An iterative multigrid regularization method for deblurring problems

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Outline

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Deblurring problem

The restored signal/image $\mathbf{f}$ is obtained solving: (in some way by regularization ...)

$$\mathbf{g} = A\mathbf{f} + \mathbf{e}$$

- $\mathbf{f}$ = true object,
- $\mathbf{g}$ = blurred and noisy object,
- A = (two-level) matrix with a Toeplitz-like structure depending on the point spread function (PSF) and the BCs.
- $\mathbf{e}$ = white Gaussian noise (we assume to know $\|\mathbf{e}\| = \delta$),

The PSF is the observation of a single point (e.g., a star in astronomy) that we assume shift invariant.

Structure of A

Given a stencil

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & a_{-1,1} & a_{0,1} & a_{1,1} & \cdots \\ \cdots & a_{-1,0} & a_{0,0} & a_{1,0} & \cdots \\ \cdots & a_{-1,-1} & a_{0,-1} & a_{1,-1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

the associated symbol is

$$z(x,y) = \sum_{j,k \in \mathbb{Z}} a_{j,k} \mathrm{e}^{\mathrm{i}(jx+ky)}$$

and the matrix

$$A = \mathcal{A}_n(z) \in \mathbb{R}^{n^2 \times n^2}$$

has a Toeplitz-like structure depending on the boundary conditions (assume that the degree of z is less than n).

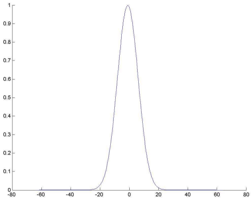
Matrix-vector product

The matrix-vector product $A\mathbf{x} = \mathcal{A}_n(z)\mathbf{x}$ can be computed by

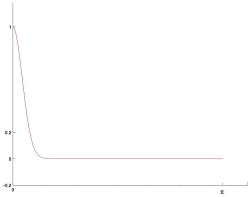
1. padding (Matlab `padarray` function) $\mathbf{x}$ with the appropriate boundary conditions
2. periodic convolution by FFT $\Longrightarrow O(n^2 \log(n))$ arithmetic cost.

Eigenvalues of a 1D PSF

- The eigenvalues of $\mathcal{A}_n(z)$ are about a uniform sampling of z.



PSF



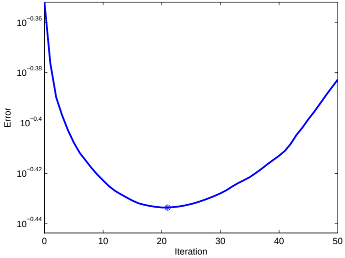
Generating function $z(x)$

- The ill-conditioned subspace is mainly constituted by the middle/high frequencies.

Outline

Iterative regularization methods

Some iterative methods (Landweber, CGLS, MR-II . . .) have regularization properties: the restoration error firstly decreases and then increases.



Reason

- They firstly reduce the algebraic error in the low frequencies (well-conditioned subspace).
- When they arrive to reduce the algebraic error in the high frequencies then the restoration error increases because of the noise.

Multigrid methods

Multigrid Idea

Project the system in a subspace, solve the resulting system in this subspace and interpolate the solution in order to improve the previous approximation.

- The Multigrid combines two iterative methods:

  Pre-Smoother: a classic iterative method,

  Coarse Grid Correction: projection, solution of the restricted problem, interpolation.

  Post-Smoother: ...
- At the lower level(s) it works on the error equation!

Deblurring and Multigrid

- For deblurring problems the ill-conditioned subspace is related to high frequencies, while the well-conditioned subspace is generated by low frequencies (signal space).
- Low-pass filter (e.g., full weighting) projects in the well-conditioned subspace (low frequencies) $\Longrightarrow$ it is slowly convergent but it can be a good iterative regularization method [D. and Serra-Capizzano, '06]).
- Intuitively: the regularization properties of the smoother are preserved since it is combined with a low-pass filter.
- Conditions on the projector such that the multigrid is a regularization method [D. and Serra-Capizzano, '08].

