A Multigrid method for restoration of images with Dirichlet boundary conditions.

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Outline

- Dirichlet Boundary Conditions
- AMG vs MGM
- Algebraic Multigrid (AMG) for DST-I and BTTB
- Noise free image
- Reconstruction without regularization
- Two Tikhonov-like regularization techniques
- Conclusion and future work

Dirichlet Boundary Conditions

- The restored image **f** is obtained from the observed image **g** solving

$$A\mathbf{f} = \mathbf{g}$$

where A is block Toeplitz - Toeplitz block ($\mathbf{BTTB}$), banded at each level.

- A BTTB is denoted by $A = T_{n,m}(z)$, explicitly:

$$T_{n,m}(z) = \begin{pmatrix} T_0(z) & T_{-1}(z) & \dots & T_{1-n}(z) \\ T_1(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & T_{-1}(z) \\ T_{n-1}(z) & \dots & T_1(z) & T_0(z) \end{pmatrix}, \quad T_j(z) = \begin{pmatrix} a_{j,0} & a_{j,-1} & \dots & a_{j,1-m} \\ a_{j,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{j,-1} \\ a_{j,m-1} & \dots & a_{j,1} & a_{j,0} \end{pmatrix},$$

where $a_{j,k}$ are the Fourier coefficients of the bivariate generating function z

$$a_{j,k} = \frac{1}{4\pi^2} \int_\Omega z(x,y) \mathrm{e}^{-\mathrm{i}(jx+ky)} \mathrm{d}x \mathrm{d}y, \qquad \mathrm{i}^2 = 1, \qquad \Omega = [-\pi, \pi] \times [-\pi, \pi].$$

Relations between DST-I and Toeplitz

- DST-I is the space containing all matrices $\tau_n(z)$ that can be diagonalized by the fast sine transforms Q_n:

$$DST\text{–}I = \{Q_n D_n Q_n : \; D_n = \mathrm{diag}_{j=1,\ldots,n}(\lambda_j)\}.$$

- Let $z(x) = a_0 + 2\sum_{k=1}^{b} a_k \cos(kx)$ be the generating function then $\lambda_j = z(\frac{j\pi}{n+1})$.
- Link between symmetric Toeplitz and DST-I:

$$\tau_n(z) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{b-1} & & & & \\ a_1 & \ddots & \ddots & & & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & & & \ddots & \\ a_{b-1} & & \ddots & \ddots & \ddots & & & a_{b-1} \\ & \ddots & & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & & \ddots & \ddots & & a_1 \\ & & & a_{b-1} & \cdots & a_1 & & a_0 \end{bmatrix}_{n\times n} - \begin{bmatrix} a_2 & a_3 & \cdots & a_{b-1} & & & \\ a_3 & & \cdot^{\cdot^{\cdot}} & & & & \\ \vdots & \cdot^{\cdot^{\cdot}} & & & & & \\ a_{b-1} & & & & & & a_{b-1} \\ & & & & & \cdot^{\cdot^{\cdot}} & \vdots \\ & & & & \cdot^{\cdot^{\cdot}} & & a_3 \\ & & & a_{b-1} & \cdots & a_3 & a_2 \end{bmatrix}_{n\times n},$$

i.e. $T_n(z) = \tau_n(z) + H_n(z)$.

AMG vs MGM

- In the MGM the projector is fixed (the linear interpolator) and the smoother is chosen to obtain a smooth error.
- In the AMG the smoother is fixed and the projector is chosen to project into the subspace where the smoother is ineffective.
- Many classical iterative methods (Richardson, Jacobi, Gauss-Seidel) have a similar spectral behavior. Therefore, for the AMG is crucial the choice of the projector.
- For PDEs the MGM is optimal: the computational cost for solve the linear system has the same order of the matrix-vector product.

Algebraic Multigrid (AMG) for DST-I

- The smoother is the relaxed Richardson: $R_i = I_{N_i} - \omega_i \tau_{n_i,n_i}(z_i)$.
- The projector at level i is defined as $P_{i+1}^{i} = U_{i+1}^{i} \cdot \tau_{n_i,n_i}(p_i)$ where

$$U_{i+1}^{i} = K_{i+1}^{i} \otimes K_{i+1}^{i}, \qquad K_{i+1}^{i} = \begin{bmatrix} 0 & 1 & 0 & & & & & \\ & & 0 & 1 & 0 & & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & 0 & 1 & 0 \end{bmatrix}_{n_{i+1}\times n_i}$$

and let $w_0 = (x_0, y_0)$ be the unique zeros of z_i of order $2q$, p_i is such that:

$$\begin{array}{lll} (1) \qquad \limsup_{w\to w_0} \left|\dfrac{p_i(\hat{w})}{z_i(w)}\right| < +\infty, & & \hat{w} \in M(w), \\ (2) \qquad 0 < \displaystyle\sum_{\hat{w}\in M(w)\cup\{w\}} p_i^2(\hat{w}), & & i = 0, \ldots, m-1, \end{array}$$

where $M(w) = \{(\pi - x,\ y),\ (x,\ \pi - y),\ (\pi - x,\ \pi - y)\}$.

