Regularization preconditioners for frame-based deconvolution

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Outline

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The model problem

Consider the solution of ill-posed equations

$$Tx = y\,, \tag{1}$$

where $T : \mathcal{X} \to \mathcal{Y}$ is a linear operator between Hilbert spaces.

- T is a compact operator, the singular values of T decay gradually to zero without a significant gap.
- Assume that problem (1) has a solution $x^\dagger$ of minimal norm.

Goal

Compute an approximation of $x^\dagger$ starting from approximate data $y^\delta \in \mathcal{Y}$, instead of the exact data $y \in \mathcal{Y}$, with

$$\|y^\delta - y\| \le \delta\,, \tag{2}$$

where $\delta \ge 0$ is the corresponding noise level.

$$y^{\delta} \quad = \quad T \quad * \quad x \quad + \quad \xi$$

- T is doubly Toeplitz, large and severely ill-conditioned (discretizzation of an integral equations of the first kind)
- y^{δ} are known measured data (blurred and noisy image)
- ξ is noise; $\|\xi\| = \delta$

$\longrightarrow$ discrete ill-posed problems (Hansen, 90's)

- The singular values of T are large in the low frequencies, decays rapidly to zero and are small in the high frequencies.
- The solution of $Tx = y^\delta$ requires some sort of regularization:

$$x = T^\dagger y^\delta = x^\dagger + T^\dagger \xi,$$

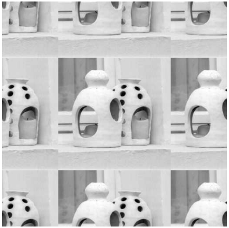
  where $\|T^\dagger \xi\|$ is large.
- Tikhonov regularization: balance the data fitting and the "explosion" of the solution

$$\min_x \{\|Tx - y^\delta\|^2 + \alpha\|x\|^2\} \iff x = (T^*T + \alpha I)^{-1} T^* y^\delta$$

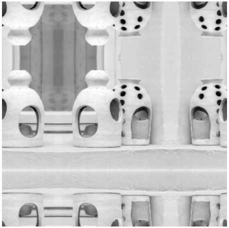
  where $\alpha > 0$ is a regularization parameter.

Boundary Conditions (BCs)

zero Dirichlet



Periodic



Reflective

Antireflective

The matrix C

Space invariant point spread function (PSF)

⇓

T has a doubly Toeplitz-like structure that carries the "correct" boundary conditions.

- doubly circulant matrix C diagonalizable by FFT, that corresponds to periodic BCs.
- The boundary conditions have a very local effect

$$T - C = E + R, \quad (3)$$

where E is of small norm and R of small rank.

Outline

Synthesis approach

- Images have a sparse representation in the wavelet domain.
- Let W^* be a wavelet or tight-frame synthesis operator ($W^*W = I$) and v the frame coefficients such that

$$x = W^* v.$$

- A regularized solution can be obtained in terms of the wavelet coefficients by

$$\min_v \{\|Kv - y^\delta\|^2 + 2\mu\|v\|_1\}$$

where $K = TW^*$.

Iterative soft-thresholding algorithm (ISTA)

- The solution of

$$\min_v \{ \|Kv - y^\delta\|^2 + 2\mu \|v\|_1 \}$$

can be computed by Iterative soft-thresholding algorithm (ISTA) [Daubechies, Defrise, De Mol, CPAM 2004]

$$\begin{cases} z^{n+1} = v^n + K^*(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{4}$$

when $\|K\| < 1$ and where S_μ denotes the soft-thresholding .

- The inner step is a Landweber iteration

- The Linearized Bregman splitting proposed in [Yin, Osher, Goldfarb, Darbon, SIIMS 2008]

$$\begin{cases} z^{n+1} = z^n + K^*(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{5}$$

  is analogous to ISTA replacing v^n with z^n at the first occurrence in the inner step.
- With $z^0 = v^0 = 0$ converges to the unique minimizer of

$$\min_{v \in \mathbb{R}^s} \left\{ \mu \|v\|_1 + \frac{1}{2}\|v\|^2 : Kv = y^\delta \right\} \tag{6}$$

  if $\|K\| < 1$.
- Common feature: slow convergence

Modified Linearized Bregman algorithm (MLBA)

- The MLBA proposed in [Cai, Osher, Shen, SIIMS 2009]

$$\begin{cases} z^{n+1} = z^n + K^* P(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{7}$$

where $z^0 = v^0 = 0$.