Other multilevel deblurring methods

1. Morigi, Reichel, Sgallari, and Shyshkov '08.
   Edge preserving prolongation solving a nonlinear PDE
2. Español and Kilmer '10.
   Haar wavelet decomposition with a residual correction by a nonlinear deblurring into the high frequencies

Common idea

Both strategies can be interpreted as a nonlinear post-smoothing step.

Transformed domain

Fourier domain vs. wavelet domain

Many recent strategies split

- deconvolution $\rightarrow$ Fourier domain
- denoising $\rightarrow$ wavelets domain

Our post-smoothing denoising

- Post-smoother: denoising (without deblurring)
- Soft-thresholding with parameter

$$\theta = \sigma\sqrt{2\log(n)/n},$$

where σ is the noise level [Donoho, '95].

Tight Frame: linear B-spline

- Low frequencies projector:

$$[1,\ 2,\ 1]/4 \Longrightarrow \text{full weighting}$$

  preserves the Toeplitz structure at the coarse level

- Exact reconstruction $F^T F = I$.
  Two high frequencies projectors:

$$\frac{\sqrt{2}}{4}[1,\ 0,\ -1], \qquad \frac{1}{4}[-1,\ 2,\ -1].$$

- 2D Tight Frame: $\Rightarrow$ 9 frames by tensor product.
- Chan, Shen, Cai, Osher, ...

Two-Grid Method

The j-th iteration for the system $A\mathbf{f} = \mathbf{g}$:

$$
\begin{array}{ll}
\textbf{(1)} & \tilde{\mathbf{f}} = \mathrm{Smooth}(A, \mathbf{f}^{(j)}, \mathbf{g}) \quad \leftarrow \text{1 step (CGLS, MR-II,...)} \\
\textbf{(2)} & \mathbf{r}_1 = \mathbf{P}(\mathbf{g} - A\tilde{\mathbf{f}}) \\
\textbf{(3)} & A_1 \approx \mathbf{P}A\mathbf{P}^T \\
\textbf{(4)} & \mathbf{e}_1 = A_1^{\dagger}\mathbf{r}_1 \\
\textbf{(5)} & \hat{\mathbf{f}} = \tilde{\mathbf{f}} + \mathbf{P}^T\mathbf{e}_1 \\
\textbf{(6)} & \mathbf{f}^{(j+1)} = F^T \mathrm{threshold}(F\hat{\mathbf{f}}, \theta) \quad \leftarrow \text{1 level}
\end{array}
$$

Multigrid (MGM): the step **(4)** becomes a recursive application of the algorithm.

2D Projector

$$P = DW$$

where $W = \mathcal{A}_n(p)$ and $D =$ downsampling.
Full-weighting $\Rightarrow P^T =$ bilinear interpolation.

$$\frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \quad \Rightarrow \quad p(x,y) = (1+\cos(x))(1+\cos(y))$$

$D = D_1 \otimes D_1$ where D_1 is

$$\text{n} \quad \mathit{even} \qquad\qquad\qquad \text{n} \quad \mathit{odd}$$

$$\begin{bmatrix} 1\ 0 & & & \\ & 1\ 0 & & \\ & & \ddots \quad \ddots & \\ & & & 1\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0\ 1\ 0 & & & \\ & 0\ 1\ 0 & & \\ & & \ddots \quad \ddots \quad \ddots & \\ & & & 0\ 1\ 0 \end{bmatrix}$$

Coarser PSFs

- The PSF has the same size of the observed image and it is centered in the middle of the image ⇒ it has many zero entries close the boundary
- The PSF at the coarser level is defined as

$$PSF_1 = PSF_{\mathrm{temp}}(1:2:end, 1:2:end)$$

where

$$PSF_{\mathrm{temp}} = \frac{1}{32}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \circledast PSF \circledast \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

by FFTs without consider boundary conditions since the PSF has many zeros at the boundary.

Coarse coefficient matrices

- Computed in a setup phase.
- Compute PSF_i and the associate symbol z_i at each level and define

$$A_i = \mathcal{A}_{n_i}(z_i).$$

This is the same strategy used in [Huckle, Staudacher '02] for multigrid methods for Toeplitz linear system.

