Boundary conditions and multiple-image re-blurring: the LBT case

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Abstract

In this paper we are concerned with deblurring problems in the case of multiple images coming from the Large Binocular Telescope (an important example of telescope of interferometric type). For this problem we are interested in checking the role of the boundary conditions in the quality of the reconstructed image. In particular we will consider reflective and anti-reflective boundary conditions and the re-blurring idea. The results of the proposed combinations are quite satisfactory when compared with classical Dirichlet or periodic boundary conditions, especially when increasing the number of acquired images by the LBT. This behavior is confirmed by a wide numerical experimentation.

Key words: Boundary conditions, multiple-images deconvolution, regularizing iterative methods, matrix algebras and Toeplitz matrices.

1 Introduction

Multiple-images deblurring gives very accurate astronomical image restorations: basically, several blurred images of the same object are acquired in or-
der to get a unique image with uniform and high resolution. Multiple-images deconvolution problems arise from recent interferometric telescopes, such as the Large Binocular Telescope (LBT) [3,7].

LBT is a ground interferometric telescope which consists of two 8.4 m mirrors placed on a common mount, with a spacing of 14.4 m between their centers (1 m = 1 meter). In that way, the interferometric technique of LBT provides images with the same resolution given by a telescope with a single 22.8 m mirror in the direction of the baseline and by a telescope with a single 8.4 m mirror in the orthogonal direction [7]. Since the resolution is much more accurate in the direction of the baseline, several images of the same object must be acquired by using different orientations. To this aim, LBT can rotate all its binocular apparatus with respect to the center of the baseline, thus it provides several different interferometric images which must be processed to obtain a high-resolution representation of the target.

The images of the same target are acquired by LBT in subsequent times. Thus, different sources of noise degrade the image formations, due to the fast changes of the atmospheric turbulence. On these grounds, the multiple-image deblurring problem can be classified as a linear ill-posed problem with data corrupted by several components of noise.

In general, obtaining an accurate model of image blurring requires essentially two main pieces of information:

I) Identification of the blur operator, called a point spread function (PSF), which is related to the continuous infinite dimensional problem and which decides the essential structure of the involved system matrix.

II) Choice of appropriate boundary conditions (BCs), assuming that the observed image is always finite, which have a substantial impact in the precision of the reconstruction especially close to the boundaries of the image (presence of ringing effects).

Concerning the role of the BCs, we notice that the formation of any single blurred image depends on the values of the true object outside the boundaries of the finite domain of the image (this region is as large as one half the support of the PSF). Even in noiseless environments, if the image does not match with the imposed BCs, then the values of the blurred image inside the domain of the picture do not allow one to recover the true original object in the same domain. Since the LBT images may contain, in principle, up to $10^8$ pixels, computational efficiency is crucial, therefore we will take it into account for the choice of the model and of the regularization method.

If the PSF is spatially invariant, as in each LBT acquisition, we have that the matrix which represents the convolution operator has a two-level structure which depends on the imposed BCs. In particular, we consider the follow-
ing four cases: zero Dirichlet BCs [2], periodic BCs [2], reflective (also called Neumann or symmetric) BCs [16], and finally, anti-reflective (AR) BCs [17,9].

In the noise-free case of (classical) single image deblurring, when the considered image is “generic”, we observed that the AR BCs are much more effective than other ones (zero Dirichlet, periodic and reflective) in terms of reconstruction quality (see [17] and especially [9,10]). Of course, if the image is “non-generic” then the situation changes: for instance, the image can show a special axial symmetry (e.g. one “quarter” of a centrally symmetric object with center in one of the 4 corners), it can be periodic (e.g. examples from astronomy), or it can be entirely contained in a constant background (e.g. examples from astronomy); in these non-generic cases, the resulting quality of the other BCs improves and reaches the same level as the AR BCs so that there is no advantage in using the latter approach.

In the case of noise, Tikhonov regularization and the conjugate gradient (CG) algorithm were used [11,10] with optimal choice of the regularization parameter obtained as in [13]. We observed that for high noise levels, as $\text{SNR} \in [1, 5]$, where SNR denotes the signal-to-noise ratio, the behavior of the different BCs becomes similar, because the boundary effects become negligible with respect to the noise contribution and therefore the choice of sophisticated BCs is useless since it cannot change the reconstruction quality. Otherwise, for $\text{SNR} > 5$, the reflective BCs are still noticeably better than the other two classical BCs (periodic and zero Dirichlet), while a negative surprise is that the AR BCs give results that deteriorate rapidly when the SNR decreases: the quality of the reconstruction is slightly better than the periodic and zero Dirichlet BCs reconstruction but it is seriously worse when compared with the reflective BCs. In [11], we have given an explanation of the latter facts and we proposed a basic modification of the classical regularization techniques (which are necessary for handling the effect of the noise), in order to exploit the quality of the AR BCs reconstruction and the efficiency of the related numerical procedures: we called this idea re-blurring. In this way, the AR choice is once again superior among the considered BCs even in presence of a sensible level of noise.

