Regularization by multigrid-type algorithms

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Outline

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Image restoration with Boundary Conditions

Using Boundary Conditions (BCs), the restored image $\mathbf{f}$ is obtained solving: (in some way ...)

$$A\mathbf{f} = \mathbf{g} + \xi$$

- $\mathbf{g}$ = blurred image,
- ξ = noise (random vector),
- A = two-level matrix depending on the point spread function (PSF) and the BCs.

The PSF is the observation of a single point (e.g., a star in astronomy) that we assume shift invariant.

Coefficient matrix structure

The matrix-vector product computed in $O(n^2 \log(n))$ ops for $n \times n$ images while the inversion costs $O(n^2 \log(n))$ ops only in the periodic case.

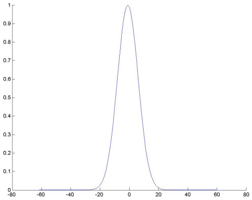
BCs	A
Dirichlet	Toeplitz
periodic	circulant
Neumann (reflective)	Toeplitz + Hankel
anti-reflective	Toeplitz + Hankel

If the PSF is symmetric with respect to each direction:

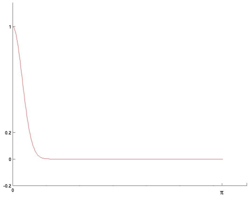
BCs	A
Neumann (reflective)	DCT III
anti-reflective	DST I + low-rank

Generating function of PSF

- The eigenvalues of $A(z)$ are about a uniform sampling of z.



PSF



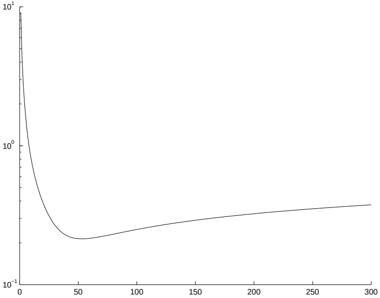
Generating function $z(x)$

- The ill-conditioned subspace is mainly constituted by the middle/high frequencies.

Semi-convergence of iterative regularization methods

Some iterative methods (CGLS, . . .) have regularization properties:

- They firstly reduce the algebraic error in the low frequencies (well-conditioned subspace).
- When they arrive to reduce the algebraic error in the high frequencies then the restoration error increases because of the noise.



Outline

Multigrid structure

Multigrid Idea

Project the system in a subspace, solve the resulting system in this subspace and interpolate the solution in order to improve the previous approximation.

Multigrid components

The Multigrid combines two iterative methods:

Smoother: a classic iterative method,

Coarse Grid Correction: projection, solution of the restricted problem, interpolation.

Multigrid

1. We use the Galerkin approach
   - $P_i = R_i^T$
   - $A_{i+1} = RA_iR_i^T$

   where R_i and P_i are the restriction and the prolongation operators at the level i, respectively.
2. The smoother is a classic iterative regularization method: in the initial iterations it is not able to reduce effectively the ill-conditioned subspace.
3. Coarser grid of size 8×8 independent of the size of the finer grid.

Image deblurring and Multigrid

- In the image deblurring the ill-conditioned subspace is related to high frequencies, while the well-conditioned subspace is generated by low frequencies.
- In order to obtain a fast convergence the algebraic multigrid projects in the high frequencies where the noise "lives" $\Longrightarrow$ noise explosion already at the first iteration (it requires Tikhonov regularization [Donatelli, NLAA, 12 (2005), pp. 715–729]).
- In this case the low-pass filter projects in the well-conditioned subspace (low frequencies) $\Longrightarrow$ it is slowly convergent but it can be a good iterative regularizer.

