Grid transfer operators for multigrid methods

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Outline

The constant coefficient case

The classic convergence analysis for multigrid methods assumes:

- d-dimensional PDE with constant coefficients

$$(-1)^q \sum_{i=1}^{d} \frac{\mathrm{d}^{2q}}{\mathrm{d}x_i^{2q}} u(x) = g(x), \qquad x \in \Omega = (0,1)^d,\ q \geq 1.$$

- Periodic boundary conditions on $\partial\Omega$ or an infinite domain.
- Discretization by centered finite difference of minimal precision on a uniform grid.

Local Fourier Analysis

The Fourier transform of the discrete differential operator is

$$\hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j \mathrm{e}^{\mathrm{i}\langle jh|\omega\rangle},$$

where $\omega \in [-\pi/h, \pi/h]^d$ denotes the frequencies for the current discretization step h and

$$l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h,\pi/h]^d} \hat{L}(\omega) \mathrm{e}^{-\mathrm{i}\langle jh|\omega\rangle} \mathrm{d}\omega.$$

The convergence result

Theorem (Convergence)

Given a constant-coefficient PDE of order m, a necessary condition for nonincreasing high frequencies arising from a coarse grid correction is

$$\gamma_r + \gamma_p \geq m, \tag{1}$$

where γ_p *and* γ_r *are the order of the prolongation and of the restriction respectively.*

Definition

A prolongation (restriction) has order γ_p if it (its transpose) leaves unchanged all polynomials of degree $\gamma_p - 1$.

More general orders

Definition

The set of all corners of x is

$$\Omega(x) = \{ y \mid y_j \in \{x_j, \pi + x_j\}, j = 1, \dots, d\}$$

and the set of the "mirror" points of x is $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$.

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Definition (P. W. Hemker 1990)

For a grid transfer operator $B \in \{R, P\}$ (B is multiplied by 2^d when $B = P$), for $x = \omega h$, $|x| \to 0$, the largest $s \geq 0$ such that

$$\hat{B}(x) = 1 + O(|x|^s),$$ is the Low Frequency (LF) order

$$\hat{B}(y) = O(|x|^s), \quad \forall y \in \mathcal{M}(x),$$ is the High Frequency (HF) order

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Remark (I. Yavneh 1998)

The same analysis for the Galerkin approach requires LF>0 .

Grid transfer operators

Interpolation operators

- Linear interpolation: $\frac{1}{2}[1 \quad 2 \quad 1]$
  LF order = HF order = 2
- cubic interpolation: $\frac{1}{16}[-1 \quad 0 \quad 9 \quad 16 \quad 9 \quad 0 \quad -1]$
  LF order = HF order = 4

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Noninterpolantion operator

- Refinement coefficients of the B-spline of order 4:
  $\frac{1}{16}[1 \quad 4 \quad 6 \quad 4 \quad 1] \qquad \Longrightarrow \qquad$ LF order = 2, HF order = 4.

Toeplitz matrices and $\hat{L}(\omega)$

- The d-level Toeplitz matrix $T_n(f)$ is such that

$$[T_n(f)]_{r,s} = a_{s-r} = a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x) e^{-\mathrm{i}\langle j|x\rangle}\, dx, \qquad r, s, j \in \mathbb{Z}^d.$$

- $f \geq 0 \quad \Leftrightarrow \quad T_n(f)$ is positive definite.

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Example

1D Laplacian: $\hat{L}(\omega) = \frac{1}{h^2}(2 - 2\cos(\omega h))$. The Toeplitz approach moves the factor $\frac{1}{h^2}$ to the rhs, thus $A_n = T_n(f)$, where $f(x) = 2 - 2\cos(x)$, which vanishes at the origin with order 2.

MGM convergence for Toeplitz matrices

- Galerkin approach: $A_k = P^T A_n P$ (A_n positive definite).
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Theorem (G. Fiorentino and S. Serra-Capizzano 1991, 1996, (2002))

Let $A_n = \mathcal{C}_n(f)$ be circulant with f having a unique zero at x^0. Defining $P = \mathcal{C}_n(p)K_n^T$, where K_n is the down-sampling and p is a trigonometric polynomial non identically zero and such that for each $x \in [-\pi, \pi)^d$

$$\limsup_{x \to x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x), \tag{2a}$$

$$\sum_{y \in \Omega(x)} p(y)^2 > 0, \tag{2b}$$

then the TGM converges in a number of iterations independent of n.

Equivalence of the two approaches

Theorem (D. 2010)

In the case of

- *constant coefficient PDEs,*
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Remark

The (2b) is equivalent to require $LF > 0$.

