Grid transfer operators for multigrid methods

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The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- \(d\)-dimensional PDE with constant coefficients

\[ (-1)^q \sum_{i=1}^{d} \frac{d^{2q}}{dx_i^{2q}} u(x) = g(x), \quad x \in \Omega = (0, 1)^d, \quad q \geq 1. \]

- Periodic boundary conditions on \(\partial \Omega\) or an infinite domain.
- Discretization by centered finite difference of minimal precision on a uniform grid.
Local Fourier Analysis

The Fourier transform of the discrete differential operator is

\[ \hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j e^{i jh|\omega|}, \]

where \( \omega \in [-\pi/h, \pi/h]^d \) denotes the frequencies for the current discretization step \( h \) and

\[ l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h, \pi/h]^d} \hat{L}(\omega)e^{-i jh|\omega|} d\omega. \]
The convergence result

Theorem (Convergence)

Given a constant-coefficient PDE of order $m$, a necessary condition for nonincreasing high frequencies arising from a coarse grid correction is

$$\gamma_r + \gamma_p \geq m,$$

(1)

where $\gamma_p$ and $\gamma_r$ are the order of the prolongation and of the restriction respectively.

Definition

A prolongation (restriction) has order $\gamma_p$ if it (its transpose) leaves unchanged all polynomials of degree $\gamma_p - 1$. 
More general orders

Definition
The set of all corners of $x$ is

$$\Omega(x) = \{ y \mid y_j \in \{ x_j, \pi + x_j \}, \ j = 1, \ldots, d \}$$

and the set of the “mirror” points of $x$ is $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$. 
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The set of all corners of \( x \) is

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Definition (P. W. Hemker 1990)
For a grid transfer operator \( B \in \{ R, P \} \) (\( B \) is multiplied by \( 2^d \) when \( B = P \)), for \( x = \omega h, |x| \to 0 \), the largest \( s \geq 0 \) such that

\[
\hat{B}(x) = 1 + O(|x|^s),
\]

\[
\hat{B}(y) = O(|x|^s), \quad \forall y \in \mathcal{M}(x),
\]

is the Low Frequency (LF) order

is the High Frequency (HF) order
New convergence conditions

Theorem

In the Convergence Theorem the order $\gamma_p$ and $\gamma_r$ are the HF order of the prolongation and of the restriction respectively.
New convergence conditions

**Theorem**

*In the Convergence Theorem the order $\gamma_p$ and $\gamma_r$ are the HF order of the prolongation and of the restriction respectively.*

**Lemma (P. W. Hemker 1990)**

*If a prolongation leaves all polynomials of degree $k-1$ invariant, then both the LF and HF orders are at least $k$.*
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In the Convergence Theorem the order $\gamma_p$ and $\gamma_r$ are the HF order of the prolongation and of the restriction respectively.

Lemma (P. W. Hemker 1990)

If a prolongation leaves all polynomials of degree $k - 1$ invariant, then both the LF and HF orders are at least $k$.

Remark (I. Yavneh 1998)

The same analysis for the Galerkin approach requires $LF > 0$. 
Grid transfer operators

Interpolation operators

- Linear interpolation: \( \frac{1}{2}[1 \ 2 \ 1] \)
  LF order = HF order = 2
- cubic interpolation: \( \frac{1}{16}[-1 \ 0 \ 9 \ 16 \ 9 \ 0 \ -1] \)
  LF order = HF order = 4
Grid transfer operators

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Noninterpolantion operator

- Refinement coefficients of the B-spline of order 4:
  $\frac{1}{16}[1 \ 4 \ 6 \ 4 \ 1] \quad \Rightarrow \quad$ LF order = 2, HF order = 4.
Multigrid methods for Toeplitz matrices

Toeplitz matrices and $\hat{L}(\omega)$

- The $d$-level Toeplitz matrix $T_n(f)$ is such that

$$ [T_n(f)]_{r,s} = a_{s-r} = aj = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x)e^{-i\langle j|x \rangle} \, dx, \quad r, s, j \in \mathbb{Z}^d. $$

- $f \geq 0 \iff T_n(f)$ is positive definite.
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**Example**

1D Laplacian: $\hat{L}(\omega) = \frac{1}{h^2} (2 - 2 \cos(\omega h))$. The Toeplitz approach moves the factor $\frac{1}{h^2}$ to the rhs, thus $A_n = T_n(f)$, where $f(x) = 2 - 2 \cos(x)$, which vanishes at the origin with order 2.
MGM convergence for Toeplitz matrices

- **Galerkin approach**: \( A_k = P^T A_n P \) (\( A_n \) positive definite).
- **Convergence analysis** for the algebra case like \( \tau \) or **circulant algebra**.
MGM convergence for Toeplitz matrices

- **Galerkin approach:** \( A_k = P^T A_n P \) (\( A_n \) positive definite).
- **Convergence analysis for the algebra case like \( \tau \) or circulant algebra.