In this paper we focus the discussion on the case of multiple-images. More precisely, we wish to test the role of the BCs and re-blurring techniques for iteratively regularized multiple-image deconvolution by CG iterations. CG is computationally attractive since, for all of the considered BCs, the matrix-vector product can be performed rapidly employing fast discrete transforms. We may anticipate that the combination of the AR BCs and of the re-blurring idea is still successful in the case of multiple images. In particular, many of the conclusions in [10], concerning the single image setting, can be reported verbatim in the case under consideration. Moreover, by increasing the number of the images observed by the LBT, the reflective and AR BCs with re-blurring lead to a noteworthy improvement in the quality of the restored image with
respect to the zero Dirichlet and periodic BCs (we note that the choice of the latter two BCs is quite popular in the astronomy community).

The paper is organized as follows: in Section 2 we define the problem of multiple image deblurring. In Section 3 we briefly discuss the different BCs and the re-blurring idea. In Section 4 we report 2D numerical experiments that confirm the effectiveness of the approach especially when reflective BCs or AR BCs with re-blurring are used. Section 5 is devoted to conclusions and final remarks.

2 Approach to multiple-image deblurring

In order to introduce the multiple-image restoration problem, we extend the classical deblurring problem of (single) blurred and noisy images with space invariant PSF. More precisely, let \( p \) denote the number of acquired images. The continuous mathematical model for the formation of the \( j \)th blurred and noisy image, \( j = 1, \ldots, p \), is described by the following integral equation (see e.g. [2]):

\[
g_j(x, y) = \int_{\mathbb{R}^2} h_j(x - \theta, y - \xi) f_\circ(\theta, \xi) \, d\theta d\xi + b_j(x, y) + \nu_j(x, y), \quad (x, y) \in [0, 1]^2,
\]

which is related to a Fredholm operator of first kind with shift-invariant kernel. Here \( f_\circ \) is the (true) input object, \( h_j \) is the shift-invariant integral kernel of the \( j \)th continuous PSF, \( b_j \) is the background associated with the \( j \)th acquisition, \( \nu_j \) is the noise which arises in the process, and \( g_j \) is the observed \( j \)th image. The values of \( g_j \) are given by the sum of the number of detected photoelectrons due to the radiation of both the convolution \( h_j \ast f_\circ \) and the background \( b_j \), with the addition of the noise \( \nu_j \). Both \( h_j \ast f_\circ \) and \( b_j \) are modelled as realizations of independent Poisson processes (whose sum is also a Poisson process), while \( \nu_j \) is the realization of a Gaussian process with zero mean (white noise) [3].

Given the \( p \) blurred and noisy images \( g_1, g_2, \ldots, g_p \), the multiple-image restoration problem is to recover a suitable approximation of the input object \( f_\circ \). Any previous equation is discretized by rectangle formulae over a uniform grid with step-size \( H \) (not very accurate discretization schemes are required since the object \( f_\circ \) is in general only piecewise regular). As a consequence, after discretization, the \( i \)th equation of the \( j \)th image is given by

\[
\tilde{g}_j(i) = \sum_{s \in \mathbb{Z}^2} f_\circ(s) h_j(i - s) + b_j(i) + \nu_j(i), \quad i \in \mathbb{Z}^2,
\]

where for \( s \in \mathbb{Z}^2 \). Here \( \tilde{g}_j(i) = g_j((i - 1)H) \), \( i = (i_1, i_2) \), \( i - 1 = (i_1 - 1, i_2 - 1) \), \( h_j(i) = h_j((i - 1)H) \), \( b_j(i) = b_j((i - 1)H) \), \( \nu_j(i) = \nu_j((i - 1)H) \), and \( f_\circ(i) \) represents an approximation of \( f_\circ((i - 1)H) \), for \( i \in \mathbb{Z}^2 \). In analogy to the
continuous setting, given \( \mathbf{h}_j, \tilde{\mathbf{g}}_j \) and some statistical knowledge of \( \mathbf{b}_j \) and \( \nu_j \), for \( j = 1, \ldots, p \), the problem is to recover the unknown “true” image \( f_0(s) \) in the window of observation described by \( s \in \{1, \ldots, n\}^2 \). Let \( \mathbf{h}_j \) be the PSF of the \( j \)th image with a support \( m \times m \), \( m \leq n \), for \( j = 1, \ldots, p \), then after a column ordering of the elements, the discretization of (1) gives rise to the following linear equation

\[
\tilde{\mathbf{g}}_j = \tilde{\mathbf{A}}_j f_0 + \mathbf{b}_j + \nu_j,
\]

where the discrete \( j \)th operator \( \tilde{\mathbf{A}}_j \) is an \( n \times (n + m - 1) \) block matrix with \( n \times (n + m - 1) \) matrix blocks formed by \( \mathbf{h}_j \) [3].