Restriction operator

$R_i = K_{N_i}\mathcal{A}_{N_i}(p)$:

- $K_{N_i} \in \mathbb{R}^{\frac{N_i}{4}\times N_i}$ is the cutting matrix that preserves the structure at the lower level.

circulant	**Toeplitz & DST − I**	**DCT − III**
$\begin{bmatrix} 1\ 0 & & & \\ & 1\ 0 & & \\ & & \ddots\ \ddots & \\ & & & 1\ 0 \end{bmatrix}$	$\begin{bmatrix} 0\ 1\ 0 & & & \\ & 0\ 1\ 0 & & \\ & & \ddots\ \ddots\ \ddots & \\ & & & 0\ 1\ 0 \end{bmatrix}$	$\begin{bmatrix} 1\ 1\ 0 & & & \\ & 1\ 1\ 0 & & \\ & & \ddots\ \ddots\ \ddots & \\ & & & 0\ 1\ 1 \end{bmatrix}$

- $p(x,y)$ is the generating function of the projector, which selects the subspace where to project the linear system.

$$2\text{D} \quad \leftrightarrow \quad p(x,y) = (1+\cos(x))(1+\cos(y))$$

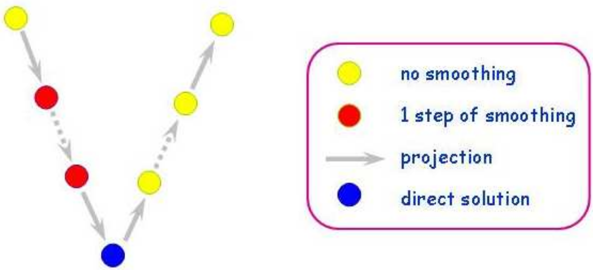
If we have an *iterative regularization method* we can improve its regularizing properties and/or accelerate its convergence using it as *smoother* in a Multigrid algorithm.

Two Level (TL) algorithm

1. No smoothing at the finer level
2. At the coarser level to apply one step of the smoother instead of to solve directly the linear system

Multigrid regularization (applying recursively the TL)

V-cycle



Using a larger number of recursive calls (e.g. W-cycle), the algorithm "works" more in the well-conditioned subspace, but it is more difficult to define an early stopping criterium.

Computational Cost

Assumptions: $n \times n$ images and $m \times m$ PSFs with $m \ll n$.

- Let $S(n)$ be the computational cost of one smoother iteration.
- The computational cost of one iteration of our multigrid regularization method with γ recursive calls is

$$C(\gamma, n) \approx \begin{cases} \frac{1}{3}S(n), & \gamma = 1 \\ S(n), & \gamma = 2 \\ 3S(n), & \gamma = 3 \end{cases}$$

- if $m \approx n$ then $S(n) = O(n^2 \log(n))$.

Remarks on the TL regularization

Theoretical analysis

- Intuitively: the regularization properties of the smoother are preserved since it is combined with a low-pass filter.
- From the filter factor analysis of the Two Level (TL) method we have:
  - a condition to choose p s.t. it is a regularizer
  - in such case it improves the regularization properties of the iterative method used as smoother.

Filter factor of the Landweber method

- Imposing P-BCs $A = \mathcal{C}_n(z)$: A is a circulant matrix of size n generated by the function z.
- $A = F_n D_n(z) F_n^H$, where $F_n = [\mathrm{e}^{\mathrm{i} j x_k}]_{k,j=0}^{n-1}/\sqrt{n}$ is the DFT matrix and $D_n(z) = \mathrm{diag}([f(x_k)]_{k=0}^{n-1})$ with $x_k = \frac{2\pi k}{n}$.
- Taking $\mathbf{x}_0 = 0$ the jth approximation of $\mathbf{f}$ is

$$\mathbf{x}_j = F_n \sum_{i=0}^{j-1} (I - D_n(|z|^2))^i D_n(\bar{z}) F_n^H \mathbf{b} \; = \; C_n(\phi_j) C_n^{-1}(z)\mathbf{b}$$

where $\phi_j(x) = 1 - (1 - |z(x)|^2)^j$, $x \in (0, 2\pi]$ is the filter factor.

Filter factor of the TL method

- For TL with Landweber as smoother $\mathbf{x}_j = B_n\mathbf{b}$ with

$$B_n = \mathcal{C}_n(p) K_n^T \mathcal{C}_{\frac{n}{2}}(\hat{g}) K_n \mathcal{C}_n(r),$$

  where $\hat{g}(x) = \frac{1-(1-|\hat{z}(x)|^2)^j}{\hat{z}(x)}$, $x \in (0, 2\pi]$, K_n is the cutting matrix and r, p and $\hat{z}$ are restriction, prolongation and PSF function at the coarser level respectively.

