Robust and optimal multi-iterative techniques for IgA collocation linear systems

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Abstract

We consider fast solvers for the large linear systems coming from the Isogeometric Analysis (IgA) collocation approximation based on B-splines of full elliptic $d$-dimensional Partial Differential Equations (PDEs). We are interested in designing iterative algorithms which are optimal and robust. The former property implies that the computational cost is linear with respect to the number of degrees of freedom (i.e. the matrix size). The latter property means that the convergence rate is completely independent of (or only mildly dependent on) all the relevant parameters: in our setting, we can mention the coefficients of the PDE, the dimensionality $d$, the geometric map $G$ describing the physical domain, the matrix size (related to the fineness parameters), and the spline degrees (associated with the IgA approximation order). Our approach is based on the spectral symbol, which describes the global eigenvalue behavior of the IgA collocation matrices. It is precisely the spectral information contained in the symbol that is exploited for the design of an optimal and robust multi-iterative solver of multigrid type. Several numerical experiments are presented and discussed in view of recent theoretical findings concerning the symbol and its properties.

Keywords: Isogeometric analysis; B-splines; Collocation method; Toeplitz matrices; Symbol; P-GMRES, multigrid, and multi-iterative methods

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1. Introduction

We consider the Isogeometric Analysis (IgA) collocation approximation based on B-splines of a full elliptic Partial Differential Equation (PDE) with non-constant coefficients and homogeneous Dirichlet boundary conditions:

\[
\begin{cases}
-\nabla \cdot K \nabla u + \alpha \cdot \nabla u + \gamma u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega$ is a bounded open domain in $\mathbb{R}^d$, $K : \overline{\Omega} \to \mathbb{R}^{d \times d}$ is a Symmetric Positive Definite (SPD) matrix of functions belonging to $C^1(\Omega) \cap C(\overline{\Omega})$, $\alpha : \overline{\Omega} \to \mathbb{R}^d$ is a vector of functions in $C(\overline{\Omega})$, $\gamma, f \in C(\overline{\Omega})$ and $\gamma \geq 0$.

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IgA is a well-established paradigm for the analysis of problems governed by PDEs [6]. Its goal is to improve the connection between numerical simulation and Computer Aided Design (CAD) systems. The main idea in IgA is to use directly the geometry provided by CAD systems and to approximate the unknown solutions of differential equations by the same type of functions. Tensor-product B-splines and their rational extension, the so-called NURBS, are the dominant technology in CAD systems used in engineering, and thus also in IgA. The Galerkin formulation has been intensively employed in this context. However, the efficiency of the Galerkin method deeply depends on the numerical quadrature rules required in the assembly of the corresponding linear systems. In contrast with the finite element context, where elementwise Gauss quadrature is known to be optimal, it is not yet completely understood how to construct efficient IgA oriented quadrature rules; see e.g. [12]. The quadrature issue motivated the use of collocation methods in IgA; see [2] and Section 2.1 for a brief description. The extremely easy and fast construction of their linear systems makes the isogeometric collocation methods very attractive and competitive, mainly for large degrees [15].

In this paper we are interested in the design of fast iterative solvers for the linear systems coming from the isogeometric collocation approximation of (1.1). The intrinsic lack of symmetry of collocation linear systems constitutes an additional difficulty with respect to the Galerkin context. Our approach is based on the spectral information of the corresponding matrices and, in fact, exploits recent results on the asymptotic spectral distribution given in [8]. In several contexts, such information turned out to be useful in the convergence analysis of (preconditioned) Krylov methods (see [3, 13] and references therein) and in the design of effective preconditioners and multigrid/multi-iterative solvers (see [1, 7, 20]). In [8], the problem (1.1) has been addressed in its full generality and it has been proved that:

1. a spectral distribution for the B-spline isogeometric collocation matrices exists and is compactly described by a symbol \( f \);
2. the symbol \( f \) has a canonical structure incorporating:
   (a) the approximation technique, identified by a finite set of polynomials in the Fourier variables \( \theta := (\theta_1, \ldots, \theta_d) \in [-\pi, \pi]^d \);
   (b) the geometry, identified by the map \( G \) in the parametric variables \( \hat{x} := (\hat{x}_1, \ldots, \hat{x}_d) \), defined on the parametric domain \( \hat{\Omega} := [0, 1]^d \);
   (c) the coefficients of the principal symbol of the PDE in (1.1), namely \( K \), in the physical variables \( x := (x_1, \ldots, x_d) \) defined on the physical domain \( \Omega \).

Most of the analytic features of the symbol derived in [8] are similar to the ones known in the Finite Difference setting [18], where this kind of analysis was originally performed. In particular, if we denote by \( p := (p_1, \ldots, p_d) \) the vector such that \( p_j \) is the spline degree in the \( \hat{x}_j \)-direction, then \( f \) is a nonnegative trigonometric polynomial in the \( \theta \)-variables, with degree \( p_j \) in the variable \( \theta_j \), and with a unique zero at \( \theta = \hat{\mathbf{0}} \) of order two. However, a surprising behavior occurs when one of the \( p \)-parameters, say \( p_j \), becomes large. In this case, the symbol \( f \) possesses infinitely many numerical zeros at the points \( \theta \) such that \( \theta_j = \pi \), and the convergence to zero is monotonic and exponential with respect to \( p_j \). This implies that small eigenvalues of the IgA collocation matrices appear related to high frequency eigenvectors, and this non-canonical source of ill-conditioning is responsible for a significant slowdown of all standard multigrid and preconditioning techniques. On the other hand, the latter information can be exploited for designing ad hoc algorithms with a convergence speed independent of the fineness parameters and of the approximation degrees \( p \).

