Arnoldi methods for image deblurring with nonsymmetric blur and anti-reflective boundary conditions

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Outline

1. Image deblurring problems
2. Boundary conditions
3. Reblurring
4. Arnoldi methods
5. Reblurring preconditioning
6. Numerical results

Image deblurring problems

$$b \quad = \quad A \quad * \quad x \quad + \quad e$$

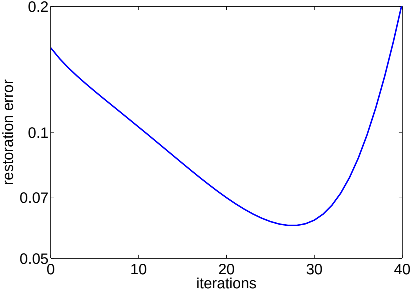
- A is large and severely ill-conditioned (space invariant blur)
- b known, measured data
- e noise, s.t. $\|e\| = \delta$

Goal: compute approximation of the noise free solution x.

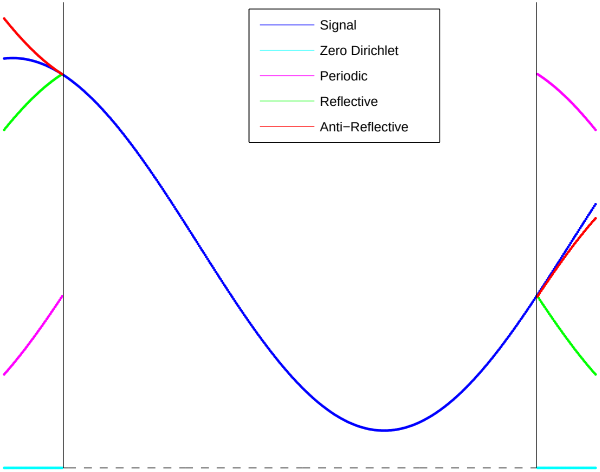
Regularization: $A^{-1}b = x + A^{-1}e$ is completely corrupted by amplification of noise.

Iterative regularization methods (semi-convergence)

- Some classical iterative methods firstly reduce the algebraic error into the low frequencies (well-conditioned subspace), when they arrive to reduce the algebraic error into the high frequencies then the restoration error increases because of the noise.
- The regularization parameter is the stopping iteration.



Boundary conditions (BCs)



BCs and structure of A

B = block, T = Toeplitz, H = Hankel, C = circulant

- Zero Dirichlet BCs: BTTB; Periodic BCs: BCCB.
- Reflective BCs: [Ng et al., SISC 1999]:
  BTTB + BTHB + BHTB + BHHB
- Antireflective BCs: [Serra-Capizzano, SISC 2003]:
  BTTB + BTHB + BHTB + BHHB + rank $4n$
  (may be that the rank $4n$ has a 1D structure inside ...).
- For symmetric blur and reflective (antireflective) BCs the spectral decomposition of A can be obtained by discrete cosine (sine) transform, but we are interested in the nonsymmetric case ...
- Syntethic BCs [Fan, Nagy, LAA 2010]: only by padding and there is not a matrix structure.

Matrix-vector product

The discrete convolution with proper BCs, i.e., the matric vector product $y = Ax$, can be easily computed as follows:

1. z = padarray(x, BCs, ...) s.t. has double size

 = padarray( , reflective)

2. circular convolution by FFT $\leftrightarrow$ w = circconv(PSF,z)
3. cut the inner part $\leftrightarrow$ y = cut(w)

w =  y = 

Problem with A^T (nonsymmetric blur)

- For Reflective BCs the matrix-vector product with A^T requires requires several FFTs.
- For Antireflective BCs the matrix-vector product with A^T requires $O(n^3)$ (due to the rank $4n$) instead of $O(n^2 \log(n))$ of the FFT.
- For Synthethic BCs A^T is not available.