Under these notations, the entire multiple-image formation process can be represented as follows. Let \( \tilde{\mathbf{A}} \) be the \( pn^2 \times (n + m - 1)^2 \) global PSF matrix defined as \( \tilde{\mathbf{A}} = (\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2, \ldots, \tilde{\mathbf{A}}_p)^\dagger \), where the symbol \( \dagger \) denotes the transposition with respect to the matrix outer structure only. Similarly, let the \( pn^2 \)-sized vectors \( \tilde{\mathbf{g}} = (\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, \ldots, \tilde{\mathbf{g}}_p)^\dagger \), \( \mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p)^\dagger \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_p)^\dagger \) be respectively the global image vector, the global background vector, and the global error vector. All the equations (2), for \( j = 1, \ldots, p \), can be collected into the following \( pn^2 \times n^2 \) linear system

\[
\tilde{\mathbf{g}} = \tilde{\mathbf{A}} f_0 + \mathbf{b} + \nu.
\]

Since the global background vector \( \mathbf{b} \) can be estimated all over the domain, we consider the data \( \mathbf{g} = \tilde{\mathbf{g}} - \mathbf{b} \) and search for a regularized least-square approximation \( \hat{f} \) of \( f_0 \) which minimizes the discrepancy functional:

\[
\| \tilde{\mathbf{A}} \hat{f} - \mathbf{g} \|^2 = \sum_{j=1}^{p} \| \tilde{\mathbf{A}}_j \hat{f} - \mathbf{g}_j \|^2.
\]

In summary, the minimization of (3) and, more precisely, the computation of the vector \( \hat{f} \) that minimizes (3) represents the discrete characterization of the LBT multiple-image reconstruction problem.

3 Boundary conditions and regularization by re-blurring

We start by observing that (2) is under-determined for all \( j = 1, \ldots, p \) since we have \( n^2 \) equations and \( (n + m - 1)^2 \) unknowns. In order to take care of this problem we can use appropriate BCs which relate the unknowns outside the windows of observation (for which we do not have any information) with the unknowns inside the windows of observation, in this way we obtain \( n^2 \) equations and \( n^2 \) unknowns.
With the imposed BCs, (3) takes the form of a new least-square problem
\[
\|A f - g\|^2_2 = \sum_{j=1}^{p} \|A_j f - g_j\|^2_2,
\]
where \(A_j\) is \(n^2 \times n^2\) and \(f\) is the inner part of \(\tilde{f}\) in the window of observation. Since the de-convolution problems is generally ill-posed (high frequency errors, i.e., components related to the noise, are greatly amplified) independently of the chosen BCs, it is evident that we have to regularize the problem. Two classical methods, i.e., Tikhonov regularization (see [2]) and iterative solvers (CG [13] or Landweber method [2]) for normal equations have been extensively used in the literature. The first one leads to the regularized system
\[
(A^T A + \mu R^T R)f = A^T g,
\]
where \(\mu > 0\) is the regularization parameter and \(R\) is chosen as the identity or as a low order differential operator and it is implemented with the same BCs in order to reduce the computational cost. The iterative solvers are, instead, implemented on
\[
A^T A f = A^T g
\]
with regularization provided by early termination.

Among the different choice of BCs, we consider the following cases which all give rise to special structures of the system matrix \(A\):

**Zero Dirichlet BCs** [2] impose a black boundary, so that the values of \(f_o\) outside the borders of the image \(f\) are all zeros. This implies an artificial discontinuity at the borders which can lead to serious ringing effects. The resulting structure of \(A\) is block Toeplitz with Toeplitz blocks (BTTB) so that the multiplication by a vector can be done in \(O(n^2 \log(n))\) complex operations (see e.g. [8]) by using two-level fast Fourier transforms (FFTs) while the solution of an associated linear system is extremely costly in general (see e.g. [15,18]).

**Periodic BCs** [2] repeat the image endlessly in all directions. The considered choice again can imply an artificial discontinuity at the borders and therefore the related ringing effects are still not negligible in general. The resulting structure of \(A\) is block circulant with circulant blocks (BCCB), so that both the matrix vector product and the solution of a linear system can be achieved by FFTs with \(O(n^2 \log(n))\) complex operations.

**Reflective BCs** [16], also known as Neumann or symmetric BCs, reflect the image like a mirror with respect to the boundaries. The latter preserves the continuity of the image but not the continuity of its normal derivative: as a consequence the ringing effects are sensibly reduced (by one order of magnitude). Moreover the resulting structure of \(A\) is block Toeplitz + Hankel with Toeplitz + Hankel blocks, so that the product is again possible.
by two-level FFTs while the solution of a linear system can be obtained in $O(n^2 \log(n))$ real operations by two-level fast cosine transforms if the PSF is doubly symmetric.