- It is the linearized Bregman splitting applied to the preconditioned equation

$$P^{1/2}Kv = P^{1/2}y^\delta.$$

- $P = (TT^* + \alpha I)^{-1} \Rightarrow$ the iteration (7) converges to the unique minimizer of

$$\min_{v \in \mathbb{R}^s} \left\{ \mu \|v\|_1 + \frac{1}{2}\|v\|^2 : v = \arg\min_{v \in \mathbb{R}^s} \|Kv - y^\delta\|_P^2 \right\} \tag{8}$$

within few iterations.

Choosing P in MLBA

- Similarly, a preconditioned version of ISTA with $P = (TT^* + \alpha_n I)^{-1}$ has been studied in [Huang, D., Chan, IPI 2013]:

$$\begin{cases} z^{n+1} = v^n + K^*(TT^* + \alpha_n I)^{-1}(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \qquad (9)$$

- $K^*(TT^* + \alpha_n I)^{-1} r^n = (K^*K + \alpha_n I)^{-1} K^* r^n \implies$ Tikhonov regularization on the residual $r^n = y^\delta - Kv^n$.
- Possible choice $\alpha_n = \alpha_0 q^n$ with $0 < q < 1$, e.g. $\alpha_n = 0.8^n$.
- The inversion of $TT^* + \alpha_n I$ could be costly $\Rightarrow$ the authors of MLBA proposed the use of $P = (CC^* + \alpha I)^{-1}$

$$\Downarrow$$

it does not provide an accurate restoration when the BCs are essential!

Outline

Preconditioned regularization

Replace the original problem $Tx = y^\delta$ with

$$P^{-1}Tx = P^{-1}y^\delta$$

such that

1. inversion of P is cheap
2. $P \approx T$ but not too much ($T^\dagger$ unbounded while P^{-1} must be bounded!)

Alert!

Preconditioners can be used to accelerate the convergence, but an imprudent choice of preconditioner may spoil the achievable quality of computed restorations.

Classical preconditioner

- Historically, the first attempt of this sort was by Hanke, Nagy, and Plemmons (1993): In that work

$$P = C_\varepsilon,$$

  where C_ε is the optimal doubly circulant approximation of T, with eigenvalues set to be one for frequencies above $1/\varepsilon$.
  Very fast, but the choice of ε is delicate and not robust.
- Subsequently, other regularizing preconditioners have been suggested: Bertero and Piana (1997), Kilmer and O'Leary (1999), Estatico (2002), Egger and Neubauer (2005), Brianzi, Di Benedetto, and Estatico (2008), D. and Hanke (2013), Dell'Acqua, D., and Estatico (2014)

- We look for a fast and accurate preconditioned Landweber-like iteration.
- The preconditioner in MLBA is analogous to the preconditioned Landweber in [Brianzi, Di Benedetto, Estatico, SISC 2009] $\Rightarrow$ in order to guarantee the convergence α could be large reducing the acceleration.
- A recent nonsymmetric strategy

$$ZTx = Zy^{\delta},$$

where Z is a regularized approximation of $T^{\dagger}$, solved by the Van Cittert iteration [Dell'Acqua, D., Estatico, JCAM 2014]:

$$x^{n+1} = x^n + Z(y^{\delta} - Tx^n) = x^n + (C^*C + \alpha I)^{-1}C^*(y^{\delta} - Tx^n)$$

$$\Downarrow$$

it can be seen as an approximated iterated Tikhonov.

Iterated Tikhonov

- The correction step

$$h^n = (TT^* + \alpha I)^{-1} T^* (y^\delta - Tx^n)$$

  is the Tikhonov solution of the error equation $Te^n = r^n$, where $r^n = y^\delta - Tx^n$ is the residual.
- The iterative refinement

$$x^{n+1} = x^n + h^n$$

  is called Iterated Tikhonov.
- In the noise free case ($\delta = 0$) we solve by Tikhonov the true error equation $Te^n = y - Tx^n$.
- In the noisy case ($\delta > 0$) the iterative refinement is correct up to the noise level δ since the residual r^n differ from the true residual $y - Tx^n$.