- $B_n = F_n \Pi_n^T W_n \Pi_n F_n^H$, where Π_n is a permutation matrix and W_n is the diagonal block matrix of size $(n/2) \times (n/2)$ with blocks of dimension 2×2. For $k = 0, \ldots, n/2-1$, the k-th diagonal block is given by

$$W_n^{(k)} = \frac{1}{2}\hat{g}(x_{2k}) \begin{bmatrix} p(x_k) \\ p(x_{(k+n/2)}) \end{bmatrix} \begin{bmatrix} r(x_k) & r(x_{(k+n/2)}) \end{bmatrix}.$$

Filter factor of the TL method 2

- The block $W_n^{(k)}$ has rank 1 and the nontrivial null eigenvalue λ_k is

$$\lambda_k = \frac{1}{2}\hat{g}(x_{2k})\left((pr)(x_k) + (pr)(x_{(k+n/2)})\right).$$

- The eigenvector associated to the null eigenvalue is

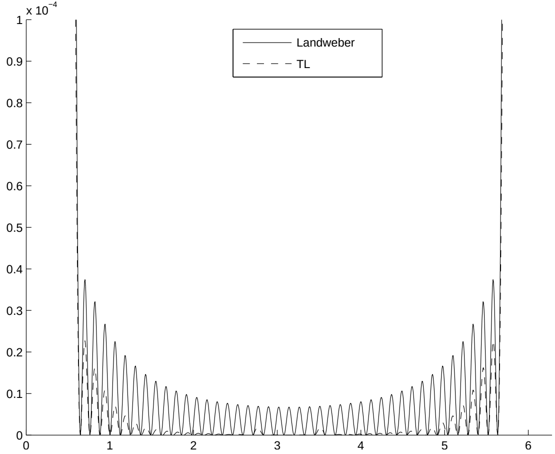
$$\frac{r(x_k)}{r(x_{(k+n/2)})} F_n^{(k+n/2)} - F_n^{(k)}.$$

  This should be an high frequency (to filtering) $\Rightarrow$ it provides a condition to choose r: e.g. nonnegative and decreasing in $[0, \pi]$.

- The eigenvector associated to λ_k defines an analogous condition for p.

Comparison TL vs Landweber

Focus on the high frequencies for the filter factors of TL and Landweber for $j = 1000$



Noise $\longrightarrow 0$?

In the noise free case the TL method does not compute the exact solution.

How to recover the high frequencies in the noise free case is a work in progress . . .

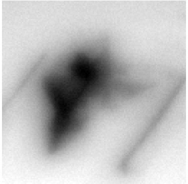
Outline

An airplane

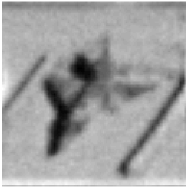
- Periodic BCs
- Gaussian PSF (A spd)
- noise $= 1\%$

Original Image

Inner part 128×128



Observed image



Restored with MGM

Restoration error: noise $= 1\%$

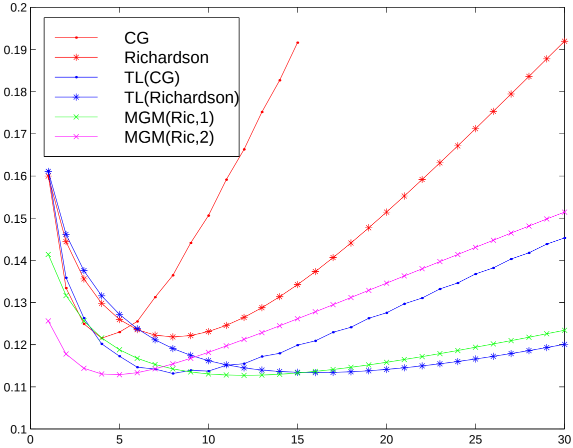
$e_j = \|\mathbf{f} - \mathbf{f}^{(j)}\|_2 / \|\mathbf{f}\|_2$ restoration error at the j-*th* iteration.