The adjoint operator

- The continuous problem behind the blurring problem is the convolution

$$[Kf](x) = \int_\Omega K(x-y)f(y)\mathrm{d}y,$$

all functions are real and the adjoint operator K^* is the correlation operator

$$[K^*f](x) = \int_\Omega K(y-x)f(y)\mathrm{d}y,$$

the only difference with K is the sign of the variable.

- Discretization with BCs:
  - $K \to A$
  - $K^* \to A'$

Reblurring (A')

- For a d-dimensional problem, changing the sign at each variable,
$$A' = JAJ,$$
where J is the d-dimensional flip matrix.
- The matrix-vector product $A'x$ can be computed like Ax flipping the PSF, i.e., rotating the PSF of 180° and than applying the same procedure of Ax.
- The Matlab Toolbox RestoreTools [Nagy et al.] implements the reblurring (new version!).

Reblurring

Algorithms for least square regularization requires A^T.
The reblurring replaces A^T with A' [D. et al., IP 2006].

$A' = A^T$? (A persymmetric?):

BCs $\diagdown$ PSF	symmetric	nonsymmetric
Dirichlet	$\curlyvee$	$\curlyvee$
Periodic	$\curlyvee$	$\curlyvee$
Reflective	$\curlyvee$	$\times$
Antireflective	$\times$	$\times$

$\times \rightarrow$ *there are not theoretical results and reblurring could fail!*

Antireflective BCs

- Symmetric PSF: Spectral filtering methods based on reblurring are regularization methods [D. and Hanke, LAA 2010].
- Nonsymmetric PSF: Reblurring could fail! (example of motion blur in [Fan and Nagy, LAA 2012] where antireflective BCs compute a poor restoration worse than reflective BCs.)

Arnoldi methods

- Arnoldi methods do not require A^T.
- Given $Ax = b$ with $A \in \mathbb{R}^{N \times N}$, k steps of the Arnoldi iteration with initial vector $v_1 = b/\|b\|$ yields the factorization

$$AV_k = V_{k+1} H_{k+1,k},$$

  where $V_k \in \mathbb{R}^{N \times k}$ satisfies $V_k^T V_k = I_k$, and $H_{k+1,k} \in \mathbb{R}^{k+1 \times k}$ is upper-Hessenberg.
- The columns of $V_k = [v_1, \ldots, v_k]$ form an orthonormal basis for the k-dimensional Krylov subspace

$$\mathcal{K}_k(A, b) = \text{span}\{b, Ab, \ldots, A^{k-1} b\}.$$

GMRES

- At the iteration k, GMRES solves the problem

$$\min_{x \in \mathcal{K}_k(A,b)} \|Ax - b\|.$$

- GMRES is an iterative regularization method [Calvetti et al., NM 2002].
- The stopping iteration is usually estimated by the Discrepancy principle as the first iteration k such that

$$\|b - Ax_k\| < \delta \quad (= \|e\|).$$

- The $\|b - Ax_k\|$ is computed in the Krylov subspace without requiring a further matrix-vector product with A.

Arnoldi-Tikhonov

- At the iteration k, GMRES solves the problem ($\mu > 0$)

$$\min_{x \in \mathcal{K}_k(A,b)} \|Ax - b\|^2 + \mu \|x\|^2.$$

- Is an hybrid method that combines the iterative regularization of GMRES with Tikhonov regularization.
- μ is chosen such that $\|Ax_{\mu,k} - b\|^2 = \delta^2$ which can be satisfied only after that GMRES satisfies the discrepancy principle (i.e., the Tikhonov regularization stabilizes the GMRES convergence).
- [Lewis and Reichel, JCAM 2009]

Range restricted Arnoldi methods

- Look for a solution in the range of A by replacing $\mathcal{K}_k(A, b)$ with $\mathcal{K}_k(A, Ab)$.
- For image deblurring it looks like a preliminary application of a low-pass filter.
- Simplest approach uses the initial vector

$$v_1 = Ab/\|Ab\|.$$

- A more numerically stable formulation starting with $v_1 = b/\|b\|$ is proposed in [Neumann et al., LAA, 2012].

