Grid transfer operators for multigrid methods

Marco Donatelli

University of Insubria
Department of Physics and Mathematics

"Due Giorni di Algebra Lineare Numerica" Bologna 6-7, March 2008

Outline

1. Convergence analysis
   - The geometric Multigrid method (MGM) for PDE
   - MGM for Toeplitz matrices
   - Equivalence of the two approaches
2. A generalization of the MGM for Toeplitz matrices
   - New Galerkin conditions
   - Numerical results
3. B-spline grid transfer operators
   - Classic grid transfer operators
   - The B-spline of order 4
   - Numerical results

MultiGrid method (MGM)

Multigrid idea

1. apply a simple iterative method (smoother),
2. project the system in the subspace where the smoother is ineffective, solve the resulting system and interpolate the solution to improve the previous approximation (CGC = coarse grid correction).

MultiGrid method (MGM)

Multigrid idea

1. apply a simple iterative method (smoother),
2. project the system in the subspace where the smoother is ineffective, solve the resulting system and interpolate the solution to improve the previous approximation (CGC = coarse grid correction).

The Galerkin approach

$A_k = RA_nP$, where A_n is the coefficient matrix and A_k is the coarse matrix. Moreover $R = P^T$, with $R =$ restriction and $P =$ prolongation.

MultiGrid method (MGM)

Multigrid idea

1. apply a simple iterative method (smoother),
2. project the system in the subspace where the smoother is ineffective, solve the resulting system and interpolate the solution to improve the previous approximation (CGC = coarse grid correction).

The Galerkin approach

$A_k = RA_nP$, where A_n is the coefficient matrix and A_k is the coarse matrix. Moreover $R = P^T$, with R = restriction and P = prolongation.

Two-grid iteration matrix (TGM): $TGM = CGC \cdot S$

- S = smoother iteration matrix,
- $CGC = I - PA_k^{-1}RA_n$.

The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- d-dimensional PDE with constant coefficients

$$(-1)^q \sum_{i=1}^{d} \frac{\mathrm{d}^{2q}}{\mathrm{d}x_i^{2q}} u(x) = g(x), \qquad x \in \Omega = (0,1)^d,\ q \geq 1.$$

The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- d-dimensional PDE with constant coefficients

$$(-1)^q \sum_{i=1}^{d} \frac{\mathrm{d}^{2q}}{\mathrm{d}x_i^{2q}} u(x) = g(x), \qquad x \in \Omega = (0,1)^d,\ q \geq 1.$$

- Periodic boundary conditions on $\partial\Omega$ or an infinite domain.

The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- d-dimensional PDE with constant coefficients

$$(-1)^q \sum_{i=1}^{d} \frac{\mathrm{d}^{2q}}{\mathrm{d}x_i^{2q}} u(x) = g(x), \qquad x \in \Omega = (0,1)^d,\ q \geq 1.$$

- Periodic boundary conditions on $\partial\Omega$ or an infinite domain.
- Discretization by centered finite difference of minimal precision on a uniform grid.
- The coarse problem is the rediscretization of the same PDE (Galerkin for black-box MGM since it is more robust).

Local Fourier Analysis

The Fourier transform of the discrete differential operator is

$$\hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j \mathrm{e}^{\mathrm{i}\langle jh|\omega\rangle},$$

where $\omega \in [-\pi/h, \pi/h]^d$ denotes the frequencies for the current discretization step h and

$$l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h,\pi/h]^d} \hat{L}(\omega)\mathrm{e}^{-\mathrm{i}\langle jh|\omega\rangle}\mathrm{d}\omega.$$

Local Fourier Analysis

The Fourier transform of the discrete differential operator is

$$\hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j \mathrm{e}^{\mathrm{i}\langle jh|\omega\rangle},$$

where $\omega \in [-\pi/h, \pi/h]^d$ denotes the frequencies for the current discretization step h and

$$l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h, \pi/h]^d} \hat{L}(\omega) \mathrm{e}^{-\mathrm{i}\langle jh|\omega\rangle} \mathrm{d}\omega.$$

Example

1D Laplacian: $[l_{-1},\ l_0,\ l_1] = \frac{1}{h^2}[-1,\ 2,\ -1]$.

