
($*$ is μ_{GCV}).

Restored images.

periodic with μ_{opt}

A_F with μ_{opt}

A_F with μ_{GCV}

Summarizing

The importance of good boundary conditions increases when the PSF has a large support and the noise is not huge.

High order BC:

- Filtering in $O(n \log(n))$ by few fast transforms.
- Work both for symmetric and nonsymmetric PSF.
- Can include some a-priori knowledge on the true object.