Regularization preconditioners for frame-based deconvolution

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Outline

Image deblurring

Frame-based regularization (synthesis approach)

Regularization preconditioners

Regularization preconditioners for linearized Bregman splitting

Numerical Results

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Image deblurring problems

$$y^\delta \quad = \quad T \quad * \quad x \quad + \quad \xi$$

- T is doubly Toeplitz, large and severely ill-conditioned (discretizzation of an integral equations of the first kind)
- y^δ are known measured data (blurred and noisy image)
- ξ is noise; $\|\xi\| = \delta$

$\longrightarrow$ discrete ill-posed problems (Hansen, 90's)

Regularization

- The singular values of T are large in the low frequencies, decays rapidly to zero and are small in the high frequencies.
- The solution of $Tx = y^\delta$ requires some sort of regularization:

$$x = T^\dagger y^\delta = x^\dagger + T^\dagger \xi,$$

where $\|T^\dagger \xi\|$ is large.

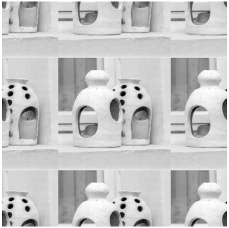
- Tikhonov regularization: balance the the data fitting and the "explosion" of the solution

$$\min_x \{\|Tx - y^\delta\|^2 + \alpha \|x\|^2\} \iff x = (T^*T + \alpha I)^{-1} T^* y^\delta$$

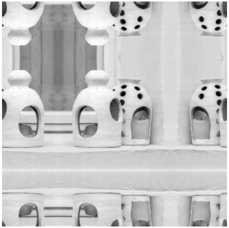
where $\alpha > 0$ is a regularization parameter.

Boundary Conditions (BCs)

zero Dirichlet



Periodic



Reflective

Antireflective

The matrix C

Space invariant point spread function (PSF)

$$\Downarrow$$

T has a doubly Toeplitz-like structure that carries the “correct” boundary conditions.

- doubly circulant matrix C diagonalizable by FFT, that corresponds to periodic BCs.
- The boundary conditions have a very local effect

$$T - C = E + R\,, \tag{1}$$

where E is of small norm and R of small rank.

Outline

Synthesis approach

- Images have a sparse representation in the wavelet domain.
- Let W^* be a wavelet or tight-frame synthesis operator ($W^*W = I$) and v the frame coefficients such that

$$x = W^*v.$$

- A regularized solution can be obtained in terms of the wavelet coefficients by

$$\min_v \{\|Kv - y^\delta\|^2 + 2\mu\|v\|_1^2\}$$

where $K = TW^*$.

Iterative soft-thresholding algorithm (ISTA)

- The solution of

$$\min_v \{\|Kv - y^\delta\|^2 + 2\mu\|v\|_1^2\}$$

can be computed by Iterative soft-thresholding algorithm (ISTA) [Daubechies, Defrise, De Mol, CPAM 2004]

$$\begin{cases} z^{n+1} = v^n + K^*(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{2}$$

when $\|K\| < 1$ and where S_μ denotes the soft-thresholding

$$[S_\mu(v)]_i = \mathrm{sgn}(v_i) \max\{|v_i| - \mu, 0\}.$$

- The inner step is a Landweber iteration

Modified Linearized Bregman algorithm (MLBA)

- The MLBA proposed in [Cai, Osher, Shen, SIIMS 2009]

$$\begin{cases} z^{n+1} = z^n + K^* P(y^\delta - Kv^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{3}$$

where $z^0 = v^0 = 0$.

- $P = I \Rightarrow$ linearized Bregman splitting (slow convergence). Analogous to ISTA replacing v^n with z^n in the inner step.
- For $P \neq I$ s.p.d. the MLBA is the linearized Bregman splitting applied to the preconditioned equation

$$P^{1/2}Kv = P^{1/2}y^\delta.$$

Choosing P in MLBA

- $P = (TT^* + \alpha I)^{-1} \Rightarrow$ the iteration (3) converges to the unique minimizer of

$$\min_{v\in\mathbb{R}^s}\left\{\mu\|v\|_1 + \frac{1}{2}\|v\|^2 : v = \arg\min_{v\in\mathbb{R}^s}\|TW^*v - y^\delta\|_P^2\right\} \quad (4)$$

  within few iterations.
- Similarly, a preconditioned version of ISTA with $P = (TT^* + \alpha_n I)^{-1}$ has been studied in [Huang, D., Chan, IPI 2013].
- The authors of MLBA proposed the use of $P = (CC^* + \alpha I)^{-1}$ which does not provide an accurate restoration when the BCs are essential!

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Regularization preconditioners

- The problem is the definition of a fast and accurate preconditioned Landweber-like iteration.
- A proposal in [Brianzi, Di Benedetto, Estatico, SISC 2009].
- A recent nonsymmetric strategy

$$ZTx = Zy^{\delta},$$

where Z is a regularized approximation of $T^{\dagger}$, solved by the Van Citter iteration [Dell'Acqua, D., Estatico, JCAM 2014]:

$$x^{n+1} = x^n + Z(y^{\delta} - Tx^n) = x^n + (CC^* + \alpha I)^{-1} C^* (y^{\delta} - Tx^n)$$

$$\Downarrow$$

it can be seen as an approximated iterated Tikhonov (AIT).