Approximated Iterated Tikhonov (AIT)

- In the noisy case, we could admit a further misfit "lower" than δ:

$$\min_{h^n}\{\|Th^n - r^n\|^2 + \alpha\|h^n\|^2\} \quad \Longrightarrow \quad \min_{h^n}\{\|Ch^n - r^n\|^2 + \alpha\|h^n\|^2\}$$

  which solution is

$$h^n = (C^*C + \alpha I)^{-1}C^*(y^\delta - Tx^n).$$

- $P = (C^*C + \alpha I)^{-1}C^*$ is a nonsymmetric preconditioner which leads to the preconditioned Van Cittert iteration

$$x^{n+1} = x^n + P(y^\delta - Tx^n).$$

- Convergence analysis based on the knowledge of the complex eigenvalues of PT and requiring a complex step length [Dell'Acqua, D., Estatico, JCAM 2014].

Nonstationary AIT

- Nonstationary preconditioning [D., Hanke, IP 2013]
- Use a sequence of regularization parameters $\{\alpha_n\}$ obtaining

$$x^{n+1} = x^n + P_n(y^\delta - Tx^n), \text{ with } P_n = (C^*C + \alpha_n I)^{-1}C^*.$$

- We need the "strong" assumption

$$\|(C - T)u\| \leq \rho \|Tu\|, \qquad \forall u \in \mathcal{X},$$

for some $0 < \rho < 1/2$.

- Compute α_n by few Newton's steps for the equation

$$\|r^n - CPr^n\| = q_n \|r^n\|,$$

where $q_n < 1$ depends on ρ and δ but it is not too small.
If C is diagonalizzable by FFT then the computational cost is linear in the number of pixels.

The Algorithm

Algorithm

Let $x^0 \in \mathcal{X}$ be given, and set $r^0 = y^\delta - Tx^0$.
Choose $\tau = (1+2\rho)/(1-2\rho)$ and fix $q \in (2\rho, 1)$.
While $\|r^n\| > \tau\delta$, let $\tau_n = \|r^n\|/\delta$, and compute α_n s.t.

$$\|r^n - Ch^n\| = q_n\|r^n\|, \qquad q_n = \max\{\, q,\, 2\rho + (1+\rho)/\tau_n \,\}. \tag{10a}$$

Then, update

$$h^n = (C^*C + \alpha_n I)^{-1} C^* r^n, \tag{10b}$$

$$x^{n+1} = x^n + h^n, \qquad r^{n+1} = y^\delta - Tx^{n+1}. \tag{10c}$$

Details

- The parameter q prevents that r_n decreases too rapidly.
- The unique α_n can be computed by Newton iteration.
- The stopping rule is almost the discrepancy principle.

Theoretical results

Theorem

The norm of the iteration error $e^n = x^\dagger - x^n$ decreases monotonically as long as

$$\|r^n\| \leq \tau\delta \leq \|r^{n-1}\|, \qquad \tau > 1 \text{ fixed.}$$

Theorem

For exact data ($\delta = 0$) the iterates x^n converges to the solution of $Tx = y$ that is closest to x^0 in the norm of $\mathcal{X}$.

Theorem

For noisy data ($\delta > 0$), as $\delta \to 0$, the approximation x^δ converges to the solution of $Tx = y$ that is closest to x^0 in the norm of $\mathcal{X}$.

Outline

AIT + linearized Bregman splitting

- Replace preconditioned Landweber with AIT in MLBA.
- Usual assumption

$$\|(C - T)u\| \leq \rho \|Tu\|, \qquad \forall u \in \mathcal{X}$$

- Further assumption

$$\|CW^*(v - S_\mu(v))\| \leq \rho\delta, \qquad \forall v \in \mathbb{R}^s, \tag{11}$$

which is equivalent to consider the soft-threshold parameter $\mu = \mu(\delta)$ and such that $\mu(\delta) \to 0$ as $\delta \to 0$.

AIT + linearized Bregman splitting – 2

Algorithm [Cai, D., Bianchi, Huang, 2016]

$$\begin{cases} z^{n+1} = z^n + WC^*(CC^* + \alpha_n I)^{-1}(y^\delta - TW^*v^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{12}$$

where the parameter (α_n, stopping iteration, etc.) are fixed as in AIT.