Computational issue

- The U_{i+1}^{i} implies that the coefficient matrix A_i at each level belong to DST-I.
- The polynomial p_i chooses the subspace where the smoother is ineffective.
- Using a Galerkin strategy ($A_{i+1} = P_{i+1}^{i} A_i (P_{i+1}^{i})^T$) the matrices A_i and P_{i+1}^{i} are computed in a Setup phase before applying the AMG.
- The A_i bandwidth tends to the double of the P_{i+1}^{i} bandwidth (which is small: it is a function of the ill-conditioning of A_0).
- Computational cost for the Setup phase.
  For each level $i = 0, \ldots, \log_2(n)$:
  - P_{i+1}^{i} is assembled with a constant cost,
  - A_{i+1} requires two convolution with a constant cost for banded matrices.

  The total cost is $O(\log_2(n))$.

Literature

1. A first AMG for DST-I matrices was proposed by Fiorentino and Serra Capizzano in 1991. In 1996 they extended it to two-level DST-I matrices.
2. In 2001 Serra Capizzano proved the Two-Grid Method (TGM) optimality also for multilevel Toeplitz matrices using the Ruge-Stüben theory.
3. Recently, using the Perron-Frobenius Theorem, we have proved (Arico', Donatelli, Serra Capizzano, SIMAX to appear) that in the 1D case the V-cycle for DST-I is optimal and now we are extending it to the multidimensional case.
4. There are several proposals for extending this AMG to Toeplitz matrices. In the case of a generating function with a zero of order two the V-cycle is optimal also for Toeplitz matrices.
5. Some authors (R. Chan, T. Huckle) proposed different MGM for Toeplitz, but no proof of optimality is given.

Level independency $\equiv$ optimality?

- With the V-cycle the answer is NO!
- The level independency is a necessary but not sufficient condition to obtain the V-cycle optimality:
  in the level independency we assume to know exactly the error at each level, while with the V-cycle the error comes from several projections.
- Counterexample:
  1. system $\tau_n(f)\mathbf{x} = \mathbf{b}$ where $f(x) = (2 - 2\cos(x))^2$
  2. $p_i = 2 + 2\cos(x) \Rightarrow$ level independency but no optimality,
  3. $p_i = (2 + 2\cos(x))^2 \Rightarrow$ optimality as well.

Number of iterations

n	level independency (2)	optimality (3)
127	283	83
255	510	83
511	899	83
1023	1541	83

AMG for BTTB

- Using the same projector defined for DST-I in (3), the TGM optimality is proved, but the matrix at each level is no longer Toeplitz.
- In (4) we have generalized the AMG for DST-I matrices to multilevel Toeplitz matrices choosing $P_{i+1}^{i} = U_{i+1}^{i}\{t\} \cdot T_{n_i,n_i}(p_i)$:
  - the polynomial p_i at each level is chosen as for the DST-I algebra,
  - the cutting matrix $U_{i+1}^{i}\{t\} = K_{i+1}^{i}\{t\} \otimes K_{i+1}^{i}\{t\}$ is such that

$$K_{i+1}^{i}\{t\} = \left[\, 0_{n_{i+1}-t}^{t} \mid K_{n_{i+1}-t}^{n_i-2t} \mid 0_{n_{i+1}-t}^{t} \,\right] \in \mathbf{R}^{(n_{i+1}-t)\times n_i},$$

  with $t = deg(p_0) - 1$.
- This cutting matrix preserves the Toeplitzness at each level by cutting the lowest possible level of information.

PSF and noise

- We add 2% of noise to the blurred image $\hat{\mathbf{g}}$ solving

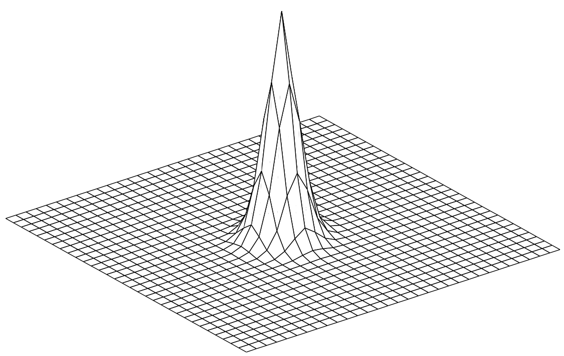
$$T_{n,n}(z)\mathbf{f} = \mathbf{g},$$

where $\mathbf{g} = \hat{\mathbf{g}} + \mathbf{n}$, $\mathbf{n} = 2\frac{\|\mathbf{g}\|_2}{\|\mathbf{a}\|_2}\mathbf{a}$ and $\mathbf{a} = \mathrm{rand}(n^2, 1)$.

