Fast transforms for high order boundary conditions*

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Abstract

We study strategies for increasing the precision in the blurring models by maintaining a complexity in the related numerical linear algebra procedures (matrix-vector product, linear system solution, computation of eigenvalues etc.) of the same order of the celebrated Fast Fourier Transform. The key idea is the choice of a suitable functional basis for representing signals and images. Starting from an analysis of the spectral decomposition of blurring matrices associated to the antireflective boundary conditions introduced in [S. Serra Capizzano, SIAM J. Sci. Comput. 25-3 pp. 1307–1325], we extend the model for preserving polynomials of higher degree and fast computations also in the nonsymmetric case.

We apply the proposed model to Tikhonov regularization with smoothing norms and the generalized cross validation for choosing the regularization parameter. A selection of numerical experiments shows the effectiveness of the proposed techniques.

Key words: matrix algebra and fast transform, Tikhonov regularization, blurring, boundary conditions

AMS Classification (2000):

1 Introduction

We consider the de-convolution problem in the case of signals where the convolution kernel is space invariant. In that case the observed signal $g : \mathcal{I} \rightarrow \mathbb{R}$, $\mathcal{I} \subset \mathbb{R}$ is expressible as

$$g(x) = \int_{\mathbb{R}} k(x-y)f(y) \, dy, \quad (1)$$

where f denotes the true signal. The approximation of the integral operator via an elementary rectangle formula over an equispaced grid with n nodes leads to a linear system with n equations.

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When imposing proper boundary conditions, the related undetermined linear system becomes square and invertible and fast filter algorithms of Tikhonov type can be employed. When talking of fast algorithms, given n the size of the related matrices, we mean an algorithm involving a constant number (independent of n) of fast trigonometric transforms (Fourier, sine, cosine, Hartley transforms) so that the overall cost is given by $O(n \log n)$ arithmetic operations.

For instance, when dealing with periodic boundary conditions, we obtain circulant matrices which are diagonalizable by using the celebrated fast Fourier transform (FFT). Unfortunately such boundary conditions are not always satisfactory from the viewpoint of the reconstruction quality. In fact, if the original signal is not periodic the presence of ringing effects given by the periodic boundary conditions spoils the precision of the reconstruction.

More accurate models are described by the reflective [11] and antireflective [13] boundary conditions, where the continuity of the signal and of its derivative are imposed, respectively. However the fast algorithms are applicable in this context only when symmetric point spread functions (PSFs) are taken into consideration.

The PSF represents the blur of a single pixel in the original signal. Therefore, since it is reasonable to expect that the global light intensity is preserved, the PSF is nothing else than a global mask having nonnegative entries and total sum equal to 1 (conservation law). Often in several applications such a PSF is symmetric and consequently the symbol associated to its mask is an even function.

Usually the antireflective boundary conditions lead to better reconstructions since linear signals are reconstructed exactly, while the periodic boundary conditions approximate badly a linear function by a discontinuous one and the reflective ones by a piece-wise linear function: in both the latter case Gibbs phenomena (called ringing effects) are observed which are especially pronounced for periodic boundary conditions. The evidence of such fact is observed in several papers in the literature, e.g. [1, 3, 4, 6, 12, 13].

Such good behavior of the antireflective boundary conditions comes directly from their definition [13], since the continuity of the first derivative of the signal was automatically imposed. From an algebraic viewpoint, the latter property can be derived from the spectral decomposition of the coefficient matrix in the associated linear system. Indeed, when considering a symmetric PSF and antireflective boundary conditions, the linear system is represented by a matrix whose eigenvalues equal to 1 (the normalization condition of the PSF coming from the conservation law) are associated to an eigenvector basis spanning all linear functions sampled over a uniform grid with n nodes, see [1, 3]. In [1], such a remark has been the starting point for defining and analyzing the antireflective transform and for designing fast algorithms for the spectral filtering of blurred and noisy signals. This algebraic interpretation is useful because it can be used for proposing generalizations that preserve the possibility of defining fast algorithms, while increasing the expected reconstruction quality especially when smooth or piece-wise smooth signals are considered.

In this paper, starting from the previous algebraic interpretation, we define

higher order boundary conditions. This can be obtained by algebraically imposing that the spanning of quadratic or cubic polynomials over a proper uniform gridding are eigenvectors related to the normalized eigenvalue 1. Our proposal improves the antireflective model when the true signal is regular enough close to the boundary. Moreover, an important property of the proposed approach is that it allows to define fast algorithms also in the case of nonsymmetric PSFs (such as the blurring caused by motion). We note that reflective and antireflective boundary conditions can resort to fast transforms only in the case of symmetric boundary conditions, while in the case of nonsymmetric PSF we have fast transforms only for periodic boundary conditions which usually provide poor restorations for nonperiodic signals.

In general, if some information on the low frequencies of the signal to be reconstructed are available, it is sufficient to impose such sampled components as eigenvectors of the blurring operator related to the eigenvalue 1 (we recall that the global spectrum will have 1 as spectral radius). In such a way these component will be maintained exactly by the filtering algorithms since they cut only the spectral components related to small eigenvalues (somehow close to zero) which are presumed to be essentially associated to the noise. In reality, the noise by its random nature of its entries will be decomposable essentially in high frequencies while the true signal is supposed to be approximated in the complementary subspace of low frequencies. Therefore, when applying filtering algorithms, if the blurring operator has non-negligible eigenvalues associated only to low frequencies (for instance low degree polynomials), then the reconstruction of the signal will be reasonably good while the noise will be efficiently reduced.