Such an approach has been successfully applied in [7] in the context of Galerkin B-spline IgA, by exploiting the asymptotic spectral analysis carried out in [10] for the simplified case where \( K \) is the identity matrix and \( \Omega = (0, 1)^d \).

We propose the following two-step strategy for solving the linear systems coming from the B-spline IgA collocation approximation of (1.1):
1. As external solver, we use a Preconditioned Generalized Minimum Residual (P-GMRES) method; see e.g. [14]. The preconditioner is chosen as the so-called Parametric Laplacian matrix (or PL-matrix), that is the matrix coming from the isogeometric collocation approximation of (1.1) in the case where $K$ is the identity matrix, $\alpha = 0$, $\gamma = 0$ and $G$ is the identity map on the parametric domain $\hat{\Omega}$. If the map $G$ is singular (i.e., the determinant of its Jacobian matrix has zeros), then we need to enrich the proposal, and the preconditioner takes the form of $D$ times the PL-matrix times $D^2$, where $D^2$ is the diagonal matrix whose diagonal elements coincide with those of the original coefficient matrix.

2. The PL-matrix itself is treated by a specific multi-iterative solver of multigrid type, in which the different basic iterations are designed by using, in a heuristic way, the spectral information contained in the symbol $f$. More precisely, the multi-iterative solver consists of a V-cycle (or W-cycle) multigrid procedure formed by:
   (a) a standard full-weighting restriction operator at each level, chosen so as to reduce the error in the low frequencies (a subspace of ill-conditioning due to the zero of the symbol at $\theta = 0$);
   (b) a few smoothing iterations of a certain P-GMRES at the finest level, designed for reducing the error in the high frequencies.

We stress that the choice of the canonical full-weighting restriction combined with a standard smoother, like Gauss-Seidel, induces a multigrid method for the PL-matrix which is optimal, but not robust, i.e., the convergence speed is independent of the matrix size but deteriorates when the $p$-parameters increase. On the other hand, the P-GMRES that we propose to use as a smoother at the finest level shows a $p$-independent convergence rate, which, however, rapidly worsens as the matrix size increases. According to the multi-iterative idea [16], the combination of these two techniques in the V-cycle (or W-cycle) multigrid procedure described above leads to a method which is optimal and robust at the same time, its convergence rate being substantially independent of both the matrix size and the $p$-parameters. Finally, the external P-GMRES for the general IgA collocation matrix coming from the approximation of the full elliptic problem (1.1) leads to a method which is again independent of the matrix size and $p$, and also of the other relevant parameters such as the coefficients $K, \alpha, \gamma$ of the PDE, the dimensionality $d$, and the geometry map $G$ describing the physical domain.

A completely similar two-step strategy has been effectively applied in the IgA Galerkin context [7]. Its extension to the collocation linear systems is not trivial due to the structural differences of the involved matrices (lack of symmetry). However, it is strongly motivated and steadily driven by the analogies between the spectral distributions of the matrices in the two cases. On this concern we remark that, in contrast with the B-spline IgA Galerkin case, the symbol in the B-spline IgA collocation case is known for the problem (1.1) in its full generality [8]. This allows a theoretical understanding of the very good performance of the strategy we are proposing here.

The paper is organized as follows. In Section 2 we describe the isogeometric collocation approach for approximating our model problem (1.1), we discuss the notion of spectral distribution, and we give the expression of the spectral symbol associated with the IgA collocation matrices. Section 3 deals with the external P-GMRES and shows that a fast (optimal and robust) solver for the general collocation matrix coming from the discretization of (1.1) is obtained once we have a fast solver for the PL-matrix. Section 4 is devoted to the description of the multi-iterative solver of multigrid type for the PL-matrix and contains several numerical experiments demonstrating its optimality and robustness. We end in Section 5 with some concluding remarks.

2. Setting and tools

This section is divided into three parts. The first part describes the problem setting and the IgA collocation method for approximating (1.1); it also reports the expression of the resulting
collocation matrices. The second part introduces the notion of spectral distribution and spectral symbol for a sequence of matrices with increasing dimension. Finally, the third part deals with the spectral symbols corresponding to our IgA collocation matrices.