- The PSF is generated by $z(x, y) \geq 0$ and is close to zero in a neighborhood of (π, π):

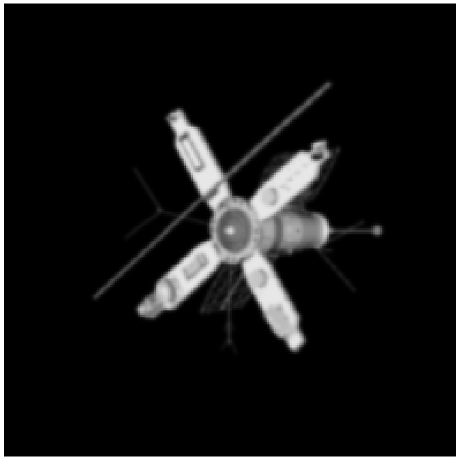
$$z(x, y) = \frac{1}{c}F(x, y)\psi(x, y)$$

where $F(x, y) = 2 + \cos(x) + \cos(y)$ is the "kernel" that vanishes in (π, π) with order 2, $\psi(x, y) > 0$ with nonnegative Fourier coefficients and c is a normalization constant.

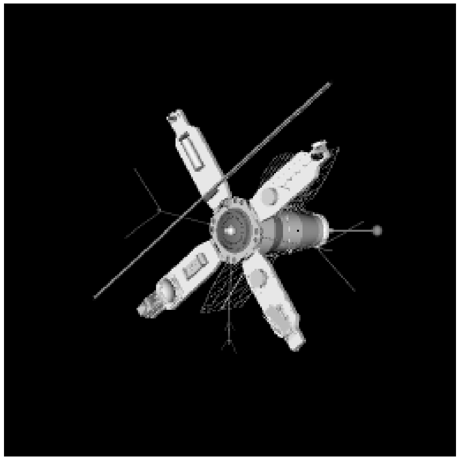


Noise free image

- Pre-smoother: two iterations of relaxed Richardson with $\omega = \max(z(x,y))^{-1}$.
  Post-smoother: two iterations of Conjugate Gradient without preconditioning.
- The $T_{n,n}(z)$ is ill-conditioned, but despite this bad spectral behavior, the proposed AMG is optimal as emphasized by the linear convergence.



Blurred image.



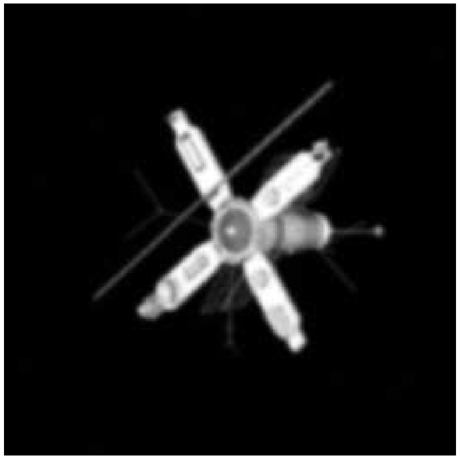
Restored image.

#(Iter.)	$\|error\|_2$
1	1.533116E-01
10	1.110547E-02
20	1.373042E-03
30	1.813615E-04
40	2.440514E-05
48	4.928051E-06

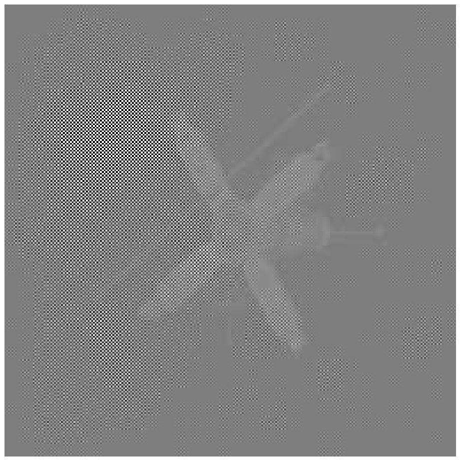
Relative error in $\| \cdot \|_2$.

Reconstruction without regularization

- Our AMG does not have any regularization property.



Blurred image with 2% of noise.



