An iterative multigrid regularization method for deblurring problems

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1 Restoration of blurred and noisy images
   The deblurring problem
   Properties of the coefficient matrix

2 Multigrid regularization
   Iterative Multigrid regularization
   Post-smoother denoising

3 Numerical results
Outline

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Deblurring problem

The restored signal/image $f$ is obtained solving: (in some way by regularization ...)

$$g = Af + e$$

- $f =$ true object,
- $g =$ blurred and noisy object,
- $A =$ (two-level) matrix with a Toeplitz-like structure depending on the point spread function (PSF) and the BCs.
- $e =$ white Gaussian noise (we assume to know $\|e\| = \delta$),

The PSF is the observation of a single point (e.g., a star in astronomy) that we assume shift invariant.
Structure of $A$

Given a stencil

\[
\begin{bmatrix}
  \vdots & \vdots & \vdots \\
  \vdots & a_{-1,1} & a_{0,1} & a_{1,1} & \vdots \\
  \vdots & a_{-1,0} & a_{0,0} & a_{1,0} & \vdots \\
  \vdots & a_{-1,-1} & a_{0,-1} & a_{1,-1} & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
\]

the associated symbol is

\[
z(x, y) = \sum_{j,k \in \mathbb{Z}} a_{j,k} e^{i(jx+ky)}
\]

and the matrix

\[
A = A_n(z) \in \mathbb{R}^{n^2 \times n^2}
\]

has a Toeplitz-like structure depending on the boundary conditions (assume that the degree of $z$ is less than $n$).
Matrix-vector product

The matrix-vector product $A\mathbf{x} = \mathcal{A}_n(z)\mathbf{x}$ can be computed by

1. padding (Matlab `padarray` function) $\mathbf{x}$ with the appropriate boundary conditions
2. periodic convolution by FFT $\Rightarrow O(n^2 \log(n))$ arithmetic cost.
Eigenvalues of a 1D PSF

- The eigenvalues of $A_n(z)$ are about a uniform sampling of $z$.

- The ill-conditioned subspace is mainly constituted by the middle/high frequencies.
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Iterative regularization methods

Some iterative methods (Landweber, CGLS, MR-II . . . ) have regularization properties: the restoration error firstly decreases and then increases.

**Reason**

- They firstly reduce the algebraic error in the low frequencies (well-conditioned subspace).
- When they arrive to reduce the algebraic error in the high frequencies then the restoration error increases because of the noise.
Multigrid methods

Multigrid Idea

Project the system in a subspace, solve the resulting system in this subspace and interpolate the solution in order to improve the previous approximation.

- The Multigrid combines two iterative methods:
  - **Pre-Smooth**: a classic iterative method,
  - **Coarse Grid Correction**: projection, solution of the restricted problem, interpolation.

- **Post-Smooth**: ...

- At the lower level(s) it works on the **error equation**!
Deblurring and Multigrid

- For deblurring problems the ill-conditioned subspace is related to high frequencies, while the well-conditioned subspace is generated by low frequencies (signal space).
- Low-pass filter (e.g., full weighting) projects in the well-conditioned subspace (low frequencies) \(\implies\) it is slowly convergent but it can be a good iterative regularization method [D. and Serra-Capizzano, ’06].
- Intuitively: the regularization properties of the smoother are preserved since it is combined with a low-pass filter.
- Conditions on the projector such that the multigrid is a regularization method [D. and Serra-Capizzano, ’08].
Other multilevel deblurring methods

1. Morigi, Reichel, Sgallari, and Shyshkov ’08.
   Edge preserving prolongation solving a nonlinear PDE

2. Español and Kilmer ’10.
   Haar wavelet decomposition with a residual correction by a nonlinear deblurring into the high frequencies

Common idea
Both strategies can be interpreted as a nonlinear post-smoothing step.
Transformed domain

Fourier domain vs. wavelet domain

Many recent strategies split

- deconvolution $\rightarrow$ Fourier domain
- denoising $\rightarrow$ wavelets domain
Our post-smoothing denoising

- **Post-smoother**: denoising (without deblurring)
- **Soft-thresholding** with parameter

\[
\theta = \sigma \sqrt{2 \log(n)/n},
\]

where \( \sigma \) is the noise level [Donoho, '95].
Tight Frame: linear B-spline

- Low frequencies projector:

\[
[1, 2, 1]/4 \implies \text{full weighting}
\]

preserves the Toeplitz structure at the coarse level

- Exact reconstruction \( F^T F = I \).

Two high frequencies projectors:

\[
\frac{\sqrt{2}}{4} [1, 0, -1], \quad \frac{1}{4} [-1, 2, -1].
\]

- 2D Tight Frame: \( \Rightarrow 9 \text{ frames by tensor product.} \)

- Chan, Shen, Cai, Osher, ...
Two-Grid Method

The $j$-th iteration for the system $A\mathbf{f} = \mathbf{g}$:

1. $\tilde{\mathbf{f}} = \text{Smooth}(A, \mathbf{f}^{(j)}, \mathbf{g}) \leftarrow 1$ step (CGLS, MR-II, ...)
2. $\mathbf{r}_1 = P(\mathbf{g} - A\tilde{\mathbf{f}})$
3. $A_1 \approx P A P^T$
4. $\mathbf{e}_1 = A_1^\dagger \mathbf{r}_1$
5. $\hat{\mathbf{f}} = \tilde{\mathbf{f}} + P^T \mathbf{e}_1$
6. $\mathbf{f}^{(j+1)} = F^T \text{threshold}(F\hat{\mathbf{f}}, \theta) \leftarrow 1$ level

Multigrid (MGM): the step (4) becomes a recursive application of the algorithm.
2D Projector

\[ P = DW \]

where \( W = A_n(p) \) and \( D = \text{downsampling} \).