Reblurring right-preconditioning

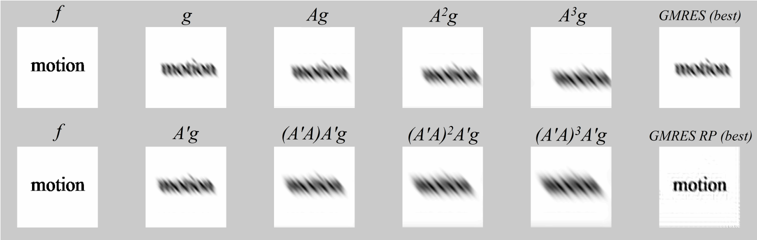
- Replace the linear system $Ax = b$ with

$$AA'y = b, \qquad x = A'y$$

- Solving by GMRES, the approximation at the step k lives in

$$\mathcal{K}_k(A'A, A'b) = \mathrm{span}\{A'b, (A'A)A'b, \ldots, (A'A)^{k-1}A'b\}.$$

- The preconditioner does not accelerate the convergence but "symmetrizes" the operator.



Numerical Experiments

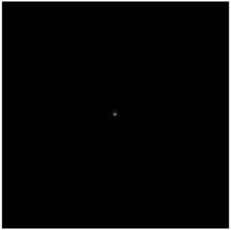
- Antireflective BCs.
- Discrepancy principle as stopping criteria.
- $\sigma =$ noise level of Gaussian noise.
- The accuracy of each restoration is measured in terms of the peak signal-to-noise ratio:

$$\mathrm{PSNR} = 10 \cdot \log_{10} \frac{mn}{\|\tilde{x} - x\|_2^2},$$

where $\tilde{x}$ is an approximation of the $m \times n$ true image x.

Test1 (small quasi-symmetric blur)

True (256×256)



PSF (196×196)

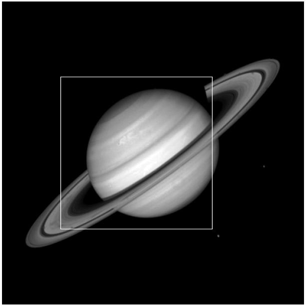
Obseverd ($\sigma = 0.2\%$)

Restoration - PSNR (iteration)

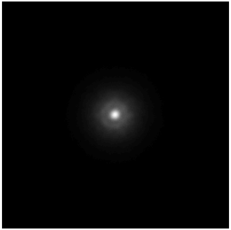
	$\sigma = 0.2\%$		$\sigma = 5\%$	
Method	max PSNR	Discrepancy	max PSNR	Discrepancy
CGLS	35.33 (41)	34.32 (28)	21.54 (5)	21.54 (5)
GMRES	35.38 (10)	35.02 (8)	20.09 (2)	18.11 (3)
RRGMES	35.31 (15)	34.65 (12)	21.70 (4)	21.70 (4)
GMRES RP	35.33 (39)	34.31 (27)	22.00 (6)	22.00 (6)
RRGMES RP	35.25 (63)	34.13 (46)	22.35 (14)	22.17 (12)

- CGLS replacing A^T with A', i.e., reblurring (works well)
- RRGMRES = range restricted GMRES (is necessary only for high noise)
- RP = reblurring right-preconditioner (is not necessary but could be useful for high noise)

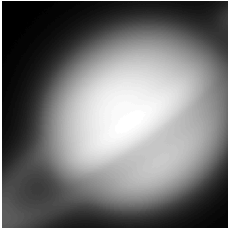
Test2 (large quasi-symmetric blur)



True (512×512)



PSF (256×256)



Obseverd ($\sigma = 0.2\%$)