**Anti-reflective BCs** [17] reflect the image with respect to the boundaries by using a central symmetry instead of an axial symmetry which characterizes the reflective case. This procedure preserves the continuity of the image and the continuity of its normal derivative as well: as a consequence the ringing effects are negligible with respect to the other BCs (see [17,11]) in the case of signals (one dimension); when considering images, the improvement, with respect to the reflective BCs, is weaker but still observable (see [17,9,10]). The associated structure of $A$ is more involved since it is block Toeplitz + Hankel with Toeplitz + Hankel blocks plus a structured low rank matrix: despite its apparently complicate structure, the product is again possible by two-level FFTs while the solution of a linear system can be obtained in $O(n^2 \log(n))$ real operations by two-level fast sine transforms if the PSF is doubly symmetric [17,9].

We summarize the main properties of the proposed BCs:

a) when the image has a black border greater than one half the support of the PSF, all the different BCs are equivalent;
b) for all BCs the matrix vector product can be performed in $O(n^2 \log(n))$ using FFT;
c) for the reflective and the AR BCs the solution of (4) can be performed in $O(n^2 \log(n))$ only if the PSF is doubly symmetric.

This last point can generates some problems in the application of the reflective or AR BCs with Tikhonov regularization for the deconvolution of problems arising from the LBT. Indeed the rotation of the binocular leads to PSFs numerically symmetric with respect to the center but not doubly symmetric. Therefore the regularization method used in the numerical experiments is CG with early termination in order to have a fast method to compare all the different BCs.

For the sake of simplicity, in the following remarks, we consider the single image case ($p = 1$ and thus $A = A_1$), but the same holds unchanged for each observation given by $A_j$ and $g_j$, for $j = 1, \ldots, p$. When the observed image is noise free, then there is a substantial gain of the reflective BCs with respect to the periodic or zero Dirichlet BCs and, analogously, there is a significant improvement when the AR BCs are used instead of the reflective BCs (see [17,9]). When using Tikhonov regularization or an iterative solver for normal equations, we observe that the coefficient matrix is a shift of $A^T A$ and that the right hand side is given by $A^T g$. When the PSF is symmetric, the only BCs that are seriously spoiled by this approach are the AR BCs: in more detail, even in presence of moderate noise, its precision becomes worse with
respect to the reflective BCs and only slightly better than the other two BCs (see Table 3.1 in [11]). The reason depends upon the matrix $A^T$ (see [11]): since the PSF is symmetric, concerning the other BCs, the matrix $A^T$ is still a blurring operator ($A^T = A$), while, in the case of the AR BCs, the matrix $A^T$ can not be interpreted as a blurring operator. A (normalized) blurring operator is characterized by nonnegative coefficients such that every row sum is equal to 1: in the case of $A^T$ with AR BCs the row sum of the first and of the last row can be substantially bigger than 1. This means that the new observed image $A^T g$ has artifacts at the borders and this reduces the quality of the reconstruction.

3.1 Computational issues with AR BCs

We start by defining the classes of matrices $\tau_n$, $S_1$, and $S_2$.

Let $Q = Q_n$ be the $n$-by-$n$ orthogonal and symmetric matrix expressed by

$$[Q]_{i,j} = \sqrt{\frac{2}{n+1}} \sin \left( \frac{ji\pi}{n+1} \right), \quad i, j = 1, \ldots, n.$$  \hfill (6)

Then we define $\tau_n$ to be the space (see [4] for algebraic and computational properties) of all the matrices that can be diagonalized by $Q$:

$$\tau_n = \{ QDQ : D \text{ is a real diagonal matrix of size } n \}. \hfill (7)$$

Now, by definition, $M \in S_1$ if

$$M = \begin{bmatrix} \alpha & \hat{M} \v v^T \\ v & w \end{bmatrix}, \hfill (8)$$

with $\alpha, \beta \in \mathbb{R}$, $v, w \in \mathbb{R}^{n-2}$ and $\hat{M} \in \tau_{n-2}$. Moreover, $M \in S_2$ if

$$M = \begin{bmatrix} \alpha \\ v M^* w \\ \beta \end{bmatrix}, \hfill (9)$$

with $\alpha, \beta \in \mathcal{S}_1$, $v, w \in \mathbb{R}^{(n-2)n \times n}$, $v = (v_j)_{j=1}^{n-2}$, $w = (w_j)_{j=1}^{n-2}$, $v_j, w_j \in \mathcal{S}_1$, $j = 1, \ldots, n-2$, and $M^* = (M^*_{i,j})_{i,j=1}^{n-2}$ such that
• $M^*$ has external $\tau_{n-2}$ structure (i.e. it is block diagonalized by $Q_{n-2} \otimes I_n$, $I_n$ being the $n$-sized identity matrix);
• every block $M^*_{i,j}$ of $M^*$ belongs to $S_1$, $i, j = 1, \ldots, n - 2$.

Notice that in the previous block matrices the presence of blanks indicates zeros or block of zeros of appropriate dimensions. Moreover every computation (matrix-vector multiplication, eigenvalue computation, linear system solution) with a matrix in the space $S_2$ has a cost at most proportional to $O(n^2 \log(n))$ real arithmetic operations (with multiplicative constant independent of the external bandwidth of $M^*$ and of the internal bandwidth of each $M^*_{i,j}$, $i, j = 1, \ldots, n - 2$, see [10]).

