An iterative multilevel regularization method

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Outline

1. Restoration of blurred and noisy signal/images
   - The model problem
   - Properties of the PSF
   - Iterative regularization methods
2. Multigrid regularization
   - Iterative Multigrid regularization
   - Edge preserving
3. Numerical experiments (signal deblurring)
4. Work in progress . . .

Outline

Deblurring problem

The restored signal/image $\mathbf{f}$ is obtained solving: (in some way by regularization ...)

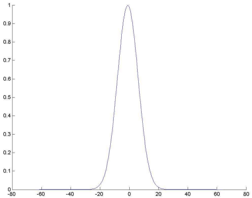
$$A\mathbf{f} = \mathbf{g} + \xi$$

- $\mathbf{g}$ = blurred object,
- ξ = noise (random vector),
- A = (two-level) matrix with a Toeplitz-like structure depending on the point spread function (PSF) and the BCs.

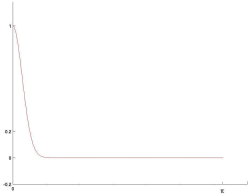
The PSF is the observation of a single point (e.g., a star in astronomy) that we assume shift invariant.

Generating function of the PSF

- The eigenvalues of $A(z)$ are about a uniform sampling of z.



PSF



Generating function $z(x)$

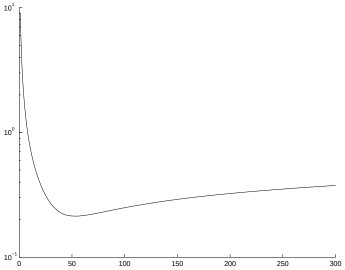
- The ill-conditioned subspace is mainly constituted by the middle/high frequencies.

Semi-convergence of iterative regularization methods

Let $\mathbf{f_o}$ the true object, it holds $A\mathbf{f_o} = \mathbf{g}$ but we "solve" $A\mathbf{f} = \mathbf{g} + \boldsymbol{\xi}$.

Some classical iterative methods (Landweber, CGLS, ...):

- firstly reduce the algebraic error into the low frequencies (well-conditioned subspace).
- when they arrive to reduce the algebraic error into the high frequencies then the restoration error increases because of the noise.



Outline

Multigrid

1. Galerkin approach for Toeplitz matrices
   - $R_i = P_i^T$
   - $A_{i+1} = P_i^T A_i P_i$
2. The smoother is an iterative regularization method: in the initial iterations it reduces the error mainly in the well-conditioned subspace.
3. Coarser grid of size 7×7.

Deblurring and Multigrid

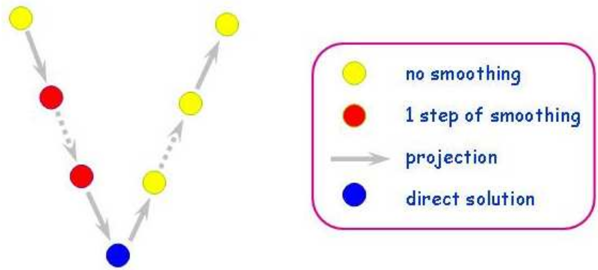
- For deblurring problems the ill-conditioned subspace is related to high frequencies, while the well-conditioned subspace is generated by low frequencies (signal space).

Deblurring and Multigrid

- For deblurring problems the ill-conditioned subspace is related to high frequencies, while the well-conditioned subspace is generated by low frequencies (signal space).
- Low-pass filter (e.g., full weighting) projects in the well-conditioned subspace (low frequencies) $\Longrightarrow$ it is slowly convergent but it can be a good iterative regularizer [D. and Serra-Capizzano, SISC, '06]).

Multigrid regularization

V-cycle



Using a larger number of recursive calls (e.g. W-cycle), the algorithm "works" more in the well-conditioned subspace, but it is more difficult to define an early stopping criterium.

Convergence analysis for Two Levels

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  - a condition to choose P s.t. it is a regularizer
  - in such case it improves the regularization properties of the iterative method used as smoother.