Given this general context, the present note is aimed to define spectral decomposition of the blurring matrix such that the related transform given by the eigenvectors is fast, the conditioning of the transform is moderate (for such an issue in connection with the antireflective transform see [5]), and the low frequencies are associated only to non-negligible eigenvalues.

The organization of the paper is as follows. Section 2 we introduce the deblurring problem investigating the spectral decomposition of the coefficient matrix for the different kinds of boundary conditions. In Section 3 we define higher order boundary conditions starting from the spectral decomposition of the antireflective matrix. Such transforms are used in Tikhonov-like procedures in Section 4. Section 5 deals with a selection of numerical tests on the deconvolution of blurred and noisy signals and images. In Section 6 the proposals are extended to a multi-dimensional setting. Finally Section 7 is devoted to concluding remarks.

2 Boundary conditions and associated coefficient matrices

In this section we introduce the objects of our analysis and we revisit the spectral decomposition of blurring matrices in the case of periodic, reflective, and antireflective boundary conditions.

Let $\mathbf{f} = (\dots, f_0, f_1, \dots, f_n, f_{n+1}, \dots)^T$ be the true signal and $\{j\}_{j=1}^n$ the set of indexes in the window of observation of the signal. Given a PSF $\mathbf{h} = (h_{-m}, \dots, h_0, \dots, h_m)$, with $2m + 1 \leq n$, we can associate to the PSF the symbol

$$z(t) = \sum_{j=-m}^m h_j e^{ijt}, \quad i = \sqrt{-1}. \quad (2)$$

2.1 Periodic and Reflective boundary conditions

Periodic boundary conditions are defined imposing

$$f_{1-j} = f_{n+1-j} \quad \text{and} \quad f_{n+j} = f_j,$$

for $j = 1, \dots, n$. The blurring matrix associated to periodic boundary conditions is diagonalized by the Fourier matrix

$$F_{ij}^{(n)} = \frac{1}{\sqrt{n}} \exp\left(\frac{-i2\pi(i-1)(j-1)}{n}\right), \quad i, j = 1, \dots, n.$$

More precisely, the blurring matrix is

$$A_P = (F^{(n)})^H \text{diag}(z(\mathbf{x})) F^{(n)}, \quad (3)$$

where $x_i = 2(i-1)\pi/n$, for $i = 1, \dots, n$. We note that the eigenvalues $\lambda_i = z(x_i)$ can be easily computed by $\lambda_i = [F^{(n)}(A_P \mathbf{e}_1)]_i / [F^{(n)} \mathbf{e}_1]_i$, where $\mathbf{e}_1$ is the first vector of the canonical base.

Reflective boundary conditions are defined imposing

$$f_{1-j} = f_j \quad \text{and} \quad f_{n+j} = f_{n+1-j},$$

for $j = 1, \dots, n$. If the PSF is symmetric, i.e., $h_{-j} = h_j$, then the blurring matrix associated to reflective boundary conditions is diagonalized by the cosine transform (see [11])

$$C_{ij}^{(n)} = \sqrt{\frac{2 - \delta_{i,1}}{n}} \cos\left(\frac{(i-1)(2j-1)\pi}{2n}\right), \quad i, j = 1, \dots, n,$$

where $\delta_{i,1} = 1$ if $i = 1$ and zero otherwise. More precisely, the blurring matrix is

$$A_R = (C^{(n)})^T \text{diag}(z(\mathbf{x})) C^{(n)}, \quad (4)$$

where $x_i = (i-1)\pi/n$, for $i = 1, \dots, n$. Like for periodic boundary conditions, the eigenvalues can be easily computed by $\lambda_i = [C^{(n)}(A \mathbf{e}_1)]_i / [C^{(n)} \mathbf{e}_1]_i$.

2.2 Antireflective boundary conditions

Antireflective boundary conditions are defined imposing (see [13])

$$f_{1-j} = 2f_1 - f_{j+1} \quad \text{and} \quad f_{n+j} = 2f_n - f_{n-j},$$

for $j = 1, \dots, n$.

Let Q be the sine transform matrix of order $n - 2$ with entries

$$Q_{ij} = \sqrt{\frac{2}{n-1}} \sin\left(\frac{ij\pi}{n-1}\right), \quad i, j = 1, \dots, n-2.$$

The antireflective transform of order n can be defined by the matrix (see [1])

$$T = \left[\begin{array}{c|c|c} \mathbf{p} & \begin{matrix} \mathbf{0} \\ Q \\ \mathbf{0} \end{matrix} & J\mathbf{p} \end{array} \right], \quad (5)$$

where

$$p_i = \sqrt{\frac{n(2n-1)}{6(n-1)}} \left(1 - \frac{i-1}{n-1}\right),$$

for $i = 1, \dots, n$ and where the permutation matrix J has nontrivial entries $J_{i, n+1-i} = 1$, $i = 1, \dots, n$. We note that $\|\mathbf{p}\|_2 = 1$; moreover J is often called flip matrix.