As shown in [11,10], we have that a matrix $B$ representing a double symmetric blurring operator with AR BCs belong to the class $S_2$. Unfortunately, it is a simple check to conclude that the structure of the matrix $B^T B$ is spoiled and we lose the $O(n^2 \log(n))$ computational cost for solving a generic system with coefficient matrix $B^T B$. The reason of this negative fact is that $B^T B \notin S_2$. More precisely, for $B \in S_2$, we have

$$B^T B = \begin{bmatrix} \alpha & a & \beta \\ b & M & c \\ \gamma & d & \delta \end{bmatrix},$$

where $a = (a_j)_{j=1}^{n-2}$, $b = (b_j)_{j=1}^{n-2}$, $c = (c_j)_{j=1}^{n-2}$, $d = (d_j)_{j=1}^{n-2}$, $M = (\tilde{M}_{i,j})_{i,j=1}^{n-2}$ with $a_j, b_j, c_j, d_j, \tilde{M}_{i,j}, \alpha, \beta, \gamma, \delta$ being expressible as the sum of a matrix belonging to $S_1$ and a matrix of rank 2. Therefore since $\tilde{M}$ has external $\tau_{n-2}$ structure, it follows that $B^T B$ can be written as the sum of a matrix in $S_2$ and a matrix of rank proportional to $n$. The cost of solving such a linear system is proportional to $n^3$ by using the Sherman-Morrison formulae (which, by the way, can be numerically unstable [14]).

In order to overcome the problem due to the multiplication by $A^T$ (which arises only with the most precise AR BCs), if the PSF is centro-symmetric, we replace $A^T$ by $A$ in regularization processes such as CG with early termination or Tikhonov regularization (in this setting if $A \in S_2$, then $A^2 \in S_2$ since $S_2$ is a matrix algebra, see Theorem 2.4, item (iii) in [10]). In the case of a general PSF, the matrix $A^T$ is replaced by $A'$ where the latter is obtained by imposing the AR BCs to a the transposed PSF. More in detail, the new PSF is constructed from the original one by rotating of $180^\circ$, where the latter procedure is the same as using transposition on the infinite dimensional operator and then imposing the AR BCs (see [10]). The above mentioned idea, called re-blurring, is briefly described in the next subsection with reference to the case of a centro-symmetric PSF for which $A' = A$.
3.2 The re-blurring idea: single image case

We start with a simple observation. In the single image case (p = 1), when the blurring operator is symmetric, the associated normal equation can be read as $A^2f = Ag$ in the case of zero-Dirichlet, periodic and reflective BCs (since $A = A^T$). Therefore the observed image $g$ is re-blurred. The re-blurring is the key of the success of the classical regularization techniques such as CG with early termination or Tikhonov regularization: indeed, the noise is also blurred and this makes the contribution of the noise less evident. To make clear the idea, we consider the case of $A$ ill-conditioned and positive definite. Thus, instead of solving (5) we can solve $Af = g$ by CG with early termination, or instead of using the Tikhonov regularization (4) we can consider the Riley approach by taking the solution of $[A + \theta Q]f = g$ with best parameter $\theta$. All the numerical experiments uniformly show that:

- if we set $R = Q = I$, where $I$ is the identity, the Tikhonov solution with best parameter is always better than the Riley solution with best parameter;
- the solution of (5) by CG with best termination is always better than the solution of $Af = g$ by CG with best termination.

The reason of that relies upon the fact that the re-blurring smoothes the noise in the right hand side and this is a straightforward consequence of a well known observation: the null space of the continuous integral operator largely intersects the noisy space so that its inversion amplifies the noise, but, for the very same argument, its direct multiplication shrinks the noise contributions (see [13]).

In conclusion, in [11,10], we proposed to replace the Tikhonov system with

$$
\left[ A^2 + \mu R^2 \right] f = Ag
$$

(10)

to be solved with the right choice of the parameter $\mu$ and to replace the classical normal equations (5) with

$$
A^2f = Ag
$$

(11)

to be solved by CG (or Landweber) with early termination. Notice that when $R$ is the identity and we consider zero Dirichlet, periodic or reflective BCs, if the PSF is symmetric, the new proposal coincides with the classical ones. Therefore the novelty solely concerns the AR BCs. As we will see in the numerical experiments, with this modification, the AR BCs are still very convenient even in presence of noise. Notice that the cost of the solution of a linear system (10) with AR BCs is of the order $n^2 \log(n)$ thanks to the structure of matrix algebra of $S_2$; see Theorem 2.4, items (ii) and (iii) in [10]. Indeed, since $A, R \in S_2$, then both $A^2$ and $R^2$ and hence $A^2 + \mu R^2$ belong to the class $S_2$ de-
fined in Subsection 3.1. Finally we make some computational remarks on the
general case. If the PSF is only symmetric with respect to an axis (not doubly
symmetric) and has support contained in square of size $m \times m$, then we can
exploit the algebra-structure only at one level and therefore the cost of solving
(10) with AR BCs is still $O(n^2 \log(n))$ with multiplicative constant depend-
ing on $m$: we notice that under this assumption the matrix $A$ is not globally
symmetric even with the other three boundary conditions. In the general case
the cost of solving (10) with AR BCs is proportional to $n^3$ with multiplicative
constant depending on $m$. On the other hand, in the case of central symme-
try of the mask (which is equivalent to the symmetry of the matrix $A$ with
Dirichlet BCs), we observe that a rotation of the observed object changes the
PSF into a doubly symmetric one; maybe this trick could be used for reducing
the cost of the Tikhonov approach in the case of reflective and AR BCs with
central symmetric PSFs.