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- Intuitively: the regularization properties of the smoother are preserved since it is combined with a low-pass filter.
- From the filter factor analysis of the Two Levels method we have:
  - a condition to choose P s.t. it is a regularizer
  - in such case it improves the regularization properties of the iterative method used as smoother.
- In the noise free case the proposed method does not compute the exact solution because it does not “work” into the high frequencies.

Other multilevel deblurring methods

1. S. Morigi, L. Reichel, F. Sgallari, and A. Shyshkov “Cascadic multiresolution methods for image deblurring” SIAM J. Imaging Sci., 1 (2008), pp. 51-74.
2. M. I. Español and M. E. Kilmer “Multilevel Approach For Signal Restoration Problems With Toeplitz Matrices” SIAM J. Sci. Comput. Volume 32, Issue 1, pp. 299-319 (2010).

Difference with our approach

They project at the coarse level the initial linear system instead of the error equation $\Longrightarrow$ oversmoothing.

Edge preserving

- [SIIMS '08]: Edge preserving prolongation solving a nonlinear PDE
- [SISC '10]: Haar wavelet decomposition with a residual correction by a nonlinear deblurring into the high frequencies

Common idea

Both strategy can be interpreted as a nonlinear post-smoothing step.

Common drawback

A regularization parameter should be estimated at each level.

Our edge preserving approach

- Post-smoother: denoising (without deblurring)
- Soft-threshold with parameter $\theta = \sigma\sqrt{2\log(n)/n}$, where σ is the noise level [Donoho, '95].

Tight Frame

- Low frequencies projector: $[1,\ 2,\ 1]/4 \Longrightarrow$ preserves the Toeplitz structure at the coarse level
- Exact reconstruction $(A^*A = I) \Longrightarrow$ two high frequencies projectors:

$$\frac{\sqrt{2}}{4}[1,\ 0,\ -1], \qquad \frac{1}{4}[-1,\ 2,\ -1].$$

The Two Level method

[*SISC* '06]

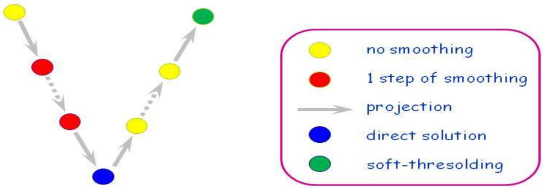
$$\begin{cases} r_k &= P^T r_n & \text{(residual restriction)} \\ A_k &= P^T A_n P & \text{(coarse matrix)} \\ e_k &= landweber(A_k, r_k, 0, 1) & \text{(regularizing smoother)} \\ \tilde{x}_n &= \tilde{x}_n + P^T e_k \end{cases}$$

denoising

$$\begin{cases} y_n &= F\tilde{x}_n & \text{(tight frame decomposition)} \\ \tilde{y}_n &= threshold(y_n, \theta) \\ x_n &= F^T \tilde{y}_n \end{cases}$$

Multigrid regularization

Post-smoother only at the finer level



- Soft-thresholding needs noise estimation that is not available at the coarse levels
- Coarse levels have low and middle frequencies components $\Longrightarrow$ not only denoising but also deblurring should be considered.

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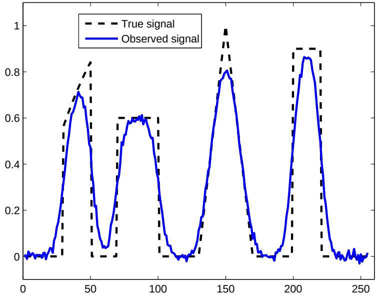
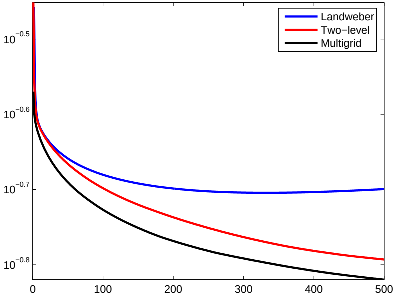
Example 1

Gaussian blur and 3% of Gaussian white noise.