If the PSF is symmetric and $2m+1 \leq n-2$, the spectral decomposition of the coefficient matrix in the case of antireflective boundary conditions is

$$A_A = T \operatorname{diag}(z(\mathbf{y})) T^{-1}, \quad (6)$$

with $\mathbf{y}$ defined as $y_i = (i-1)\pi/(n-1)$ for $i = 1, \dots, n-1$ and $y_n = 0$. The eigenvalues of A can be computed in $O(n \log n)$ real operations resorting to the discrete sine transform (see [2]).

Concerning the inverse antireflective transform T^{-1} , in [1] we have given its expression and the resulting form is analogous to that of the direct transform T . As a matter of fact, given an algorithm for the direct transform, a procedure for computing the inverse transform needs only to have a fast way for multiply T^{-1} by a vector.

Remark 1 Observe that $\mathcal{S}_l = \operatorname{span}\{\mathbf{p}, J\mathbf{p}\}$ is the subspace spanned by equispaced samplings of linear functions. In that case its linear complement is given by $\mathcal{S}_l^C = \operatorname{span}\{\sin(j\mathbf{x})\}_{j=1}^{n-2}$, with $x_i = (i-1)\pi/(n-1)$, $i = 1, \dots, n$. Unfortunately such a linear complement is not orthogonal and consequently the related transform cannot be unitary, as long as we maintain such a trigonometric basis useful for the fast computations. Up to standard normalization factors this choice leads to the antireflective transform (5).

Remark 2 Implementing filtering methods, like Tikhonov, $\mathcal{S}_l$ is about fully preserved since the associated eigenvalues are $z(0) = 1$.

3 Higher order boundary conditions

Starting from Remarks 1 and 2, we define higher order boundary conditions which represent the main contribution of this work. The approach in Section 2 defines accurate boundary conditions imposing a prescribed regularity to the true signal f . The study of the spectral decomposition of the associated coefficient matrices is a subsequent step for defining fast and stable filtering methods. In this section, we define higher order boundary conditions starting from the eigenspace, i.e., the signal components, that we wish to preserve.

We start by imposing to preserve $\mathcal{S}_l$ and by suggesting other choices for $\mathcal{S}_l^C$. By the way, the request of giving fast algorithms suggests the use of a cosine or exponential basis in place of that of sine functions, both for the direct and inverse transforms. To preserve polynomials of low degree and at the same time to resort to fast trigonometric transforms, we need a transform with a structure analogous to (5). Therefore we need the cosine transform and the Fourier matrix of order $n - 2$. We define $F = F^{(n-2)}$ and $C = C^{(n-2)}$, explicitly

$$C_{ij} = \sqrt{\frac{2 - \delta_{i,1}}{n-2}} \cos\left(\frac{(i-1)(2j-1)\pi}{2n-4}\right)$$

and

$$F_{ij} = \frac{1}{\sqrt{n-2}} \exp\left(\frac{-i2\pi(i-1)(j-1)}{n-2}\right),$$

for $i, j = 1, \dots, n-2$.

We note that the first column of C^T and of F^H are a sampling of the constant function. Hence the span of the columns of C^T or F^H has a nontrivial intersection with $\mathcal{S}_l$. Accordingly, we choose the two vectors for completing these trigonometric basis as a uniform sampling of a quadratic function instead of a linear function. More precisely, instead of $\mathcal{S}_l$ we consider $\mathcal{S}_q = \text{span}\{\mathbf{q}, J\mathbf{q}\}$, where $\mathbf{q}$ is a uniform sampling of a quadratic function in an interval that will be fixed later.

The interval and the sampling grid for the basis functions of our transform are fixed according to the following remark.

Remark 3 *Up to normalization, the j th column of Q is $\sin(j\mathbf{x})$, where $x_i = i\pi/(n-1)$, for $i = 1, \dots, n-2$. Extending the sampling grid such that the j th frequency is extended by continuity, we add the grid points $x_0 = 0$ and $x_{n-1} = \pi$. Since $\sin(jx_0) = \sin(jx_{n-1}) = 0$ for all j , we obtain exactly the two zero vectors in the first and the last row of T , i.e., the $(j+1)$ th column of T is the j th column of Q extended in x_0 and x_n , for $j = 1, \dots, n-2$. The first and the last column of T are the sampling of linear functions at the same equispaced points $x_i \in [0, \pi]$, $i = 0, \dots, n-1$.*

3.1 The case of symmetric PSF

Firstly, we consider a symmetric PSF. In such case we can use the cosine basis. Up to normalization, the $(j+1)$ th column of C^T is $\cos(j\mathbf{x})$, where $x_i = (2i -$