3.3 The re-blurring idea: multiple image case

The re-blurring idea can be generalized in many ways (see [11] for a discus-
sion): here we are interested in the case of multiple images and then now we
discuss how to interpret the re-blurring idea in this context. As previously dis-
cussed, the coefficient matrix $A$ has size $pn^2 \times n^2$. Therefore, while the normal
equations (or the shifted normal equations needed by the Tikhonov approach)
can be formed leading to the matrix $A^T A = \sum_{j=1}^{p} A_j^T A_j$, a pure re-blurring
approach is not possible since the matrix $A^2$ is not well defined. The most
natural direction for interpreting the re-blurring idea in the multiple-image
setting is to consider the matrix

$$A^\dagger A = \sum_{j=1}^{p} A_j^2$$

instead of $A^T A$. In this way, for every single image we are making re-blurring
and we have the right global dimensions to perform the product between
matrices. It will be shown, through numerical tests in the next section, that
this process is effective in smoothing the noise as in the one-image case.

4 Numerical experiments

In this section, we compare and analyze LBT reconstructions generated with
BCs of the previous section.

The true $256 \times 256$ image is a part of Saturn with its rings, developed by CI-
CLOPS and Space Science Institute, Colorado (used with courtesy of NASA/JPL-Caltech/Space Science Institute [1]). To generate the blurred image data, we convolve each PSF with this original image, and then extract the subimage having dimensions 192 × 192 from the center of the large 256 × 256 image. The 256 × 256 full image and the 192 × 192 internal part are shown in Fig. 1. Our aim is to restore this 192 × 192 subimage, from the knowledge of the p blurred and noisy images inside the same domain, by considering different BCs. All computations were done using IDL 5.4 (Interactive Data Language), with floating point precision of $10^{-8}$.

The PSFs are generated by IDL routines developed with the AirY Software Package for LBT multiple-image restoration [6]. In the first row of Fig. 2, three PSFs arising from the LBT interferometric blurring are shown, corresponding to three different orientations of the apparatus. The images in the second row
Figure 3. Logarithm of the 2D Fourier Components of $0^\circ$ PSF and $45^\circ$ PSFs and sum of the four PSFs, for $p = 4$ (the data of each image is scaled)

are the associated blurred images, without noise. Several oscillations appear close to the edge of any blurred image in the same direction of the PSF. The blurred image on the left has vertical oscillations (see the internal rings in the center of the figure), the blurred image in the center has oscillations with $45^\circ$ of slope (see the rings in the upper-right part of the figure), and the blurred image on the right has horizontal oscillations (see the rings on the top or the bottom of the figure).

The Fourier components of the PSFs are shown in Figure 3. In particular, the logarithm of the absolute value of the Fourier widths of the first two point spread functions (the $0^\circ$PSF and $45^\circ$PSF) are reported in the left side and in the center, respectively (the data is scaled to use the full range of gray levels). These pictures are useful for the analysis of the kernel of the related transfer operator (1). We can observe that in each one of these two pictures there is a large subset of the $(u, v)$-plane where the Fourier components vanish (see the dark zones). Basically, this means that the linear subspace of the invisible object is “large”, since the related components of the true image cannot be transferred into the blurred image. On the other hand, the third image of Figure 3 shows the logarithm of the absolute value of the Fourier widths of the global PSF, for $p = 4$. The global PSF is related to the matrix $A$ of (5) and is generated by the sum of the four singular PSFs. As it can be seen, the support of the Fourier Transform of the global PSF is much larger (the vanishing subset of the $(u, v)$-plane in the right is much smaller than the two images on the left). This means that the subspace of the invisible object is now reduced, which gives rise to an improvement of the resolution capabilities of LBT, when a large number of acquisitions is used [7].

After the convolution between the true object and the PSF, the Poisson process of acquisition of both blurred image and background is simulated, and the Gaussian white noise is added by using another appropriate IDL routine of AirY. According to (2), the result of the procedure is the LBT blurred and noisy image related to the considered PSF. One example of a blurred image (without and with noise), and related noise, is shown in Fig. 4. Notice that there is more noise in the region of the image with high values (that is, close the
surface of Saturn and close the rings) than in the empty region in the center of the figure. The latter observation is due to the fact that the numerical model simulates the Poisson process related to the detection of photo-electronic radiations performed by the LBT, whose mean and variance are roughly equal to the intensity of the signal. In this way, in high intensity regions, the distance between the exact, i.e., deterministic blurred image and the real, i.e., stochastic blurred image is higher than in regions of low intensity.