Signals



Restoration error vs iterations

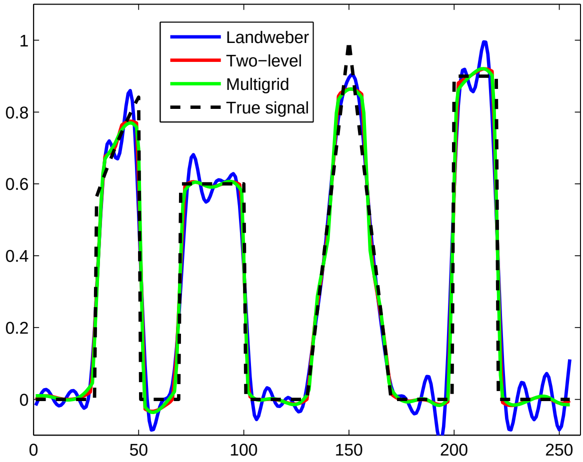
Restoration error

$e_j = \|\mathbf{f_o} - \mathbf{f}^{(j)}\|_2 / \|\mathbf{f_o}\|_2$ restoration error at the j-*th* iteration.

Minimum restoration error

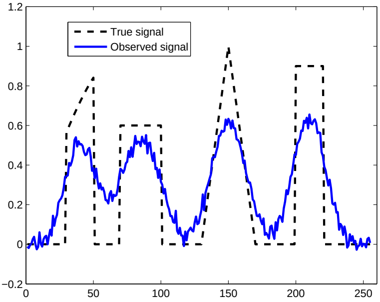
Method	$\min_{j=1,\dots} (e_j)$	$\arg\min_{j=1,\dots} (e_j)$
Landweber	0.1976	331
Two-level	0.1611	500
Multigrid	0.1514	500

Restored signals

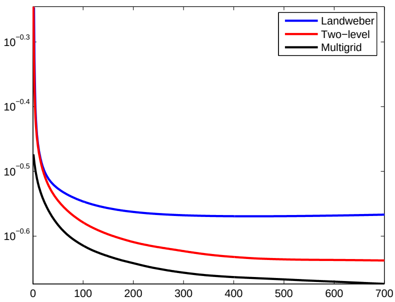


Example 2

Larger Gaussian blur and 7% of Gaussian white noise.



Signals



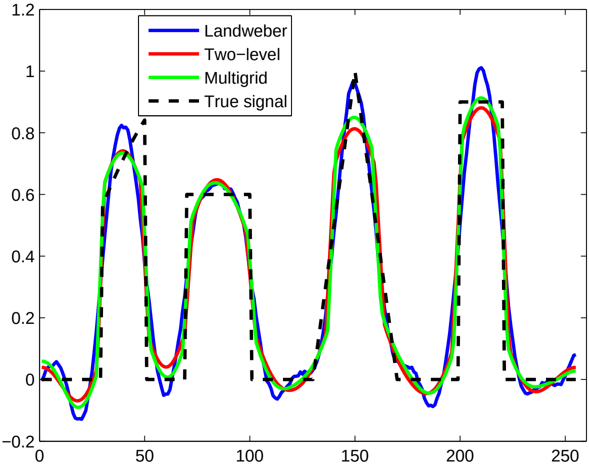
Restoration error vs iterations

Restoration error

Minimum restoration error

Method	$\min_{j=1,\ldots}(e_j)$	$\arg\min_{j=1,\ldots}(e_j)$
Landweber	0.2696	438
Two-level	0.2303	700
Multigrid	0.2119	700

Restored signals



Outline

Work in progress . . .

2D case (image deblurring)

- Tight frame by tensor product $\Longrightarrow$ 9 Frame
- For some frame only threshold or also deblurring?

Other investigations

- Faster iterative regularization smoothers
- Other boundary conditions
- High order wavelets instead of tight frame?