Restored image after 51 iterations.

#(Iter.)	$\|error\|_2$
1	1.174721E+01
10	1.187055E+01
20	1.187349E+01
30	1.187383E+01
40	1.187387E+01
51	1.187387E+01

Relative error in $\|\cdot\|_2$.

- It approximates the solution in all the frequency space and not only in a low frequency subspace: it is disturbed by noise at each iteration.
- At the first level the coarse grid correction projects the problem in the subspace generated by the high frequencies where the noise lives.

Two regularization techniques

- Tikhonov-like regularization:

$$(T_{n,n}(z)T_{n,n}(z) + \mu I)\mathbf{f} = T_{n,n}(z)\mathbf{g},$$

but $T_{n,n}(z)^2$ is not Toeplitz, then instead of the previous system we solve

$$T_{n,n}(z^2 + \mu)\tilde{\mathbf{f}} = T_{n,n}(z)\mathbf{g}.$$

- Riley's regularization:

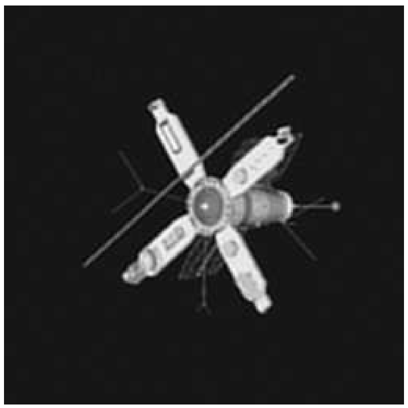
$$(T_{n,n}(z) + \theta I)\mathbf{f} = \mathbf{g},$$

which is equivalent to

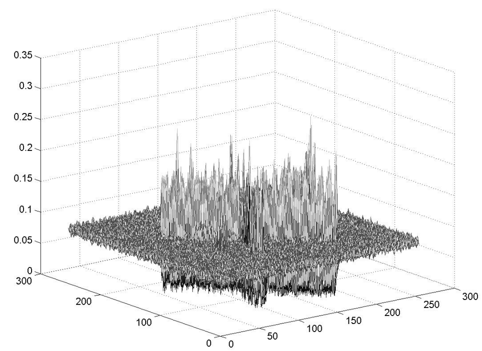
$$\min_{\mathbf{f}\in\mathbb{R}^N} \left\{ \left\| T_{n,n}(z)^{\frac{1}{2}}\mathbf{f} - T_{n,n}(z)^{-\frac{1}{2}}\mathbf{g} \right\|_2^2 + \theta \left\| \mathbf{f} \right\|_2^2 \right\}.$$

Experimental comparison

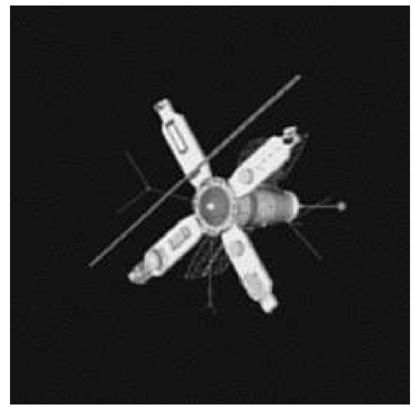
- Tikhonov-like | Riley



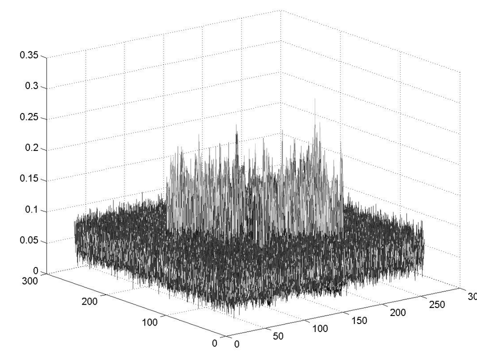
Restored image μ_{opt}.



Error.



Restored image θ_{opt}.



Error.

- Number of iterations for AMG and Geometric Multigrid using the Riley's strategy:

θ	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$p(x) = (2 - 2\cos(x_1))(2 - 2\cos(x_2))$ our projector	7	27	47	51
$p(x) = (2 + 2\cos(x_1))(2 + 2\cos(x_2))$ linear interpolator	6	23	114	817

$\Longrightarrow$ R. Chan, T.Chan et al.
T. Huckle et al.

Conclusion and future work

- Our AMG does not have any regularization property.
- To test our AMG with a PSF generated by a Gaussian function.
- To extend the work to other Boundary Conditions (Periodic, Reflective, Anti-Reflective).
- Will it be possible to define a regularization property or a regularization parameter estimation inside the AMG?