For a comparison of the reconstructions among different BCs, we solve each LBT re-blurred system \( A^\dagger A f = A^\dagger g \) by means of CG without any preconditioning techniques. We remark that \( A^\dagger = A^\tau \) for all BCs except that for the AR BCs since the PSFs are center-symmetric. We consider CG with optimal termination, within 100 iterations. It is worth noticing that CG is widely used for inverse problems since it is an iterative regularization method, which basically means that the first iterations restore components of the data with low noise (i.e., low frequencies components) [13].

We test the different reconstructions corresponding to the following BCs:

- **BC1** = zero Dirichlet,
- **BC2** = Periodic,
- **BC3** = Reflective,
- **BC4** = Anti-Reflective,
- **BC5** = Anti-Reflective-corner.

For the anti-reflection around the corner see [9]. In all tests, we consider the cases of \( p = 2, 4 \) and 8 multiple-images. In addition, we consider several levels of noise with \( \text{SNR} \in [10, +\infty] \), where \( \text{SNR} = +\infty \) indicates no noise.

Some of the best restorations with \( p = 8 \) acquired images are shown in Fig. 5 for SNR=50 and in Fig. 6 for SNR=25. We can observe that, in both the instances, the reflective and the two AR restorations are much better than the zero Dirichlet and periodic ones. Moreover the restorations with AR BCs give the best results: the white Saturn’s surface on the left has lower artifacts even with respect to the reflective case, and the details inside the rings are more...
Figure 5. Best restored images, with different BCs for SNR=50 and $p = 8$.

Figure 6. Best restored images, with different BCs for SNR=25 and $p = 8$.

accurate (in particular, notice in Fig. 5 the dark small zones in the upper and lower borders of restoration with BC3, which are absent with BC4 and BC5, and the better separation between bright and dark rings obtained again by BC4 and BC5).

The values of the minimum relative restoration errors $RRE = \frac{\|f^{(i)} - f_0\|_2}{\|f_0\|_2}$,
Table 1

<table>
<thead>
<tr>
<th>SNR</th>
<th>P</th>
<th>BC1 (Zero)</th>
<th>BC2 (Per.)</th>
<th>BC3 (Refl.)</th>
<th>BC4 (AR)</th>
<th>BC5 (AR-c)</th>
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</thead>
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</table>

Best relative restoration errors within 100 iterations of CG method.

where \( f^{(i)} \) is the restored image of the \( i \)th CG iteration, and the corresponding number \( i \) of iterations are shown in Table 1. This table confirms that the choice of the BCs strongly affects the restoration, since the RREs of the last columns are quite smaller than the RREs of the first columns (for instance, RRE decreases from about 0.17 to 0.07 in the case of SNR=50). In addition, the poor restoration process due to Dirichlet and periodic BCs overcomes the contribution of the noise on the data: in the first two columns of Table 1, the RREs and the number of iterations are approximately constant (RRE about \( 0.17 ÷ 0.18 \)). In the other cases, with reflective and especially anti-reflective BCs, the results are better, and they effectively degrade with respect to increasing levels of noise. Indeed, the reflective and AR BCs give rise to good restoration processes, which are now affected by the level of noise on the data. For instance, with SNR=50, BC3 gives about RRE=0.09, BC4 and BC5 give RRE=0.07, while, with SNR=10, BC3, BC4 and BC5 give about RRE=0.15.

Two examples of the convergence histories for all the considered BCs are shown in Figure 7 (SNR=50 and SNR=25, \( p = 8 \)). These graphs are related to the best restorations of Figure 5 and Figure 6, that is, each minimum value of the graphs correspond to those outputs. We can observe that both BC1 and BC2 give always very poor results, with respect to the other three BCs. Moreover,
AR BCs are better than reflective BCs, especially when the noise is low (here SNR=50). Moreover, graphs of Figure 7 show that the convergence of BC4 and BC5 is quite flat, which turns out to simplify the choice of the iteration where stopping the restoration process.