$1)\pi/(2n-4)$, for $i = 1, \dots, n-2$. Extending the grid by continuity, we add $x_0 = -\pi/(2n-4)$ and $x_{n-1} = (2n-3)\pi/(2n-4)$. With this extended grid we can define the basis functions as a n points uniform sampling of the interval

$$[a, b] = \left[-\frac{\pi}{2n-4}, \frac{(2n-3)\pi}{2n-4} \right], \quad (7)$$

where the grid points are

$$x_i = (2i-1)\pi/(2n-4), \quad i = 0, \dots, n-1. \quad (8)$$

We fix $\mathbf{q} = \tilde{\mathbf{q}}/\|\tilde{\mathbf{q}}\|_2$, where $[\tilde{\mathbf{q}}]_{i+1} = (b-x_i)^2$, $i = 0, \dots, n-1$. The fast transform associated to $\mathcal{S}_q$ and C^T can be defined as follows

$$T_C = \left[\begin{array}{c|c|c} \mathbf{q} & \begin{array}{c} \mathbf{c}_a^T \\ C^T \\ \mathbf{c}_b^T \end{array} & J\mathbf{q} \end{array} \right], \quad (9)$$

where $[\mathbf{c}_a]_j = \sqrt{\frac{2-\delta_{j,1}}{n-2}} \cos((j-1)a)$ and $[\mathbf{c}_b]_j = \sqrt{\frac{2-\delta_{j,1}}{n-2}} \cos((j-1)b) = (-1)^{j-1}[\mathbf{c}_a]_j$ since $b = \pi - a$, for $j = 1, \dots, n-2$.

It remains to define the eigenvalues associated to T_C . Since we want to preserve $\mathcal{S}_q$, similarly to what was done for $\mathbf{p}$ in the case of the antireflective boundary conditions, we associate to $\mathbf{q}$ and $J\mathbf{q}$ the eigenvalue $z(0) = 1$. Concerning the other frequencies, since they are defined by the cosine transform, we consider the eigenvalues of the reflective matrix in (4), but of order $n-2$.

In conclusion, for the case of a symmetric PSF, we define a new blurring matrix using the following spectral decomposition

$$A_C = T_C \text{diag}(z(\mathbf{x})) T_C^{-1}, \quad (10)$$

where $x_i = (i-2)\pi/(n-2)$, for $i = 2, \dots, n-1$, and $x_1 = x_n = 0$.

We note that $z(x_1) = z(x_n) = 1$, while the eigenvalues $z(x_i)$, for $i = 2, \dots, n-1$, are the same of A_R of order $n-2$ and hence they can be computed in $O(n \log n)$ by a discrete cosine transform. The product of T_C by a vector can be computed mainly resorting to the inverse discrete cosine transform. The inverse of T_C will be studied in Subsection 3.3, where we will show that the product of T_C^{-1} by a vector can be computed mainly resorting to a discrete cosine transform. Therefore the spectral decomposition (10) can be used to define fast filtering methods in the case of symmetric PSFs. Moreover, we expect an improved restoration with respect to the antireflective model since A_C preserves uniform samplings of quadratic functions while A_R preserves only uniform samplings of linear functions.

3.2 The case of nonsymmetric PSF

In the case of nonsymmetric PSF we can use the exponential basis. Up to normalization, the $(j+1)$ th column of F^H is $\exp(ij\mathbf{x})$, where $i = \sqrt{-1}$ and

$x_i = (i-1)2\pi/(n-2)$, for $i = 1, \dots, n-2$. Extending the grid by continuity, we add $x_0 = -2\pi/(n-2)$ and $x_{n-1} = 2\pi$. With this extended grid, we can define the basis functions as a n points uniform sampling of the interval

$$[a, b] = [-2\pi/(n-2), 2\pi], \quad (11)$$

where the grid points are

$$x_i = (i-1)2\pi/(n-2), \quad i = 0, \dots, n-1. \quad (12)$$

We note that the interval and the grid points in the nonsymmetric case are different with respect to the symmetric case (compare (11) with (7) and (12) with (8)). Therefore, defining $\mathbf{q} = \tilde{\mathbf{q}}/\|\tilde{\mathbf{q}}\|_2$, where $[\tilde{\mathbf{q}}]_{i+1} = (b - x_i)^2$, $i = 0, \dots, n-1$, it is different from which obtained in the symmetric case in the previous subsection. The fast transform associated to $\mathcal{S}_q$ and F^H can be defined as follows

$$T_F = \left[\begin{array}{c|c} \mathbf{q} & \begin{array}{c} \mathbf{c}_a^T \\ F^H \\ \mathbf{c}_b^T \end{array} \end{array} \right] J\mathbf{q}, \quad (13)$$

where now $[\mathbf{c}_a]_j = \exp(i(j-1)a)/\sqrt{n-2}$ and $[\mathbf{c}_b]_j = \exp(i(j-1)b)/\sqrt{n-2} = 1/\sqrt{n-2}$, for $j = 1, \dots, n-2$.

It remains to define the eigenvalues associated to T_F . Similarly to what done for T_C , we associate to $\mathbf{q}$ and $J\mathbf{q}$ the eigenvalue $z(0) = 1$, while for the other frequencies we consider the eigenvalues of the circulant matrix in (3), but of order $n-2$.

Consequently, in the case of a generic PSF, we define a new blurring matrix using the following spectral decomposition

$$A_F = T_F \text{diag}(z(\mathbf{x})) T_F^{-1}, \quad (14)$$

where $x_i = (i-2)2\pi/n$, for $i = 2, \dots, n-1$, and $x_1 = x_n = 0$.