Minimum restoration error

Method	$\min_{j=1,\ldots} (e_j)$	$\arg\min_{j=1,\ldots} (e_j)$
CG	0.1215	4
Richardson	0.1218	8
TL(CG)	0.1132	8
TL(Rich)	0.1134	16
MGM(Rich, 1)	0.1127	12
MGM(Rich, 2)	0.1129	5
CGLS	0.1135	178
Landweber	0.1135	352

Restoration error: noise $= 1\%$

Relative error vs. number of iterations



Noise $= 10\%$

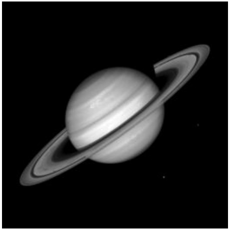
For CG and Richardson it is better to resort to normal equations.

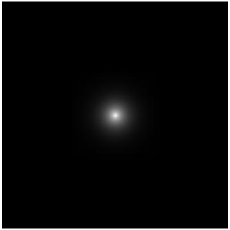
Minimum restoration error

Method	$\min_{j=1,\ldots}(e_j)$	$\arg\min_{j=1,\ldots}(e_j)$
CGLS	0.1625	30
Landweber	0.1630	59
TL(CGLS)	0.1611	48
TL(Landweber)	0.1613	97
MGM(Landweber,1)	0.1618	69
MGM(Landweber,2)	0.1621	26
MGM(Rich,1)	0.1648	3
MGM(Rich,2)	0.1630	1

An astronomic example with nonnegativity constraint

- Periodic BCs and dark background (exact)
- Gaussian PSF + noise = 5%


Original Image

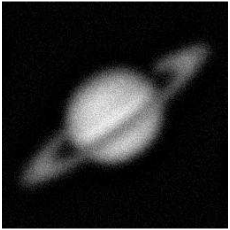

PSF


Observed image

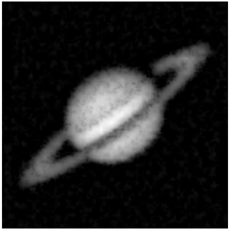
Minimum restoration error

Method	$\min_{j=1,\ldots}(e_j)$	$\arg\min_{j=1,\ldots}(e_j)$
CG^+	0.2268	4
$Rich^+$	0.2298	9
$MGM(Rich^+,1)^+$	0.1556	27
$MGM(Rich^+,2)^+$	0.1530	12
$Landweber^+$	0.1419	2735
$CGLS^+$	0.1419	885
$MGM(CGLS^+,2)^+$	0.1389	109

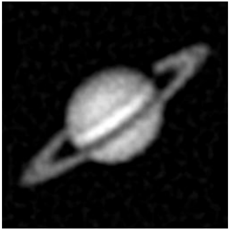
Restored images



Rich$^+$ (9 iter.)

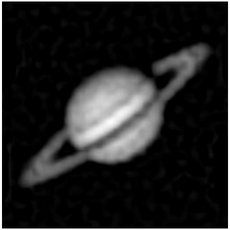


MGM(Rich$^+$,2)$^+$ (12 iter.)

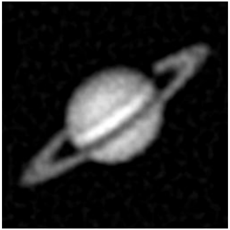


CGLS$^+$ (885 iter.)

Restored images



MGM(CGLS$^+$,2)$^+$
(109 iter.)



CGLS$^+$ (885 iter.)

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Summarizing . . . multigrid regularization method

- It is a general framework which can be used to improve the regularization properties of an iterative regularizing method.
- It leads to a smaller relative error and a flatter error curve with respect to the smoother applied alone.
- It is fast and usually it obtains a good restored image also without resorting to normal equations.
- It can be combined with other techniques and it can lead to several generalizations (e.g., nonnegativity constraints).

Reference

M. Donatelli and S. Serra Capizzano, *On the regularizing power of multigrid-type algorithms*, SIAM J. Sci. Comput., 27–6 (2006) pp. 2053–2076.