The convergence result

Theorem

Given a constant-coefficient PDE of order m, a necessary condition for non increasing the high frequencies arising from a CGC with a TGM it is

$$\gamma_r + \gamma_p \geq m, \tag{1}$$

where γ_p and γ_r are the order of the prolongation and of the restriction respectively.

Definition

A prolongation (restriction) has order γ_p if it (its transpose) leaves unchanged all polynomials of order at least γ_p.

More general orders

Definition

The set of all corners of x is

$$\Omega(x) = \{\, y \,|\, y_j \in \{x_j, \pi + x_j\}, \, j = 1, \ldots, d\}$$

and the set of the “mirror” points of x is $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$.

More general orders

Definition

The set of all corners of x is

$$\Omega(x) = \{\, y \mid y_j \in \{x_j,\, \pi + x_j\},\, j = 1, \ldots, d\}$$

and the set of the “mirror” points of x is $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$.

Definition (P. W. Hemker 1990)

For a grid transfer operator $B \in \{R, P\}$ (B is multiplied by 2^d when $B = P$), for $x = \omega h$, $|x| \to 0$, the largest $s \geq 0$ such that

$$\hat{B}(x) = 1 + O(|x|^s), \qquad \text{is the Low Frequency order (LF)}$$
$$\hat{B}(y) = O(|x|^s), \quad \forall y \in \mathcal{M}(x), \qquad \text{is the High Frequency order (HF)}$$

Toeplitz matrices and $\hat{L}(\omega)$

- The d-level Toeplitz matrix $T_n(f)$ is such that

$$[T_n(f)]_{r,s} = a_{s-r} = a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x) e^{-\mathrm{i}\langle j|x\rangle}\, dx, \qquad r,s,j \in \mathbb{Z}^d.$$

- $f \geq 0 \quad \Leftrightarrow \quad T_n(f)$ is positive definite.

Toeplitz matrices and $\hat{L}(\omega)$

- The d-level Toeplitz matrix $T_n(f)$ is such that

$$[T_n(f)]_{r,s} = a_{s-r} = a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x) e^{-\mathrm{i}\langle j|x\rangle}\, dx, \qquad r,s,j \in \mathbb{Z}^d.$$

- $f \geq 0 \quad \Leftrightarrow \quad T_n(f)$ is positive definite.
- The changing of variable $x = \omega h \Rightarrow a_j = l_j$ and $f(x) = \hat{L}(\omega)$.

Toeplitz matrices and $\hat{L}(\omega)$

- The d-level Toeplitz matrix $T_n(f)$ is such that

$$[T_n(f)]_{r,s} = a_{s-r} = a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x) e^{-\mathrm{i}\langle j|x\rangle}\, dx, \qquad r,s,j \in \mathbb{Z}^d.$$

- $f \geq 0 \quad \Leftrightarrow \quad T_n(f)$ is positive definite.
- The changing of variable $x = \omega h \Rightarrow a_j = l_j$ and $f(x) = \hat{L}(\omega)$.

Example

1D Laplacian: $\hat{L}(\omega) = \frac{1}{h^2}(2 - 2\cos(\omega h))$. The Toeplitz approach moves the factor $\frac{1}{h^2}$ to the rhs, thus $A_n = T_n(f)$, where $f(x) = 2 - 2\cos(x)$.

- For a factor $\frac{1}{h^{2q}}$ the $f(x)$ vanishes at the origin with order $2q$.

MGM convergence for Toeplitz matrices

- Galerkin approach: $A_k = P^T A_n P$.

MGM convergence for Toeplitz matrices

- Galerkin approach: $A_k = P^T A_n P$.
- Convergence analysis for the algebra case like τ or circulant algebra.

MGM convergence for Toeplitz matrices

- Galerkin approach: $A_k = P^T A_n P$.
- Convergence analysis for the algebra case like τ or circulant algebra.