Theorem

For noisy data ($\delta > 0$), as $\delta \to 0$, the approximation x^δ converges to the solution of $Tx = y$ that is closest to z^0 in the ℓ_2-norm.

In the applications, if an estimation of the best α is available, we can fix $\alpha_n = \alpha_{\rm opt}$, but there is not any convergence result.

Outline

- W by piecewise linear B-spline framelets. In 1D, given the masks

$$b_0 = \frac{1}{4}[1,\ 2,\ 1], \quad b_1 = \frac{\sqrt{2}}{4}[1,\ 0,\ -1], \qquad b_2 = \frac{1}{4}[-1,\ 2,\ -1]$$

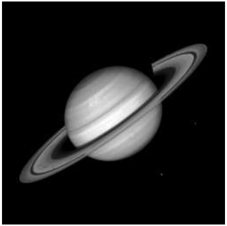
  the associated matrices are defined by imposing reflective BCs. The nine 2D filters are obtained by tensor product.
- $\rho = 10^{-4}$, $q = 0.5$.
- $\mathrm{PSNR} = 20 \log_{10} \frac{255 \cdot n}{\|x - \tilde{x}\|}$, with $\tilde{x}$ the computed approximation.
- Best regularization parameter μ and α when necessary are estimated by trial and error for every method in order to achieve the max PSNR.

Compared methods

- MLBA: iteration (7) by Cai, Osher, and Shen.
- AIT–Breg: our nonstationary iteration (12).
- AIT–Breg–opt: our iteration (12) with a stationary $\alpha_n = \alpha_{\text{opt}}$ chosen by hand like in MLBA.
- FA–MD, TV–MD: ADMM [Almeida, Figueiredo, IEEE 2013] for Frame-based Analysis and Total Variation, respectively (rectangular matrix).
- FTVd: extension of FTVd in [Wang et al. SIIMS 2008] to deal with boundary artifacts [Bai et al., 2014] (enlarging of the domain).

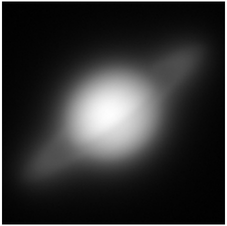
Example 1 (Saturn)

$\nu = 1\,\%$, Zero BCs.



True image

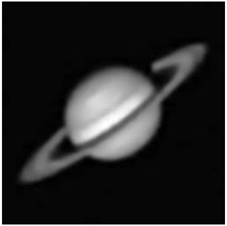
PSF



Observed image

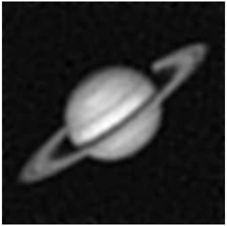
Restorations

Method	PSNR	CPU time
AIT–Breg	31.25	10.32
AIT–Breg–opt	31.49	16.56
MLBA	30.97	200.99
FA–MD	30.87	90.85
TV–MD	31.17	47.61
FTVd:	30.50	1.75



MLBA

AIT–Breg

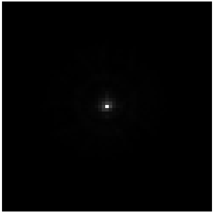


FTVd

Example 2 (Boat)

$\nu = 1\,\%$, Antireflective BCs.

True image



PSF

Observed image

Restorations

Method	PSNR	CPU time
AIT–Breg	29.77	19.57
AIT–Breg–opt	30.17	3.67
MLBA	29.43	34.26
FA–MD	29.61	15.95
TV–MD	29.87	16.74
FTVd:	28.95	0.73

MLBA

AIT–Breg–opt

TV–MD

Conclusions and future work

- Regularization preconditioners can be used to speed up the convergence of nonlinear methods without losing the quality of the restoration.
- Under the assumption that an approximation C of T is available, our new scheme turns out to be fast and stable.
- The choice of ρ reflect how much we trust in the previous approximation and in practice it can be small enough.
- Our scheme does not require T^*.
- The proposed regularizing preconditioners could be combined also with ISTA, with the analysis approach, etc.

References

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