ANTIREFLECTIVE BOUNDARY CONDITIONS, RE-BLURRING AND FAST DE-BLURRING METHODS

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Abstract

We consider the classical deblurring problem of noisy and blurred signals or images in the case of space-invariant point spread function (PSF) imposing the boundary conditions (BCs). Our focus is on the antireflective BCs since they reduce substantially artifacts called ringing effects with respect to other classical choices holding the same computational cost.

The Boundary Conditions

Basically, the mathematical model of image blurring with spatial invariant kernel is the following Fredholm operator of first kind

\[ g(x, y) = \int K(x - \theta, y - \varphi) f(\theta, \varphi) d\theta d\varphi + \nu(x, y), \]

where \( f \) is the (true) input object, \( K \) is the integral kernel of the operator, also called point spread function, \( \nu \) is the noise, and \( g \) is the observed image.

In the discrete case the observed image is \( n \times n \) (for simplicity we assume square images) and the PSF is \( p \times p \). It gives rise to the corresponding matrix operator \( g = K n \times n + \nu \) that is under-determined since the matrix \( K \) has size \( n^2 \times n^2 \) with \( n = p - 1 \). An attractive technique both from the quality of the restored images and computational point of view is the use of appropriate boundary conditions: linear or affine relations between the unknowns outside the field of view (FOV) and the unknowns inside the FOV.

The more common type of BCs and related properties are summarized in the following table for the 1D case.

<table>
<thead>
<tr>
<th>Type</th>
<th>Description outside the FOV</th>
<th>Matrix structure</th>
<th>Matrix-vector product</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Dirichlet</td>
<td>zero pad</td>
<td>Toeplitz</td>
<td>( O(n \log n) ) complex op.</td>
<td>[1]</td>
</tr>
<tr>
<td>Periodic</td>
<td>circular convolution</td>
<td>Circulant</td>
<td>( O(n \log n) ) complex op.</td>
<td></td>
</tr>
<tr>
<td>Reflective</td>
<td>reflection like a mirror</td>
<td>Toeplitz + Hankel</td>
<td>symmetric PSF ( O(\log n) ) real op.</td>
<td>[3]</td>
</tr>
<tr>
<td>Anti-reflective</td>
<td>reflection through central symmetry</td>
<td>Toeplitz + Hankel</td>
<td>symmetric PSF ( O(\log n) ) real op.</td>
<td>[4]</td>
</tr>
</tbody>
</table>

The generalization to the-dimensional case can be done resorting to tensor product computations. The computational cost is related to the algebraic actual size of the involved objects (for instance we have \( O(n \log n) \) for the periodic BCs, if the object is \( n \times n \times n \times n \).

The Antireflective BCs (AR-BCs)

For the sake of simplicity we explain the imposition of AR-BCs in the 1D case. We emphasize that in the 2D there is not a unique choice at the vertices of the image (see [2]).

Let \( L_n \) be \( f_0, f_1, \ldots, f_n, f_{n+1}, \ldots \) and \( L_n' \) be \( \cdots, 0, f_{-n}, \ldots, f_{-1} \), \( f_0, \ldots, f_n \). Then \( K_l \) becomes \( A \) if \( A \) is Toeplitz-Hankel plus 2 rank correction matrix, where the correction is placed at the first and last columns. Furthermore, in the case of symmetric PSF, \( A \) belongs to a matrix algebra denoted by \( S_n \) such that there exists an algorithm that solves the linear system \( A \mathbf{x} = \mathbf{g} \) in \( O(\log n) \) operations mainly using the discrete fast sine transforms (DST-bs).

The matrix algebra \( S_n \) can be introduced as follows (see [2]). Let \( Q \) be the \( n \times n \) orthogonal and symmetric matrix expressed by \( Q_{ij} = \sqrt{1-\cos(j \pi/n)} \), \( i = 1, \ldots, n \), then we define \( \tau = Q D Q^T \) is a real diagonal matrix of size \( n \). Now, by definition, \( M \in S_n \) if

\[ M = \begin{bmatrix} \alpha & \nu \\ v^T & \beta \end{bmatrix} \]

with \( \alpha, \beta \in \mathbb{R} \), \( v, w \in \mathbb{R}^{n-2} \) and \( M \in \tau \).