Consequences of such equivalence

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Consequences of such equivalence

1. For the Galerkin approach, the analysis with circulant matrices is more general since it includes also non differential problems, like for instance some integral problems.
2. Allow to define a MGM for Toeplitz matrices with $R \neq P$.
3. Allow to compare the grid transfer operators used in the two approaches.

Grid transfer operators for Toeplitz matrices

- $p(x) = \prod_{j=1}^{d}(1 + \cos(x_j - x_j^{(0)}))^q$ for $f(x^{(0)}) = 0$ with order $2q$.

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- In the PDE setting $x^{(0)} = 0$ and $p(x)$ can be generalized as

$$\varphi_m(x) = 2^{-dm} \prod_{j=1}^{d} \left(1 + \mathrm{e}^{-\mathrm{i}x_j}\right)^m .$$

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- The grid transfer operators with HF$= m$ can be obtained by $\varphi_m(x)\psi_m(x)$ such that $\psi_m(y) \neq 0$ for all $y \in \mathcal{M}(0)$ and $\psi_m(0) = 1$.

B-spline refinement coefficients

- The coefficients of φ_m are the refinement coefficients of the B-spline of order m in the MRA.
- $\phi_m(x) = \varphi_m(x)\mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2} \rfloor}$ defines centered B-spline.

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The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^m \phi_m$ in the 1D case.

m	h_{-2}	h_{-1}	h_0	h_1	h_2
1		1	1		
2		1	2	1	
3	1	3	3	1	
4	1	4	6	4	1

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- $m = 2q \Rightarrow$ vertex centered discretization.
- $m = 2q + 1 \Rightarrow$ cell centered discretization.

Grid transfer operators comparison

We consider the following PDE

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a(x)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)\right) = g(x), & x \in (0,1), \\ \text{homogeneous boundary conditions} \end{cases}$$

with nonconstant $a(x)$.

- It has order $m = 4$.
- Smoother: Gauss-Seidel
- Galerkin approach
- The condition for V-cycle is at least $\gamma_r + \gamma_p > m$.

Iteration numbers

V-cycle iteration numbers varying the problem size n and $a(x) = (x - 0.5)^2$.

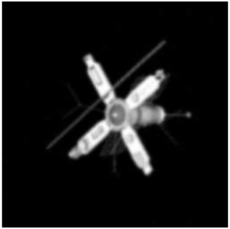
restriction prolongation	ϕ_2 ϕ_2	ϕ_2 ϕ_4	ϕ_2 g_c	ϕ_4 ϕ_4	ϕ_4 g_c
n	# iterations				
15	15	10	10	9	9
31	33	13	17	10	11
63	61	17	24	13	11
127	101	26	27	17	13
255	155	35	29	20	16
511	221	44	36	24	19
1023	284	53	46	27	22

- g_c = cubic interpolation
- For the choices $(\phi_2,\ g_c)$ and $(\phi_4,\ \phi_4)$, the coarse matrices have the same bandwidth.

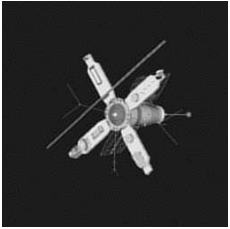
An image deblurring problem

Riley regularization:
$T_{n,n}(z+\theta)f = g$

Observed image
2% of noise



MGM
reconstruction



θ	10^{-2}	10^{-3}	10^{-4}
Linear interpolation	45	286	>1000
Toeplitz (*)	49	73	73

(*) $p(x,y) = (2-2\cos(x))^2(2-2\cos(y))^2$

Summary

- In the case of elliptic PDEs and Galerkin approach, the LFA and the theory for Toeplitz matrices give the same conditions.
- The LFA does not requires $R = P^T \implies R \neq P^T$ also for Toeplitz matrices.
- The convergence theory for Toeplitz matrices includes other applications (e.g. image deblurring).
- Introduce a class of grid transfer operators associated to B-spline.

Reference

M. DONATELLI, *An algebraic generalization of local Fourier analysis for grid transfer operators in multigrid based on Toeplitz matrices*, Numer. Linear Algebra Appl., 17 (2010), pp. 179–197.

About V-cycle . . .

V-cycle optimality with Ruge-Stüben theory and Perron-Frobenius theorem.

Theorem (A. Aricò, D., and S. Serra-Capizzano, 2004)

TGM $\Rightarrow$ *V-cycle*

$$\limsup_{x \to x^0} \left| \frac{p(y)^{\cancel{2}}}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x),$$

[Multidimensional case A. Aricò and M. D., 2007]

Remark

This is equivalent to $\gamma_p \geq m$ ($\gamma_p + \gamma_r \geq 2m$) instead of $2\gamma_p > m$ ($\gamma_p + \gamma_r > m$). This condition is also necessary [A. Napov and Y. Notay, NM 2011].