Theorem (S. Serra-Capizzano and G. Fiorentino 1991, 1996)

Let $A_n = \mathcal{C}_n(f)$ be circulant with f having a unique zero at x^0. Defining $P = \mathcal{C}_n(p)K_n^T$, where K_n is the down-sampling and p is a trigonometric polynomial non identically zero and such that for each $x \in [-\pi, \pi)^d$

$$\limsup_{x \to x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x), \tag{2a}$$

$$\sum_{y \in \Omega(x)} p(y)^2 > 0, \tag{2b}$$

then the TGM converges in a number of iteration independent of n.

Equivalence of the two approaches

Theorem

In the case of

- *constant coefficient PDE,*
- *periodic boundary conditions,*
- $R = P^T$,

the two conditions (1) and (2a) are equivalents.

Equivalence of the two approaches

Theorem

In the case of

- *constant coefficient PDE,*
- *periodic boundary conditions,*
- $R = P^T$,

the two conditions (1) and (2a) are equivalents.

Remark

The (2b) is equivalent to require $LF > 0$ that is necessary for an effective MGM (A. Brandt 1994) and arises from the same analysis for the Galerkin approach (I. Yavneh 1998).

Consequences of such equivalence

1. For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance integral problems of the first kind.

Consequences of such equivalence

1. For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance integral problems of the first kind.
2. Allow to define a MGM for Toeplitz matrices with $R \neq P$.

Consequences of such equivalence

1. For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance integral problems of the first kind.
2. Allow to define a MGM for Toeplitz matrices with $R \neq P$.
3. Give a comparison of the grid transfer operators used in the two approaches. More specifically, we will give a geometrical interpretation of the prolongations used for Toeplitz matrices when the generating function vanishes at the origin.

How to generalize the Galerkin condition

- $A_k = P^T A_n P$ is Toeplitz only for a prolongation of order at most 2.

How to generalize the Galerkin condition

- $A_k = P^T A_n P$ is Toeplitz only for a prolongation of order at most 2.
- If P has order greater than 2, then $A_k = \text{Toeplitz} + L$, where L is a low rank matrix.
- The rank of L affects both the implementation, the computational cost and the convergence.

How to generalize the Galerkin condition

- $A_k = P^T A_n P$ is Toeplitz only for a prolongation of order at most 2.
- If P has order greater than 2, then $A_k =$ Toeplitz $+ L$, where L is a low rank matrix.
- The rank of L affects both the implementation, the computational cost and the convergence.
- To reduce the rank of L we can generalize the Galerkin approach:
  1. $A_k = R A_n P$ with $R \neq P$,
  2. A_k positive definite (the symbols of R and P both even or odd).

TGM conditions

Theoretical problem: If $r \neq p$ the *CGC* is again a projector, but it is not longer orthogonal with respect to the scalar product $<\mathbf{y}, \mathbf{x}>_{A_n} = \mathbf{y}^H A_n \mathbf{x}$.

TGM conditions

Theoretical problem: If $r \neq p$ the CGC is again a projector, but it is not longer orthogonal with respect to the scalar product $<\mathbf{y}, \mathbf{x}>_{A_n} = \mathbf{y}^H A_n \mathbf{x}$.

TGM conditions (conjecture) *Let $A_n = \mathcal{C}_n(f)$ with f having a unique zero at x^0. Defining $R = K_n \mathcal{C}_n(r)$ and $P = \mathcal{C}_n(p) K_n^T$ such that for each $x \in [-\pi, \pi)^d$*

$$\limsup_{x \to x^0} \left| \frac{r(y)p(y)}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x), \tag{3a}$$

$$\sum_{y \in \Omega(x)} r(y)p(y) \neq 0, \tag{3b}$$

then defining $A_k = R A_n P$ the TGM is optimal.

Equivalence result

The previous conjecture is motivated by the following

Theorem

In the case of

- *constant coefficient elliptic PDE,*
- *periodic boundary conditions,*

the two conditions (1) and (3a) are equivalents.