Nonstationary AIT

- $(TT^* + \alpha I)^{-1}T^*(y^\delta - Tx^n)$ is the Tikhonov solution of the error equation in the noise free case ($\delta = 0$).
- In the noisy case ($\delta > 0$) we can admit a further misfit approximating T with C in Tikhonov obtaining $(CC^* + \alpha I)^{-1}C^*(y^\delta - Tx^n)$.
- Such misfit can be measured by the assumption

$$\|(C - T)u\| \leq \rho \|Tu\|, \qquad \forall\, u \in \mathbb{R}^s.$$

- Estimating the parameter α at each step n (nonstationary) by

$$\|r^n - C(CC^* + \alpha_n I)^{-1}C^* r^n\| = q_n \|r^n\|$$

where $r^n = y^\delta - Tx^n$ and $q_n < 1$ depends on ρ but it is not too small, we can prove convergence, monotonicity of the error, regularization property, etc.

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AIT + linearized Bregman splitting

- Replace preconditioned Landweber with AIT in MLBA.
- Usual assumption

$$\|(C - T)u\| \leq \rho \, \|Tu\| \,, \qquad \forall \, u \in \mathbb{R}^s, .$$

- Further assumption

$$\|CW^*(v - S_\mu(v))\| \leq \rho\delta, \qquad \forall \, v \in \mathbb{R}^s, \tag{5}$$

which is equivalent to consider the soft-threshold parameter $\mu = \mu(\delta)$ and such that $\mu(\delta) \to 0$ as $\delta \to 0$.

AIT + Bregman splitting – 2

Algorithm [Cai, D., Bianchi, Huang, 2016]

$$\begin{cases} z^{n+1} = z^n + WC^*(CC^* + \alpha_n I)^{-1}(y^\delta - TW^* v^n), \\ v^{n+1} = S_\mu(z^{n+1}), \end{cases} \tag{6}$$

where the parameter (α_n, stopping iteration, etc.) are fixed as in AIT.

Theorem
For noisy data ($\delta > 0$), as $\delta \to 0$, the approximation x^δ converges to the solution of $Tx = y$ that is closest to x_0.

If an estimation of the best α is available, we can fix $\alpha_n = \alpha_{\mathrm{opt}}$.

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Numerical Results

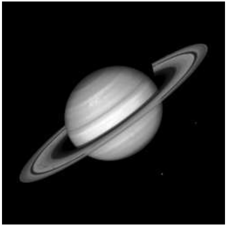
- W by linear B-spline.
- $\rho = 10^{-4}$.
- $\mathrm{PSNR} = 20 \log_{10} \frac{255 \cdot n}{\|x - \tilde{x}\|}$, with $\tilde{x}$ the computed approximation.
- Best regularization parameter by hand for every method.

Compared methods

- MLBA: iteration (3) by Cai, Osher, and Shen.
- AIT–Breg: our nonstationary iteration (6).
- AIT–Breg–opt: our iteration (6) with a stationary $\alpha_n = \alpha_{\mathrm{opt}}$ chosen by hand like in MLBA.
- FA–MD, TV–MD: ADMM [Almeida, Figueiredo, IEEE 2013] for Frame-based Analysis and Total Variation, respectively.
- FTVd: extension of FTVd in [Wang et al. SIIMS 2008] to deal with boundary artifacts [Bai et al., 2014].

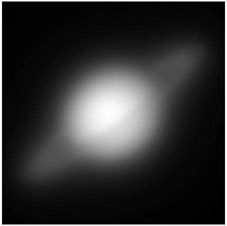
Example 3 (Saturn)

$\nu = 1\,\%$, Zero BCs.



True image

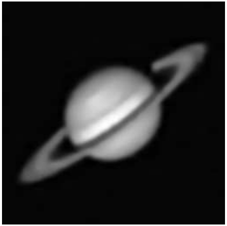
PSF



Observed image

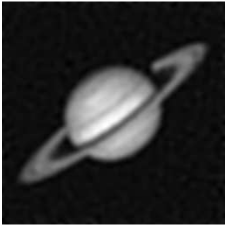
Restorations

Method	PSNR	CPU time
AIT–Breg	31.25	10.32
AIT–Breg–opt	31.49	16.56
MLBA	30.97	200.99
FA–MD	30.87	90.85
TV–MD	31.17	47.61
FTVd:	30.50	1.75



MLBA

AIT–Breg

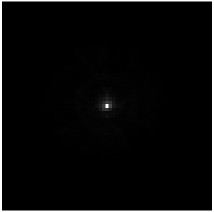


FTVd

Example 4 (Boat)

$\nu = 1\,\%$, Antireflective BCs.

True image



PSF

Observed image

Restorations

Method	PSNR	CPU time
AIT–Breg	29.77	19.57
AIT–Breg–opt	30.17	3.67
MLBA	29.43	34.26
FA–MD	29.61	15.95
TV–MD	29.87	16.74
FTVd:	28.95	0.73

MLBA

AIT–Breg–opt

TV–MD

Conclusions and future work

- Under the assumption that an approximation C of T is available, our new scheme turns out to be fast and stable.
- The choice of ρ reflect how much we trust in the previous approximation and in practice it can be small enough.
- Our scheme does not require T^*.
- The proposed regularizing preconditioners could be combined also with ISTA, with the analysis approach, etc.

References

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