Garlerkin strategy $\mathcal{A}_{n_i}(z_i) = PA_{i-1}P^T$ if

- $n = 2^\beta$ and periodic boundary conditions
- $n = 2^\beta - 1$ and zero Dirichlet boundary conditions

otherwise they differ for a low rank matrix.

Outline

Numerical results

- RestoreTools Matlab Toolbox [Nagy '07]
- Stopping rule is the discrepancy principle:

$$\|r_n\| < 1.01\,\delta$$

where r_n is computed after the presmoothing step at the finer level. It should be better stop some iteration later ...

- Post-smoother: linear B-spline soft-thresholding with parameter

$$\frac{\delta}{\|\mathbf{g}\|}\sqrt{\frac{2\log(n)}{n}}$$

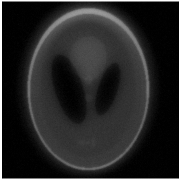
.

Example 1

- Black border $\Rightarrow A = T_n(z)$ (zero Dirichlet boundary conditions)
- nonsymmetric PSF
- $\sigma = \delta/\|\mathbf{g}\| = 0.07$ of white Gaussian noise
- W-MGM: multigrid without postsmoother, W-cycle, and without presmoothing at the finer level as proposed in [D., Serra Capizzano, '06]

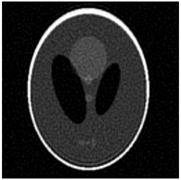
Best restorations (minimum error)

Observed image

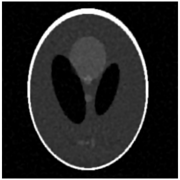


CGLS: 0.2641 − it.:7

W − MGM: 0.26284 − it.:11

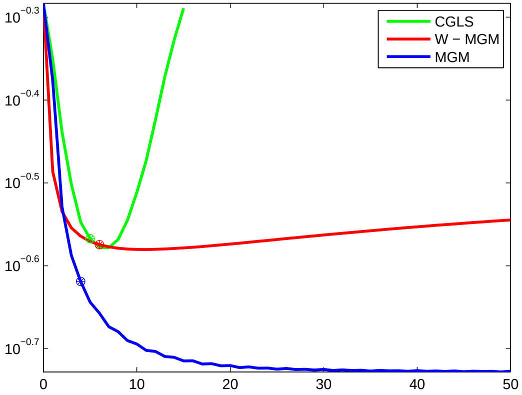


MGM: 0.18712 − it.:49



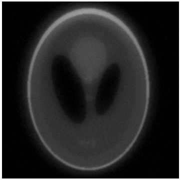
Relative restoration error

Restoration error $= \frac{\|\tilde{\mathbf{f}}-\mathbf{f}\|}{\|\mathbf{f}\|}$, where $\tilde{\mathbf{f}}$ is the restored image.
The circle is the discrepancy principle stopping iterations