We note that $z(x_1) = z(x_n) = 1$, while the eigenvalues $z(x_i)$, for $i = 2, \dots, n-1$, are the same of A_P of order $n-2$ and hence they can be computed in $O(n \log n)$ by a fast Fourier transform. The product of T_F by a vector can be computed essentially resorting to the inverse fast Fourier transform. The inverse of T_F will be studied together with the inverse of T_C in the next Subsection, where we will show that the product of T_F^{-1} by a vector can be computed by using the fast Fourier transform. Therefore the spectral decomposition (14) can be used to define fast filtering methods also in the case of nonsymmetric PSFs.

3.3 The inverse transform

In this subsection we show that the inverse of T_C and the inverse of T_F are fast transforms. This means that the associated matrix vector product can be performed mainly via a suitable trigonometric transform.

Theorem 1 *Let be given the $n \times n$ matrix*

$$T_X = \left[\begin{array}{c|c|c} \mathbf{q} & \begin{matrix} \mathbf{c}_a^T \\ X^{-1} \\ \mathbf{c}_b^T \end{matrix} & J\mathbf{q} \end{array} \right], \quad (15)$$

where J is the flip matrix and X is a discrete trigonometric transform such that $JXJ = X$. Then $T_X^{-1}\mathbf{y}$ can be computed in $O(n \log(n))$ for all $\mathbf{y} \in \mathbb{C}^n$.

We note that

$$T_X = \tilde{T}_X + [\mathbf{e}_1 | \mathbf{e}_n] \begin{bmatrix} 0 & \mathbf{c}_a^T & 0 \\ 0 & \mathbf{c}_b^T & 0 \end{bmatrix}, \quad (16)$$

where

$$\tilde{T}_X = \left[\begin{array}{c|c|c} \mathbf{q} & \begin{matrix} \mathbf{0}^T \\ X^{-1} \\ \mathbf{0}^T \end{matrix} & J\mathbf{q} \end{array} \right],$$

is easy to invert. Therefore T_X^{-1} can be computed by the Sherman-Morrison-Woodbury formula.

We compute $\tilde{T}_X^{-1}$. Since $\mathbf{q}_n = 0$ the first and the last row can be decoupled and we look for $\tilde{T}_X^{-1}$ of the form

$$\tilde{T}_X^{-1} = \left[\begin{array}{c|c|c} \alpha & \mathbf{0}^T & 0 \\ \mathbf{v} & X & J\mathbf{v} \\ 0 & \mathbf{0}^T & \alpha \end{array} \right],$$

Fixing $\mathbf{q} = [q_1, \hat{\mathbf{q}}^T, 0]^T$, by direct computation $\alpha = 1/q_1$ and $\mathbf{v} = -X\hat{\mathbf{q}}/q_1$. Therefore, $\mathbf{v}$ can be computed in $O(n \log n)$ by a trigonometric transform. For the implementation it can be explicitly computed and inserted into the code.

Given $A \in \mathbb{C}^{n \times n}$ and $U, V \in \mathbb{C}^{n \times k}$, the Sherman-Morrison-Woodbury formula is [9]:

$$(A + UV^H)^{-1} = A^{-1} - A^{-1}U(I + V^H A^{-1}U)^{-1}V^H A^{-1}. \quad (17)$$

It can be very useful for computing the inverse of $A + UV^H$ when $k \ll n$, taking into account the possible instability. Applying the formula (17) to (16) we obtain

$$\begin{aligned} T_X^{-1} &= \tilde{T}_X^{-1} - \tilde{T}_X^{-1}[\mathbf{e}_1 | \mathbf{e}_n] \left(I + \begin{bmatrix} 0 & \mathbf{c}_a^T & 0 \\ 0 & \mathbf{c}_b^T & 0 \end{bmatrix} \tilde{T}_X^{-1}[\mathbf{e}_1 | \mathbf{e}_n] \right)^{-1} \begin{bmatrix} 0 & \mathbf{c}_a^T & 0 \\ 0 & \mathbf{c}_b^T & 0 \end{bmatrix} \tilde{T}_X^{-1} \\ &= \tilde{T}_X^{-1} - \begin{bmatrix} \alpha & 0 \\ \mathbf{v} & J\mathbf{v} \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} 1 + \mathbf{c}_a^T \mathbf{v} & \mathbf{c}_a^T J\mathbf{v} \\ \mathbf{c}_b^T \mathbf{v} & 1 + \mathbf{c}_b^T J\mathbf{v} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c}_a^T \mathbf{v} & \mathbf{c}_a^T X & \mathbf{c}_a^T J\mathbf{v} \\ \mathbf{c}_b^T \mathbf{v} & \mathbf{c}_b^T X & \mathbf{c}_b^T J\mathbf{v} \end{bmatrix}. \end{aligned}$$

We note that $\mathbf{c}_a^T X$ and $\mathbf{c}_b^T X$ can be computed in $O(n \log(n))$ and moreover they can be explicitly computed and inserted into the implementation like done for the vector $\mathbf{v}$. In this way the matrix vector product for T_X^{-1} requires a fast discrete trigonometric transform of $O(n \log(n))$ plus few lower order operations between vectors.