MGM implementation

Theorem

Let $A_n = \mathcal{C}_n(f)$, $P = \mathcal{C}_n(p)K_n^T$, $R = K_n\mathcal{C}_n(r)$, with f, p, r trigonometric polynomials, p and r satisfying the conditions (3). *Then*

1. $A_{n/2} = RA_nP = \mathcal{C}_{n/2}(\hat{f})$ *where*

$$\hat{f}(x) = \frac{1}{2^d} \sum_{y \in \Omega(x/2)} r(y)f(y)p(y), \qquad x \in [-\pi, \pi)^d. \tag{4}$$

2. *if $x^0 \in [-\pi, \pi)^d$ is a zero of f, then $y^0 = 2x^0 \bmod 2\pi$ is a zero of $\hat{f}$. The order of y^0 for $\hat{f}$ is exactly the same as the one of x^0 for f.*

TGM: numerical results

- Smoother = weighted Richardson
- $A_n = T_n(f)$ with $f(x) = (2 + 2\cos(x))^3$
- $z(x) = (2 + 2\cos(x - x_0))^{\frac{\delta_z}{2}}$, $\delta_z = 2j$, $z \in \{r, p\}$

TGM: numerical results

- Smoother = weighted Richardson
- $A_n = T_n(f)$ with $f(x) = (2 + 2\cos(x))^3$
- $z(x) = (2 + 2\cos(x - x_0))^{\frac{\delta_z}{2}}$, $\delta_z = 2j$, $z \in \{r, p\}$

TGM iteration numbers varying the orders δ_r and δ_p.

n	$\delta_r = 2$ $\delta_p = 2$	$\delta_r = 2$ $\delta_p = 4$	$\delta_r = 4$ $\delta_p = 4$
15	219	65	51
31	607	72	52
63	1501	76	51
127	> 2000	77	50
255	> 2000	78	49

MGM: numerical results

$A_n = T_n(f)$ with $f(x) = (2 + 2\cos(x))^2$.

1. For $\delta_r = \delta_p = 2$, $A_{n^{(i)}} = T_n^{(i)}(\tilde{z})$, where $\tilde{z}(x) = (2 - 2\cos(x))^2$.
2. For $\delta_r = 2$ and $\delta_p = 4$ we have $A_{n^{(i)}} = 2^{\mathrm{i}} T_n^{(i)}(\tilde{z}) + c_i e_1 e_1^T + c_i e_n e_n^T$.
3. For $\delta_r = \delta_p = 4$, $A_{n^{(i)}} =$ Toeplitz $+$ 4 rank correction, moreover the bandwidth of the Toeplitz part is not longer 5 but it becomes 7.

W-cycle iteration numbers varying the orders δ_r and δ_p.

n	$\delta_r = 2$, $\delta_p = 2$	$\delta_r = 2$, $\delta_p = 4$	$\delta_r = 4$, $\delta_p = 4$
31	25	23	22
63	32	23	21
127	35	23	21
255	37	23	20
511	37	23	20

Interpolation operators

- 1D case:
  - even components: the solution computed in the coarse grid
  - odd components: interpolation

Interpolation operators

- 1D case:
  - even components: the solution computed in the coarse grid
  - odd components: interpolation
- Interpolation operators:
  - linear interpolation: $\frac{1}{2}[1 \quad 2 \quad 1]$
  - cubic interpolation: $\frac{1}{16}[-1 \quad 0 \quad 9 \quad 16 \quad 9 \quad 0 \quad -1]$

Interpolation operators

- 1D case:
  - even components: the solution computed in the coarse grid
  - odd components: interpolation
- Interpolation operators:
  - linear interpolation: $\frac{1}{2}[1 \quad 2 \quad 1]$
  - cubic interpolation: $\frac{1}{16}[-1 \quad 0 \quad 9 \quad 16 \quad 9 \quad 0 \quad -1]$
- *d*-dimensional case: tensor product

Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d}(1 + \cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.

Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d}(1 + \cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.
- For PDE $x^{(0)} = 0$ and $p(x)$ can be generalized as

$$\varphi_m(x) = 2^{-dm} \prod_{j=1}^{d} \left(1 + \mathrm{e}^{-\mathrm{i}x_j}\right)^m .$$

Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d}(1 + \cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.
- For PDE $x^{(0)} = 0$ and $p(x)$ can be generalized as

$$\varphi_m(x) = 2^{-dm} \prod_{j=1}^{d} \left(1 + \mathrm{e}^{-\mathrm{i}x_j}\right)^m .$$

- φ_m has HF$= m$ and LF$= 2$.

Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d}(1+\cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.
- For PDE $x^{(0)} = 0$ and $p(x)$ can be generalized as

$$\varphi_m(x) = 2^{-dm} \prod_{j=1}^{d} \left(1 + \mathrm{e}^{-\mathrm{i}x_j}\right)^m .$$

- φ_m has HF$= m$ and LF$= 2$.
- Grid transfer operator with HF$= m$ can be obtained from $\varphi_m(x)\psi_m(x)$ such that $\psi_m(y) \neq 0$ for all $y \in \mathcal{M}(0)$ and $\psi_m(0) = 1$.

B-spline refinement coefficients

- The coefficients of φ_m are the refinement coefficients of the B-spline of order m in the MRA.
- $\phi_m(x) = \varphi_m(x)\mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.

B-spline refinement coefficients

- The coefficients of φ_m are the refinement coefficients of the B-spline of order m in the MRA.
- $\phi_m(x) = \varphi_m(x)\mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.

The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^m\phi_m$ in the 1D case.

m	h_{-2}	h_{-1}	h_0	h_1	h_2
1		1	1		
2		1	2	1	
3	1	3	3	1	
4	1	4	6	4	1

B-spline refinement coefficients

- The coefficients of φ_m are the refinement coefficients of the B-spline of order m in the MRA.
- $\phi_m(x) = \varphi_m(x)\mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.

The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^m\phi_m$ in the 1D case.

m	h_{-2}	h_{-1}	h_0	h_1	h_2
1		1	1		
2		1	2	1	
3	1	3	3	1	
4	1	4	6	4	1

- $m = 2q \Rightarrow$ vertex centered discretization.
- $m = 2q + 1 \Rightarrow$ cell centered discretization.

Interpolation and B-spline

- $q = 1 \Rightarrow$ linear interpolation.

Interpolation and B-spline

- $q = 1 \Rightarrow$ linear interpolation.
- The cubic interpolation has $HF = LF = 4$ and its generating function is $g_c(x) = \varphi_4(x)(2 - \cos(x))$ with stencil $\frac{1}{32}[-1\ 0\ 9\ 16\ 9\ 0\ -1]$.

Interpolation and B-spline

- $q = 1 \Rightarrow$ linear interpolation.
- The cubic interpolation has $HF = LF = 4$ and its generating function is $g_c(x) = \varphi_4(x)(2 - \cos(x))$ with stencil $\frac{1}{32}[-1\ 0\ 9\ 16\ 9\ 0\ \ -1]$.
- $\phi_4(x) = (1 + \cos(x))^2/4$ with stencil $\frac{1}{8}[1\ 4\ 6\ 4\ 1]$.

Interpolation and B-spline

- $q = 1 \Rightarrow$ linear interpolation.
- The cubic interpolation has $HF = LF = 4$ and its generating function is $g_c(x) = \varphi_4(x)(2 - \cos(x))$ with stencil $\frac{1}{32}[-1\ \ 0\ \ 9\ \ 16\ \ 9\ \ 0\ \ -1]$.
- $\phi_4(x) = (1 + \cos(x))^2/4$ with stencil $\frac{1}{8}[1\ \ 4\ \ 6\ \ 4\ \ 1]$.
- ϕ_4 with respect to ϕ_2 leaves unchanged the odd components but it reinforces those even with a quadratic approximation:

$$y_j = \begin{cases} (x_k + x_{k+1})/2, & j = 2k+1, \\ (x_{k-1} + 6x_k + x_{k+1})/8, & j = 2k, \end{cases} \qquad k = 1, \dots, n.$$

Interpolation and B-spline

- $q = 1 \Rightarrow$ linear interpolation.
- The cubic interpolation has $HF = LF = 4$ and its generating function is $g_c(x) = \varphi_4(x)(2 - \cos(x))$ with stencil $\frac{1}{32}[-1\ 0\ 9\ 16\ 9\ 0\ \ -1]$.
- $\phi_4(x) = (1 + \cos(x))^2/4$ with stencil $\frac{1}{8}[1\ 4\ 6\ 4\ 1]$.
- ϕ_4 with respect to ϕ_2 leaves unchanged the odd components but it reinforces those even with a quadratic approximation:

$$y_j = \begin{cases} (x_k + x_{k+1})/2, & j = 2k+1, \\ (x_{k-1} + 6x_k + x_{k+1})/8, & j = 2k, \end{cases} \qquad k = 1, \dots, n.$$

- Fix $y_{2k} = x_k$ assumes that the coarse problem is a well representation of the fine problem and that it is well solved (TGM).