The previous considerations can be generalized to the-dimensional case obtaining the \( S_d \) algebra.

The normal equations and re-blurring

Regulation methods extensively used in the literature (e.g., Tikhonov regularization, CGNE, Lanczos-) are usually applied to the normal equations \( A^* A \mathbf{x} = A^* \mathbf{g} \) with \( A \in \mathbb{R}^{d \times d} \) in the image deblurring. For the first three kinds of BCs of Table 1, the algebraic transposition of the coefficient matrix can be equivalently obtained by a 180° rotation of the PSF. This fails only in the case of AR-BCs and indeed they lose their symmetry with respect to the other BCs when applied to the normal equations.

Therefore, instead of dealing with the normal equations, we work with \( A^* A = A^* A \) where \( A \) is the matrix obtained imposing the current BCs to the transpose of the PSF (the only difference is in the AR-BCs case).

With this new formulation the AR-BCs become again the better choice regarding the quality of the restored image. This technique is known as re-blurring [see [4]]. The strategy is useful from a computational point of view as well, since \( A^* A \neq A \) in general, while \( A A^* \in S_n \) keeping the \( O(\log(n))^2 \) computational cost for solving the linear system in the case of centro-symmetric PSF.

Numerical experiments

We test the BCs deblurring techniques for the following two shift-invariant 256 x 256 PSFs.

(1) Gaussian PSF.

The matrix \( A \) is the Knoeckner product of two symmetric banded Toeplitz matrices \( T_{n \times n} = \sigma \cos(\pi x/m) \), \( \sigma(\pi x/m) \) for \( j \neq k \leq 0 \), and \( 0 \) elsewhere. \( k_j(\pi x/m) \) is the Gaussian distribution with zero mean and standard deviation \( \sigma \), and the points \( x_j = 1, \ldots, m \) are equally spaced in \([-2, 2]\). The PSF have been proposed by L. Klein in prototype of image restoration problems.

(2) Experimental PSF.

The matrix \( A \) is the symmetric version, that is, \( A = (B + B^T)/2 \), of the widely used experimental 256 x 256 blurring matrix \( B \) developed by US Air Force Phillips Laboratory, Lasers and Imaging Directorate, Kirtland Air Force Base, New Mexico.

The true data \( f \) is either the actor in the middle of Fig. 2 or an image of "blocks of flats". The noise on the blurred images is Gaussian with zero mean. We recover the 192 x 192 internal portion of the true data from the knowledge of the blurred and noisy image in the same area.

Table 2 shows the best relative restoration errors among the first 100 iterations of the CG method. On the left, both the two input images with signal to noise ratio (SNR) equals to 25 are considered.

On the right, the image of the face of actor for different levels of noise on the blurred data is considered. Although some positive effects arise in all cases, the choice of the antireflective BCs is important mainly if the noise on the data is low, that is, for high values of SNR.

Figure 3 shows the three optimal deblurred images, for different BCs.

Table 2 Best relative restoration errors within 100 iterations of CG method applied to \( A^* A = A \), (SNR = 25).

<table>
<thead>
<tr>
<th>BC</th>
<th>Periodic</th>
<th>Reflective</th>
<th>Anti-Reflective</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR = 100</td>
<td>0.0782</td>
<td>0.1806</td>
<td>0.2036</td>
</tr>
<tr>
<td></td>
<td>0.1354</td>
<td>0.2871</td>
<td>0.3164</td>
</tr>
<tr>
<td></td>
<td>0.2093</td>
<td>0.4011</td>
<td>0.4307</td>
</tr>
<tr>
<td></td>
<td>0.3004</td>
<td>0.5911</td>
<td>0.6203</td>
</tr>
<tr>
<td></td>
<td>0.4104</td>
<td>0.8130</td>
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</tr>
<tr>
<td></td>
<td>0.5393</td>
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<tr>
<td></td>
<td>0.6942</td>
<td>1.4723</td>
<td>1.4939</td>
</tr>
<tr>
<td></td>
<td>0.8940</td>
<td>1.8105</td>
<td>1.8321</td>
</tr>
<tr>
<td></td>
<td>1.1354</td>
<td>2.2765</td>
<td>2.2981</td>
</tr>
</tbody>
</table>

Figure 3 shows the three optimal deblurred images, for different BCs.

References