From Theorem 1, it follows that the product of T_C^{-1} and T_F^{-1} , by a vector can be computed in $O(n \log(n))$ and hence they are fast transforms.

4 Tikhonov regularization with fast transforms

We consider the Tikhonov regularization, where the regularized solution is computed as the solution of the following minimization problem

$$\min_{\mathbf{f} \in \mathbb{R}^n} \{ \|\mathbf{g} - A\mathbf{f}\|_2^2 + \mu \|L\mathbf{f}\|_2^2 \}, \quad \mu > 0, \quad (18)$$

where, μ is the regularization parameter that should be properly chosen, $\mathbf{g}$ is the observed signal, A is the coefficient matrix and L is a matrix such that $\text{Null}(A) \cap \text{Null}(L) = 0$ (see [7]). The matrix L is usually the identity matrix or an approximation of partial derivatives.

It is convenient to define L using the same boundary conditions of A in order to obtain fast algorithms. For instance, L equal to the Laplacian with antireflective boundary conditions is

$$L_A = \begin{bmatrix} 0 & \dots & 0 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 & -1 \\ 0 & \dots & 0 \end{bmatrix}. \quad (19)$$

We note that $\dim(\text{Null}(L)) = 2$. However, $\text{Null}(L) = \mathcal{S}_l$ and hence $\text{Null}(A) \cap \text{Null}(L) = 0$. We have $L_A = T_X \text{diag}(s(\mathbf{y})) T_X^{-1}$, for $s(x) = (2 - 2\cos(x))$ and $\mathbf{y}$ defined according to (6) (note that $s(y_1) = s(y_n) = s(0) = 0$).

Using the approach in Section 3, for high order boundary conditions the Laplacian matrix can be defined similarly by

$$\begin{aligned} L_C &= T_C \text{diag}(s(\mathbf{y})) T_C^{-1}, \\ L_F &= T_F \text{diag}(s(\mathbf{x})) T_F^{-1}, \end{aligned}$$

where the grid points $\mathbf{y}$ and $\mathbf{x}$ are defined according to (10) and (14) respectively.

4.1 Tikhonov regularization and reblurring

The minimization problem (18) is equivalent to the normal equations

$$(A^T A + \mu L^T L) \mathbf{f} = A^T \mathbf{g}. \quad (20)$$

Regarding the antireflective algebra, in [6] it was observed that the transposition operation destroys the algebra structure and leads to worse restorations with respect to reflective boundary conditions. To overcome this problem, in [4] the authors proposed the reblurring which replaces the transposition with the correlation operation. Moreover, it was shown that the latter is equivalent to compute the solution of a discrete problem obtained by a proper discretization of a continuous regularized problem. Practically, we replace A^T with A' obtained imposing the same boundary conditions to the coefficient matrix arising from the PSF rotated by 180 degrees. The matrices defined in (5), (10), and (14) can

be denoted by $A_X(z)$, $X \in \{A, C, F\}$ since they are univocally defined by the function z when the transform T_X is fixed. With this notation $A'_X(z) = A_X(\bar{z})$ since the rotation of the PSF exchange h_j with h_{-j} in (2), which corresponds to take $\bar{z}$. Therefore, in the case of periodic boundary conditions $A' = A^T$, but this is not true in general for the other boundary conditions. If the PSF is symmetric then $A' = A$ for every boundary conditions.

In the following we use the reblurring approach and hence we replace (20) with

$$(A'A + \mu L'L)\mathbf{f}_{\text{reg}} = A'\mathbf{g}. \quad (21)$$

If we use the same boundary conditions for A and L , the (21) can be written as $A_X(|z|^2 + \mu|s|^2)\mathbf{f}_{\text{reg}} = A_X(\bar{z})\mathbf{g}$. In [5] it is proved that for antireflective boundary conditions (21) defines a regularization method when $L = I$ and the PSF is symmetric.

If the spectral decomposition of A is $A = T_X D T_X^{-1}$, where $D = \text{diag}(\mathbf{d})$, and $L = T_X \text{diag}(\mathbf{s}) T_X^{-1}$, then the spectral filter solution in (21) is given by

$$\mathbf{f}_{\text{reg}} = T_X \Phi D^{-1} T_X^{-1} \mathbf{g}, \quad \Phi = \text{diag}_{i=1, \dots, n} \left(\frac{|d_i|^2}{|d_i|^2 + \mu|s_i|^2} \right) \quad (22)$$

4.2 GCV for the estimation of the regularization parameter

A largely used method for estimating the regularization parameter μ is the generalized cross validation (GCV) [8]. For the method in (22), GCV determines the regularizing parameter μ that minimized the GCV functional

$$G(\mu) = \frac{\|\mathbf{g} - A\mathbf{f}_{\text{reg}}\|_2^2}{\text{trace}(I - AT_X \Phi D^{-1} T_X^{-1})^2}, \quad (23)$$

where $\mathbf{f}_{\text{reg}}$ is defined in (22). For A_P (periodic boundary conditions) and A_R (reflective boundary conditions with a symmetric PSF), the equation (21) is exactly (20). In this case, in [10] it is proven that

$$G(\mu) = \frac{\sum_{i=1}^n (\sigma_i \hat{g}_i)^2}{(\sum_{i=1}^n \sigma_i)^2}, \quad (24)$$

where $\sigma_i = |s_i|^2 / (|d_i|^2 + \mu|s_i|^2)$ and $\hat{\mathbf{g}} = T_X^{-1} \mathbf{g}$, for T_X^{-1} equal to $F^{(n)}$ and $C^{(n)}$ respectively.