Let \( A_n = \mathcal{C}_n(f) \) be circulant with \( f \) having a unique zero at \( x^0 \). Defining \( P = \mathcal{C}_n(p) K_n^T \), where \( K_n \) is the down-sampling and \( p \) is a trigonometric polynomial non identically zero and such that for each \( x \in [-\pi, \pi)^d \)

\[
\limsup_{x \to x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \quad \forall y \in \mathcal{M}(x), \tag{2a}
\]

\[
\sum_{y \in \Omega(x)} p(y)^2 > 0, \tag{2b}
\]

then the TGM converges in a number of iterations independent of \( n \).
Equivalence of the two approaches

Theorem (D. 2010)

In the case of

- constant coefficient PDEs,
- periodic boundary conditions,
- \( R = P^T \),

the two conditions (1) (HF order) and (2a) are equivalent.
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Remark

The (2b) is equivalent to require $LF > 0$. 
Consequences of such equivalence

1 For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance some integral problems.
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2. Allow to define a MGM for Toeplitz matrices with $R \neq P$. 
Consequences of such equivalence

1. For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance some integral problems.

2. Allow to define a MGM for Toeplitz matrices with $R \neq P$.

3. Allow to compare the grid transfer operators used in the two approaches.
Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d} (1 + \cos(x_j - x_j^{(0)}))^{q}$ for $f(x^{(0)}) = 0$ with order $2q$. 
Grid transfer operators for Toeplitz matrices

- \( p(x) = \prod_{j=1}^{d} (1 + \cos(x_j - x_j^{(0)}))^q \) for \( f(x^{(0)}) = 0 \) with order \( 2q \).

- In the PDE setting \( x^{(0)} = 0 \) and \( p(x) \) can be generalized as

\[
\varphi_m(x) = 2^{-dm} \prod_{j=1}^{d} (1 + e^{-ix_j})^m.
\]

\( \varphi_m \) has \( \text{HF} = m \) and \( \text{LF} = 2 \).
Consequences

Grid transfer operators for Toeplitz matrices

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\]

\( \varphi_m \) has HF = \( m \) and LF = 2.

- The grid transfer operators with HF = \( m \) can be obtained by \( \varphi_m(x)\psi_m(x) \) such that \( \psi_m(y) \neq 0 \) for all \( y \in \mathcal{M}(0) \) and \( \psi_m(0) = 1 \).
B-spline refinement coefficients

- The coefficients of $\varphi_m$ are the refinement coefficients of the B-spline of order $m$ in the MRA.
- $\phi_m(x) = \varphi_m(x) e^{i x \lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.
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The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^m \phi_m$ in the 1D case.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$h_{-2}$</th>
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<th>$h_1$</th>
<th>$h_2$</th>
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<tr>
<td>1</td>
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<td>3</td>
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<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>6</td>
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<td>1</td>
</tr>
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</tbody>
</table>

- $m = 2q \Rightarrow$ vertex centered discretization.
- $m = 2q + 1 \Rightarrow$ cell centered discretization.
Numerical results

Grid transfer operators comparison

We consider the following PDE

\[
\begin{aligned}
\frac{d^2}{dx^2} \left( a(x) \frac{d^2}{dx^2} u(x) \right) &= g(x), \quad x \in (0, 1), \\
\text{homogeneous boundary conditions}
\end{aligned}
\]

with nonconstant \( a(x) \).

- It has order \( m = 4 \).
- Smoother: Gauss-Seidel
- Galerkin approach
- The condition for \( V \)-cycle is at least \( \gamma_r + \gamma_p > m \).
Numerical results

**Iteration numbers**

V-cycle iteration numbers varying the problem size $n$ and $a(x) = (x - 0.5)^2$.

<table>
<thead>
<tr>
<th>$n$</th>
<th># iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>15 10 10 9 9</td>
</tr>
<tr>
<td>31</td>
<td>33 13 17 10 11</td>
</tr>
<tr>
<td>63</td>
<td>61 17 24 13 11</td>
</tr>
<tr>
<td>127</td>
<td>101 26 27 17 13</td>
</tr>
<tr>
<td>255</td>
<td>155 35 29 20 16</td>
</tr>
<tr>
<td>511</td>
<td>221 44 36 24 19</td>
</tr>
<tr>
<td>1023</td>
<td>284 53 46 27 22</td>
</tr>
</tbody>
</table>

- $g_c$ = cubic interpolation
- For the choices $(\phi_2, g_c)$ and $(\phi_4, \phi_4)$, the coarse matrices have the same bandwidth.
An image deblurring problem

Riley regularization:

\[ T_{n,n}(z + \theta)f = g \]

Observe image

2% of noise

MGM reconstruction

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( 10^{-2} )</th>
<th>( 10^{-3} )</th>
<th>( 10^{-4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear interpolation</td>
<td>45</td>
<td>286</td>
<td>&gt; 1000</td>
</tr>
<tr>
<td>Toeplitz (*)</td>
<td>49</td>
<td>73</td>
<td>73</td>
</tr>
</tbody>
</table>

\( (*) \ p(x, y) = (2 - 2 \cos(x))^2(2 - 2 \cos(y))^2 \)
Summary

- In the case of elliptic PDEs and Galerkin approach, the LFA and the theory for Toeplitz matrices give the same conditions.
- The LFA does not require $R = P^T \implies R \neq P^T$ also for Toeplitz matrices.
- The convergence theory for Toeplitz matrices includes other applications (e.g. image deblurring).
- Introduce a class of grid transfer operators associated to B-spline.

Reference

Numerical results

About V-cycle . . .

$V$-cycle optimality with Ruge-Stüben theory and Perron-Frobenius theorem.

**Theorem (A. Aricò, D., and S. Serra-Capizzano, 2004)**

$TGM \Rightarrow V$-cycle

$$\limsup_{x \to x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \quad \forall y \in \mathcal{M}(x),$$

[Multidimensional case A. Aricò and M. D., 2007]

**Remark**

*This is equivalent to $\gamma_p \geq m \ (\gamma_p + \gamma_r \geq 2m)$ instead of $2\gamma_p > m \ (\gamma_p + \gamma_r > m)$. This condition is also necessary [A. Napov and Y. Notay, NM 2011].*