Full-weighting \( \Rightarrow P^T = \text{bilinear interpolation} \).

\[
\frac{1}{16} \begin{bmatrix}
1 & 2 & 1 \\
2 & 4 & 2 \\
1 & 2 & 1 \\
\end{bmatrix} \Rightarrow p(x, y) = (1 + \cos(x))(1 + \cos(y))
\]

\( D = D_1 \otimes D_1 \) where \( D_1 \) is

\[
\begin{array}{c c c}
\text{n even} & \text{n odd} \\
\begin{bmatrix}
1 & 0 & 0 & \ldots & 1 \\
1 & 0 & 0 & \ldots & 0 \\
\end{bmatrix} & \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 1 \\
\end{bmatrix}
\end{array}
\]
Coarser PSFs

- The PSF has the same size of the observed image and it is centered in the middle of the image $\Rightarrow$ it has many zero entries close the boundary.
- The PSF at the coarser level is defined as

$$PSF_1 = PSF_{\text{temp}}(1:2:\text{end}, 1:2:\text{end})$$

where

$$PSF_{\text{temp}} = \frac{1}{32} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \ast PSF \ast \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

by FFTs without consider boundary conditions since the PSF has many zeros at the boundary.
Coarse coefficient matrices

- Computed in a setup phase.
- Compute $PSF_i$ and the associate symbol $z_i$ at each level and define

$$A_i = A_{n_i}(z_i).$$

This is the same strategy used in [Huckle, Staudacher '02] for multigrid methods for Toeplitz linear system.

Garlerkin strategy $A_{n_i}(z_i) = PA_{i-1}P^T$ if

- $n = 2^\beta$ and periodic boundary conditions
- $n = 2^\beta - 1$ and zero Dirichlet boundary conditions

otherwise they differ for a low rank matrix.
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Numerical results

- **RestoreTools Matlab Toolbox** [Nagy ’07]
- Stopping rule is the **discrepancy principle**:
  \[
  \| r_n \| < 1.01 \delta
  \]
  where \( r_n \) is computed after the presmoothing step at the finer level. It should be better stop some iteration later . . .
- **Post-smoother**: linear B-spline soft-thresholding with parameter
  \[
  \frac{\delta}{\| g \|} \sqrt{\frac{2 \log(n)}{n}}
  \]
Example 1

- Black border $\Rightarrow A = T_n(z)$ (zero Dirichlet boundary conditions)
- nonsymmetric PSF
- $\sigma = \delta/\|g\| = 0.07$ of white Gaussian noise
- **W-MGM**: multigrid without postsmoother, W-cycle, and without presmoothing at the finer level as proposed in [D., Serra Capizzano, '06]
Best restorations (minimum error)

- Observed image
- CGLS: 0.2641 – it.:7
- W – MGM: 0.26284 – it.:11
- MGM: 0.18712 – it.:49
Relative restoration error

Restoration error $= \frac{\|\tilde{f} - f\|}{\|f\|}$, where $\tilde{f}$ is the restored image. The circle is the discrepancy principle stopping iterations.
Restorations at the discrepancy principle stopping iteration

Observed image

CGLS: 0.2709 – it.:5

W – MGM: 0.26637 – it.:6

MGM: 0.24048 – it.:4
Numerical results

PCGLS - Relative restoration error

![Graph showing relative restoration error for PCGLS and MGM methods](image)
Numerical results

PCGLS - Best restorations (minimum error)

True image

PCGLS: 0.25928 – it.:4

Observed image

MGM: 0.23271 – it.:30
PCGLS - Restorations at the discrepancy principle stopping iteration

True image

PCGLS: 0.26425 – it.:1

Observed image

MGM: 0.24902 – it.:1
Example 2

- Reflective boundary conditions [Ng, Chan, Tang, 1999]
- Nonsymmetric PSF

\[ A_{n_i}(z_i) \neq PA_{i-1}P^T! \]

- \( \sigma = 0.02 \) of white Gaussian noise
Numerical results

Best restorations (minimum error)

True image

Observed image

CGLS: 0.14554 – it.:10

MGM: 0.13545 – it.:50
Relative restoration error

![Graph showing relative restoration error for CGLS and MGM methods. The x-axis represents iterations from 0 to 50, and the y-axis represents the relative restoration error on a logarithmic scale. The graph compares the performance of CGLS and MGM methods, with CGLS showing a lower error rate at higher iterations.]
Restorations at the discrepancy principle stopping iteration

True image

Observed image

CGLS: 0.14859 – it.:7

MGM: 0.14411 – it.:10
Conclusions

- The multigrid regularization can be easily combined with a soft-thresholding denoising obtaining and iterative regularization method with a stable error curve.
- No parameters to estimate at each level but only at the finer level.

Work in progress . . .

- Proof of convergence and stability
- Relations with other approaches (analysis, balanced, etc.)
- Pre-smoother no $l_2$-norm.