Here we show that, even if for (23) the (24) does not hold in general because T_X can be not unitary, the two expression in (24) reach the minimum at the same point.

Lemma 1 *Let $G(\mu)$ be defined in (23) and $\mu_{\text{GCV}} = \arg \min_{\mu > 0} G(\mu)$, then*

$$\mu_{\text{GCV}} = \arg \min_{\mu > 0} \frac{\sum_{i=1}^n (\sigma_i \hat{g}_i)^2}{(\sum_{i=1}^n \sigma_i)^2},$$

where $\sigma_i = |s_i|^2 / (|d_i|^2 + \mu|s_i|^2)$ and $\hat{\mathbf{g}} = T_X^{-1} \mathbf{g}$.

Figure 1: - - - true signal, — observed signal with Gaussian blur and 0.1% of noise. The vertical lines denote the field of view.

Fixing $c_\mu = \text{trace}(I - AT_X\Phi D^{-1}T_X^{-1})$, we have

$$\begin{aligned}
 \mu_{\text{GCV}} &= \arg \min_{\mu > 0} \frac{\|\mathbf{g} - A\mathbf{f}_{\text{reg}}\|_2^2}{c_\mu^2} \\
 &= \arg \min_{\mu > 0} \left\| \frac{T_X(I - \Phi)T_X^{-1}\mathbf{g}}{c_\mu} \right\|_2^2 \\
 &= \arg \min_{\mu > 0} \left\| \frac{(I - \Phi)\hat{\mathbf{g}}}{c_\mu} \right\|_2^2,
 \end{aligned} \tag{25}$$

where the last equality holds thanks to the invertibility of T_X .

Moreover, it holds

$$I - \phi = \text{diag}_{i=1, \dots, n}(\mu\sigma_i) \tag{26}$$

and

$$c_\mu = \text{trace}(I - \Phi). \tag{27}$$

Finally, replacing (26) and (27) in (25) we obtain the thesis.

5 Numerical experiments

We present some signal deblurring problems. The restorations are obtained with Tikhonov regularization using (22) with smoothing operator L equal to the Laplacian. The code is implemented with Matalab 7.0.

In the first example the observed signal is affected from a Gaussian blur and 0.1% of Gaussian noise. True and observed signals are shown in Figure 1. We consider a low level of noise because in such case the restoration error is mainly affected from the error of the boundary conditions model. Since the PSF is symmetric, we compare our blurring matrix A_C with reflective and antireflective boundary conditions.

In Figure 2 we show the relative restoration errors (RRE) defined as $\|\mathbf{f} - \mathbf{f}_{\text{reg}}\|_2 / \|\mathbf{f}\|_2$, where $\mathbf{f}$ is the true signal. In such figure it is evident that A_C

Figure 2: RRE: — A_C , - - - antireflective, - · - reflective (* denotes values corresponding to μ_{GCV}).

provides restorations with a lower RRE with respect to antireflective boundary conditions, which are already known to be more precise than reflective boundary conditions. Moreover, the RRE curve varying the regularization parameter μ is flatter with respect to the other boundary conditions. This allows a better estimation of the regularization parameter using the GCV. The value μ_{GCV} that gives the minimum of the GCV functional in (24) is reported in Figure 2 with a ‘*’. It is evident that in the case of A_C the RRE obtained with μ_{GCV} is closer to the minimum with respect to antireflective boundary conditions. The minimum RRE is 0.135 for A_C while it is 0.177 for the antireflective boundary conditions. Moreover, for A_C we obtain $\mu_{GCV} = 5.6 \times 10^{-5}$ which gives a RRE equal to 0.135, while for antireflective boundary conditions $\mu_{GCV} = 1.6 \times 10^{-4}$ gives a RRE equal to 0.502.

The quality of the restoration is validated also from the visual evidence of the restored signals. In Figure 3 are shown the restored signal corresponding to μ_{opt} , which is the value of the regularization parameter μ corresponding to the minimum RRE, and to μ_{GCV} . We note that A_C gives a better restoration especially for preserving jumps in the signal. On the other hand this implies a slightly loss of the smoothness of the restored signal. Eventually, using μ_{GCV} our proposal with A_C gives a good enough restoration while this is not true for the antireflective boundary conditions.

The second example is a moving PSF with a 1% of Gaussian noise. True and observed signals are shown in Figure 4. Since the PSF is nonsymmetric we consider A_F instead of A_C . Moreover, since antireflective and reflective boundary conditions lead to matrices that can not be diagonalized by fast transforms, we can compare A_F only with periodic boundary conditions. From Figure 5 and 6 we note that the same considerations done in the previous example hold unchanged. Indeed the minimum RRE is 0.091 for A_F while it is 0.198 for the periodic boundary conditions. Moreover, for A_F we obtain $\mu_{GCV} = 1.7 \times 10^{-3}$ which gives a RRE equal to 0.096, while for periodic boundary conditions $\mu_{GCV} = 2.7 \times 10^{-3}$ gives a RRE equal to 0.704.