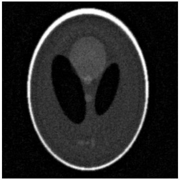


Restorations at the discrepancy principle stopping iteration

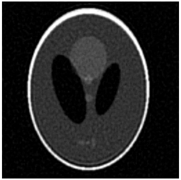
Observed image



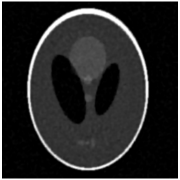
CGLS: 0.2709 − it.:5



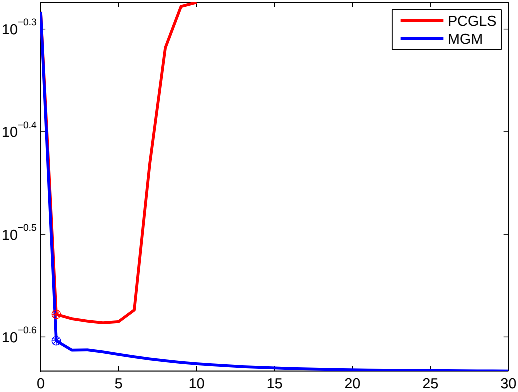
W − MGM: 0.26637 − it.:6



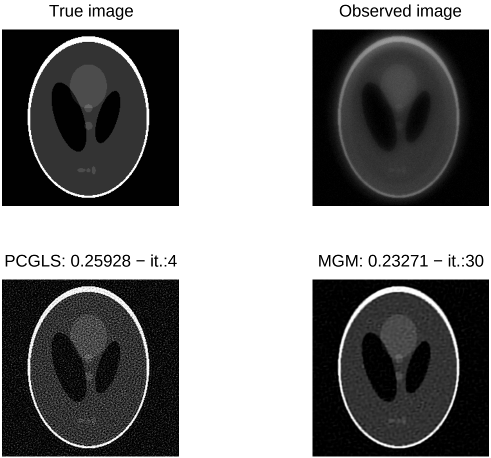
MGM: 0.24048 − it.:4



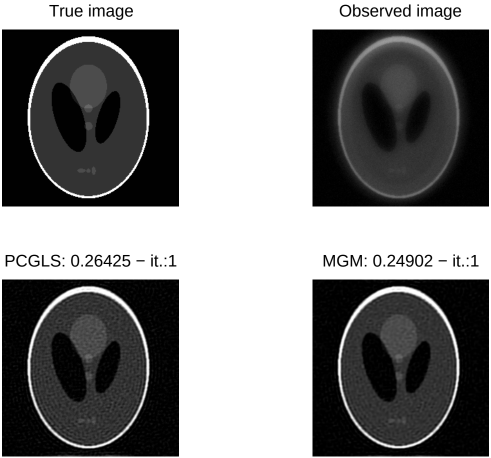
PCGLS - Relative restoration error



PCGLS - Best restorations (minimum error)



PCGLS - Restorations at the discrepancy principle stopping iteration



Example 2

- Reflective boundary conditions [Ng, Chan, Tang, 1999]
- nonsymmetric PSF

$$\Downarrow$$

$$\mathcal{A}_{n_i}(z_i) \neq PA_{i-1}P^T!$$

- $\sigma = 0.02$ of white Gaussian noise

Best restorations (minimum error)

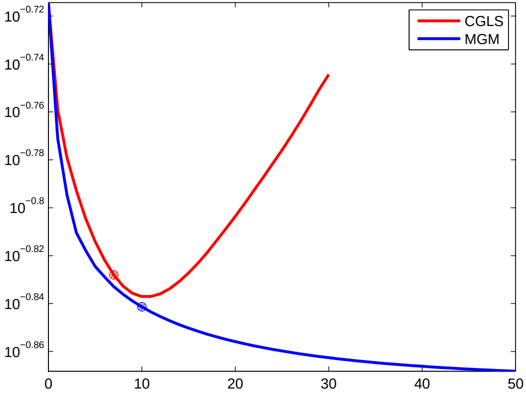
True image

Observed image

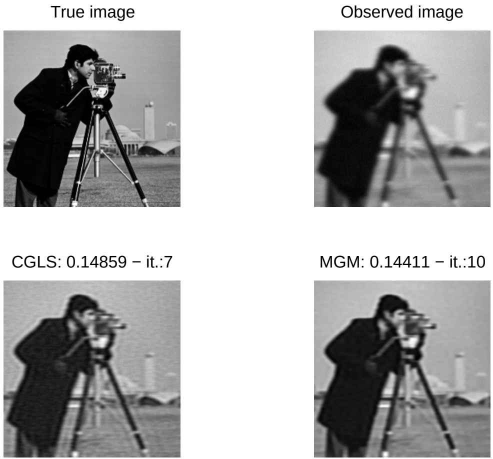
CGLS: 0.14554 − it.:10

MGM: 0.13545 − it.:50

Relative restoration error



Restorations at the discrepancy principle stopping iteration



Conclusions

- The multigrid regularization can be easily combined with a soft-thresholding denoising obtaining and iterative regularization method with a stable error curve.
- No parameters to estimate at each level but only at the finer level.

Work in progress ...

- Proof of convergence and stability
- Relations with other approaches (analysis, balanced, etc.)
- Pre-smoother no l_2-norm.