Now we study how the number \( p \) of acquisitions affects the restoration process (see Fig. 8, 9). The case of several acquisitions \( p = 8 \) does not allow us to obtain restorations with absolute values of RREs much lower than the case of \( p = 2 \), although a visible improvement arises. For instance, Table 1 shows that the restoration from \( p = 2 \) blurred images with SNR=25 gives RRE=0.1125 for the reflective BCs, whereas RRE=0.1056 if \( p = 8 \), in the same setting. In the same table, using the anti-reflective BCs, the case of SNR=50 gives RRE=0.0750 for \( p = 2 \), RRE=0.0701 for \( p = 4 \), and RRE=0.0687 for \( p = 8 \). The gain in the absolute values of RREs is reasonable and indeed the relative improvement is about 7-8% (for instance, in the latter case \((0.0750−0.0687)/0.0750 ≈ 8\%\)). It should be noted that even if these RREs are not so different, the details of the restored images are really better if a large number of acquisition is used. This fact can be observed in Figures 8 and 9, where the restorations with \( p = 2, 4, \) and 8 images are shown, for the periodic, reflective and the AR BCs (here we do not show the restorations with AR-corner BCs, the latter being very similar to the restorations with AR BCs). For instance, in Fig. 8 the details inside the rings of the restorations with 8 blurred images for BC3 and BC4 are better than the corresponding restorations with 2 images. This confirms the relative improvement of 8\%, by considering 8 acquired images instead of 2. Moreover, for \( p = 8 \), the restoration with BC4 is of slightly better quality than the one with BC3. In Fig. 9, the higher noise (SNR=25) degrades the restorations, but again usage of several blurred images improves the accuracy of the details. Indeed, as already mentioned, in this case the kernel of the global PSF becomes very small (see Figure 3). Finally, we stress the poor performances of zero Dirichlet and periodic BCs for which we do not observe any improvement using 8 images instead of 2 as Table 1 clearly shows: refer also to Fig. 8 and Fig. 9 where the different behavior of periodic, reflective,
Figure 8. Best restored images, with different number of acquired blurred images (SNR=50).

and AR BCs is reported.

Since our study involves the application of BCs, we give some results about the role of the BCs when the blurring operator has a large support, by using enlarged versions of the PSFs generated by AirY. In other words, in these final tests, the 128 × 128 central part of each PSF is expanded to 256 × 256 by linear interpolation. In addition, we observe that these tests simulate the case of LBT restoration of half sized images. One example of blurred image (without and with noise) related to an enlarged PSF, with related noise, is shown in Figure 10 (cf. Fig.4). Notice that the blurring effect is higher than the one in the previous case.

In Table 2, for these enlarged PSFs, the same quantities as in Table 1 are reported. The best restorations with $p = 8$ acquired images are shown in Fig.
Figure 9. Best restored images, with different number of acquired blurred images (SNR=25).

Figure 10. Blurred image and noise (SNR=25) for the enlarged PSFs.
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<tr>
<th>SNR</th>
<th>P</th>
<th>BC1 (Zero)</th>
<th>BC2 (Per.)</th>
<th>BC3 (Refl.)</th>
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Table 2
Best relative restoration errors within 100 iterations of CG for the enlarged PSFs.

Figure 11. Best restored images, with different BCs (SNR=25, \( p = 8 \)), for the enlarged PSFs.
The best restorations varying the number of acquired images ($p = 2, 4, 8$) are shown in Fig. 12. These results follow the same trend of the previous one for the smaller PSFs, however we remark that enlarging the support of the PSFs increases the importance of the choice of BCs. Indeed, the gap between each kind of BCs grows with the accuracy of the choice: this fact is evident in Table 2. Furthermore, also the improvement in the quality of the restored images increases in percentage more than with the smaller PSFs when the number $p$ of acquired images increases. For instance, when passing from $p = 2$ to $p = 8$ acquired images and when considering the AR BCs with SNR=10, we obtain a relative gain of about 10% in the case of large PSFs and of about 7% in the case of small PSFs.

Graphs of the point-wise restoration errors $f_o - f^{(i)}$ are shown in Fig. 13, where $p = 2$ and the input data is without noise and the enlarged PSFs are
considered. The restoration errors of Dirichlet and periodic BCs are high all over the domain in the first case and close to the boundaries in the second case. Conversely, the restorations of the reflective and AR BCs are of much better quality (good detection of the details). In particular, the AR BCs give the minimal restoration errors, which are more uniform in magnitude than in all the others cases.

5 Conclusions

In this paper we considered a basic modification (re-blurring) of the normal equation approach (5) to be solved by CG with early termination. By using this modification, the reflective and AR BCs choices are straightforward to implement and still convenient, from the quality reconstruction viewpoint, among the considered BCs even in presence of high levels of noise.

Some more remarks:

- The reflective and the AR BCs yield much better results with respect to zero Dirichlet and periodic BCs; in particular, by increasing the number of images, the reflective and the AR BCs lead to improvements in the quality of the results and especially in the detection of the details. For moderate levels of noise it should be noted that the AR BCs overcome the reflective
BCs but for high levels of noise they are effectively equivalent.

- For $p > 2$ the PSFs have central symmetry but they are not doubly symmetric and therefore the linear systems with reflective and AR BCs are no longer in a matrix algebra related to fast transforms. However the matrix vector product is still possible in $O(n^2 \log(n))$ arithmetic operations due to the Toeplitz plus Hankel structure. Therefore, for the more precise BCs, the iterative regularization is computationally more attractive in comparison with the Tikhonov approach. In this respect, a future work should include the analysis of the GMRES method and of special V-cycle algorithms for which recent results indicate good regularization features (see [5,12]).

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References