The B-spline of order 4

Let $B_i^{(n)}(t) = \binom{n}{i}(1-t)^i t^{n-i}$, $t \in [0,1]$, $i = 1, \ldots, n$, be the Bernstein polynomial of order n. Given the quadratic rational Bezier curve

$$C(t) = \frac{\sum_{i=0}^{2} \omega_i b_i B_i^{(2)}(t)}{\sum_{i=0}^{2} \omega_i B_i^{(2)}(t)},$$

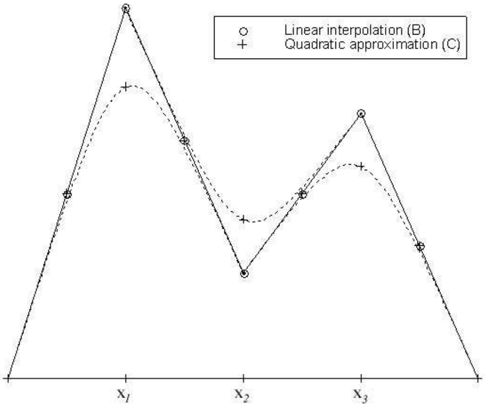
- $b_i = x_{k+i-1}$ for $i = 0, 1, 2$ (control points)
- $\omega_1 = 3/2$ and $\omega_0 = \omega_2 = 1/2$ (weights)

then

$$C\left(\frac{1}{2}\right) = \frac{x_{k-1} + 6x_k + x_{k+1}}{8}.$$

Quadratic approximation

Computation of the 5 points at the finer grid using the linear interpolation and the quadratic approximation



MGM vs. wavelets

- The factorization $g(x) = \varphi_m(x)\psi_m(x)$ is the same used to define the Daubechies wavelets (they was used for a TGM for Toeplitz matrices by L. Cheng et al. 2003 obtaining a projector with HF=4).

MGM vs. wavelets

- The factorization $g(x) = \varphi_m(x)\psi_m(x)$ is the same used to define the Daubechies wavelets (they was used for a TGM for Toeplitz matrices by L. Cheng et al. 2003 obtaining a projector with HF=4).
- The B-spline are not orthogonal, but they satisfy the quasi-interpolant Strang-Fix condition, i.e. they can well approximate "sufficiently" smooth functions.
- The orthogonality is not crucial since the MGM is an iterative method. Moreover, we would a basis for the low frequencies (the orthogonal space of the range of the smoother) but it is not exactly known or too expensive to compute.

Numerical results

We consider the following PDE

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a(x)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)\right) = g(x), \qquad x \in (0,1), \\ u(0) = u(1) = 0 \end{cases}$$

with nonconstant $a(x)$.

- It has order $m = 4$.
- V-cycle that is cheaper than the W-cycle in parallel implementations
- Smoother: Gauss-Seidel
- The condition for V-cycle is at least $\gamma_r + \gamma_p > m$.

Iteration numbers

V-cycle iteration numbers varying problem size n and $a(x) = (x - 0.5)^2$.

restriction / prolongation	ϕ_2 / ϕ_2	ϕ_2 / ϕ_4	ϕ_2 / g_c	ϕ_4 / ϕ_4	ϕ_4 / g_c
n	# iterations				
15	15	10	10	9	9
31	33	13	17	10	11
63	61	17	24	13	11
127	101	26	27	17	13
255	155	35	29	20	16
511	221	44	36	24	19
1023	284	53	46	27	22

- g_c = cubic interpolation
- For the choices $(\phi_2,\ g_c)$ and $(\phi_4,\ \phi_4)$ the coarse matrices have the same bandwidth.

Conclusions

constant coefficients PDE + Galerkin approach
geometric MGM ≡ MGM for Toeplitz matrices

↙ ↓ ↘

MGM for Toeplitz matrices with $R \neq P$	B-spline grid transfer operators	?