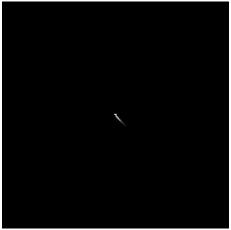
Restoration - PSNR (iteration)

	$\sigma = 0.2\%$		$\sigma = 5\%$	
Method	max PSNR	Discrepancy	max PSNR	Discrepancy
CGLS	25.90 (200)	25.90 (200)	23.88 (100)	23.88 (100)
GMRES	27.27 (9)	22.38 (16)	19.13 (2)	9.08 (6)
RRGMES	31.29 (26)	31.23 (23)	24.37 (10)	24.37 (10)
GMRES RP	31.07 (76)	30.88 (60)	24.81 (23)	24.53 (19)
RRGMES RP	31.04 (118)	30.72 (84)	24.91 (41)	24.60 (35)

- CGLS is too slow
- RRGMRES = is necessary also for low noise
- reblurring right-preconditioner is not necessary but could be useful for high noise

Test3 (Motion blur - strongly non-symmetric)

True (256×256)



PSF (196×196)

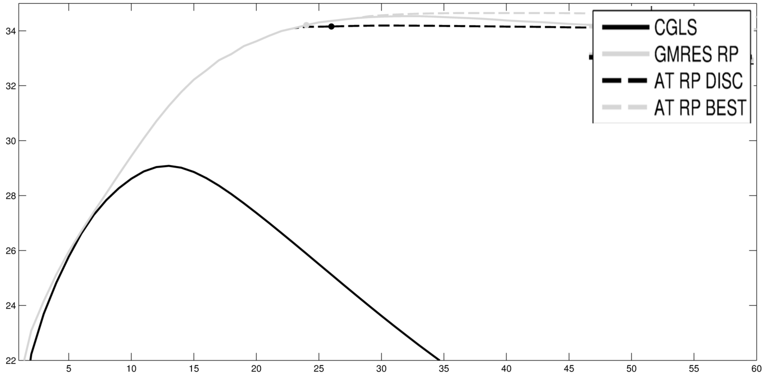
Obseverd ($\sigma = 0.2\%$)

Restoration - PSNR (iteration)

	$\sigma = 0.2\%$		$\sigma = 5\%$	
Method	max PSNR	Discrepancy	max PSNR	Discrepancy
CGLS	29.08 (13)	#	23.13 (4)	23.13 (4)
GMRES	26.37 (18)	#	19.57 (3)	17.27 (7)
RRGMES	21.99 (25)	#	18.36 (9)	#
GMRES RP	34.53 (32)	34.21 (24)	23.82 (4)	23.68 (3)
RRGMES RP	34.21 (52)	33.96 (42)	23.68 (8)	23.59 (7)

- $\# \leftrightarrow$ fails!
- CGLS with reblurring works better than GMRES
- RRGMRES is worst than GMRES: the therapy is worse than the illness!
- With reblurring right-preconditioning the range restricted strategy is not longer necessary.

PSNR vs iterations ($\sigma = 0.2\%$)



Restored images

CGLS (BEST)
PSNR = 29.08 (13)

GMRES RP
PSNR = 34.21 (24)

AT RP DISC
PSNR = 34.16 (26)

CGLS restoration has again some blur and artifacts (ringing effects) at the boundary.

Conclusions

- For accurate BCs A^T is not available or the matrix vector is computationally too expensive.
- PSF symmetric or quasi-symmetric: reblurring (for least square methods) and range restricted (for Arnoldi methods) strategies works well.
- PSF strongly nonsymmetric (like motion): the previous two strategies fail and the only approach that works is Arnoldi methods with reblurring right-preconditioning.
- Arnoldi methods with reblurring right-preconditioning:
  - require 2 matrix-vector products at each iteration;
  - are robust for high levels of noise (also for quasi-symmetric PSF);
  - range restricted strategy is not longer necessary.