Figure 3: Restored signals: — A_C , $\cdots$ antireflective, - - - true signal.

Figure 4: — true signal, - - - observed signal with moving blur and 1% of noise. The vertical lines denote the field of view.

Figure 5: RRE: — A_F , - - - periodic (* denotes values corresponding to μ_{GCV}).

Figure 6: Restored signals: — A_F , $\cdots$ periodic, - - - true signal.

6 The multidimensional case

The multidimensional transform can be defined by tensor product. Thus

$$T_X^{(d)} = T_{X,\mathbf{n}}^{(d)} = T_{X,n_1} \otimes \cdots \otimes T_{X,n_d}$$

d times, where $\mathbf{n} = (n_1, \dots, n_d)$ and $T_{X,m}$ is the transform T_X of order m . For a 2D array of size $n \times m$, this is easily implemented doing m 1D transforms of size n for each column and then n 1D transforms of size m for each row.

The computation of the eigenvalues is more involved. The strategy is the same described in [2] for computing the eigenvalues of antireflective matrices. The algorithm in Section 3.2.1 in [2] can be applied to our proposal replacing the discrete sine transform by the cosine or the Fourier transform. More in details, in the 2D case:

1. Compute two 1D PSF summing the rows and the columns of the 2D PSF.
2. Apply two 1D transforms T_X for computing the eigenvalues that correspond to the frequencies indexed as the edges of the image (the vertical edges are associated to the PSF obtained summing the columns and the horizontal to the other PSF).
3. Apply a 2D cosine or Fourier transform for computing the eigenvalues indexed as the inner part of the image.

6.1 Image deblurring

We consider the deblurring problem with an out of focus blur and 0.1% of Gaussian noise. The true and the observed images are shown in Figure 7. The restored images are obtained with the smoothing operator $L = I$.

We note that A_C gives a better restoration with respect to antireflective and reflective boundary conditions (see Figure 8). In Table 1 is shown the RRE for μ_{opt} and μ_{GCV} , while in Figures 9 and 10 we have the restored images for the considered boundary conditions and the two choices of μ .

Figure 7: True image (left) and observed image with out of focus blurring and 0.1% of Gaussian noise (right)

Figure 8: RRE: — A_C , - - - antireflective, - · - reflective (* denotes values corresponding to μ_{GCV}).

	μ_{opt}	μ_{GCV}
reflective	0.0647	0.0723
antireflective	0.0570	0.0602
A_C	0.0564	0.0579

Table 1: RRE for the restoration of the observed image in Figure 7.

reflective

antireflective

A_C

Figure 9: Restored images for μ_{opt}

reflective

antireflective

A_C

Figure 10: Restored images for μ_{GCV} .

Figure 11: On the left the observed image with a nonsymmetric PSF. On the right the RRE for — A_F and - - - periodic boundary conditions (* denotes values corresponding to μ_{GCV}).

periodic with μ_{opt}

A_F with μ_{opt}

A_F with μ_{GCV}

Figure 12: Restored images for the observed image in Figure 11.

Even if there is not a large reduction of the RRE, the images restored with A_C show lesser ringing effects with respect to the antireflective boundary conditions at least in the south-west corner of the image.

For a general image the use of A_C instead of antireflective boundary conditions leads to negligible improvement if the image is not smooth enough at the boundary or if the noise level is so high to dominate the approximation error in the restoration.

For concluding, we consider a nonsymmetric PSF. The observed image in Figure 11 is affected from an out of focus combined with a moving blur. Since the PSF is nonsymmetric we compare A_F with periodic boundary conditions like in Section 5. In Figure 11 the RRE for A_F is significantly lower than the RRE of periodic boundary conditions. Indeed, in Figure 12 it is possible to note that in the case of periodic boundary conditions, the ringing effects at the edges (in the direction of the motion) damage completely the restoration also for μ_{opt} . Moreover, the GCV give a good estimation of the regularization parameter only in the case of A_F as it is evident in the plot of Figure 11.

7 Conclusions

In Section 3 we have given a framework to construct precise models for deconvolution problems using fast trigonometric transforms. The same idea could be applied to different problems having a shift invariant kernel. Indeed, if we have information on the signal to restore, the space $\mathcal{S}_l$ can be replaced with other functions that we want to preserve. Moreover, it could be possible to achieve also higher order, even if the numerical results show that for images deblurring problems this approach does not give substantial improvements.

The introduced fast transforms was applied in connection with Tikhonov regularization and the reblurring approach. However, they could be useful also for more sophisticated regularization methods like Total Variation for instance.

The analysis of the Tikhonov regularization in Section 4 is useful also for the antireflective boundary conditions. Indeed, it was not previously considered in the literature the case of $L \neq I$ and the choice of the regularization parameter μ using the GCV.

Since the proposed transforms are not orthogonal, they was applied in connection with the reblurring approach, but the theoretical analysis of the regularizing properties of such approach exist only in the case of antireflective boundary conditions and symmetric kernel (see [5]). Therefore, a more detailed analysis, especially in the multidimensional case with a nonsymmetric kernel, should be done in the future.

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