2.1. Problem setting, IgA collocation methods and matrices

The problem (1.1) can be reformulated as follows:

\[
\begin{cases}
-1(K\circ Hu)1^T + \beta \cdot \nabla u + \gamma u = f, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \(1 := (1, \ldots, 1)\), \(Hu\) denotes the Hessian of \(u\), i.e.

\[(Hu)_{ij} := \frac{\partial^2 u}{\partial x_i \partial x_j},\]

and \(\circ\) denotes the componentwise Hadamard matrix product [4]. Moreover, \(\beta\) collects the coefficients of the first order derivatives in (1.1), namely

\[\beta_j := \alpha_j - d \sum_{i=1}^{d} \frac{\partial \kappa_{ij}}{\partial x_i},\]

with \(\kappa_{ij}\) the entries of the matrix \(K := [\kappa_{ij}]_{i,j=1}^{d}\).

We consider the approximation of the solution of the problem (2.1) by the standard collocation approach, as explained briefly in the following. Let \(W\) be a finite dimensional vector space of sufficiently smooth functions defined on \(\Omega\) and vanishing on the boundary \(\partial \Omega\). We call \(W\) the approximation space. Then, we introduce a set of \(N := \dim W\) collocation points in \(\Omega\),

\[\{\tau_i \in \Omega, \ i = 1, \ldots, N\},\]

and we look for a function \(u_W \in W\) such that

\[-1(K(\tau_i) \circ Hu_W(\tau_i))1^T + \beta(\tau_i) \cdot \nabla u_W(\tau_i) + \gamma(\tau_i)u_W(\tau_i) = f(\tau_i), \ \forall \tau_i.\]

If we fix a basis \(\{\varphi_1, \ldots, \varphi_N\}\) for \(W\), then the collocation problem (2.2) is equivalent to solving the linear system

\[Au = b,\]

where

\[A := [-1(K(\tau_i) \circ H\varphi_j(\tau_i))1^T + \beta(\tau_i) \cdot \nabla \varphi_j(\tau_i) + \gamma(\tau_i)\varphi_j(\tau_i)]_{i,j=1}^{N} \in \mathbb{R}^{N \times N}\]

is the collocation matrix and \(b := [f(\tau_i)]_{i=1}^{N}\). Once we find \(u := [u_1 \cdots u_N]^T\), we know \(u_W = \sum_{j=1}^{N} u_j \varphi_j\).

Let us now describe the isogeometric collocation approach. Let

\[\{\hat{\varphi}_1, \ldots, \hat{\varphi}_{N+N_b}\}\]

be a set of basis functions defined on the parametric domain \(\hat{\Omega} := [0, 1]^d\), and assume that the physical domain \(\Omega\) in (2.1) can be described by a global geometry function \(G\) expressed in terms of the functions \(\hat{\varphi}_i\) as follows:

\[G : \hat{\Omega} \to \Omega, \ G(\hat{x}) := \sum_{i=1}^{N+N_b} \hat{\varphi}_i(\hat{x}) p_i, \ p_i \in \mathbb{R}^d.\]
We assume that the map $G$ is invertible in $\hat{\Omega}$ and $G(\partial\hat{\Omega}) = \partial\Omega$. If $\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\}$ is defined as the subset of the functions in (2.5) which vanish on the boundary $\partial\hat{\Omega}$, then the approximation space $W$ is defined as the vector space spanned by

$$\varphi_i(x) := \hat{\varphi}_i(G^{-1}(x)) = \hat{\varphi}_i(\hat{x}), \quad i = 1, \ldots, N, \quad x = G(\hat{x}). \quad (2.7)$$

Moreover, we introduce a set of collocation points in the parametric domain $\hat{\Omega}$,

$$\{\hat{\tau}_i \in \hat{\Omega}, \quad i = 1, \ldots, N\}, \quad (2.8)$$

and we define the collocation points in the physical domain $\Omega$ as follows:

$$\tau_i := G(\hat{\tau}_i), \quad i = 1, \ldots, N. \quad (2.9)$$

In the isogeometric collocation approach, we solve the linear system (2.3) with the basis functions and the collocation points given by (2.7) and (2.9), respectively. In the most common formulation of IgA, the functions $\hat{\varphi}_i$ in (2.5) are tensor-product B-splines or NURBS. Nonetheless, other kinds of functions can be used as well.

In this paper, the role of the functions $\hat{\varphi}_i$ in (2.5) will be played by tensor-product B-splines over uniform knot sequences. In addition, we do not confine ourselves to the isoparametric approach, since the geometry map $G$ will be allowed to be any function from $\hat{\Omega}$ to $\Omega$, not necessarily expressed in terms of the basis (2.5) as in (2.6). For the choice of the collocation points, which is crucial for the stability and good behavior of the discrete problem, we follow [2]: our collocation points $\hat{\tau}_i$ in (2.8) are chosen as the Greville abscissae corresponding to the used B-splines.

We now provide the explicit construction of our basis functions $\hat{\varphi}_i$ and of our collocation points $\hat{\tau}_i$. For $p, n \geq 2$, let us consider the uniform knot sequence

$$t_1 = \cdots = t_{p+1} = 0 < t_{p+2} < \cdots < t_{p+n} < 1 = t_{p+n+1} = \cdots = t_{2p+n+1},$$

where

$$t_{p+i+1} := \frac{i}{n}, \quad i = 0, \ldots, n.$$

The uniform B-splines $N_{i,[p]} : [0, 1] \to \mathbb{R}$ of degree $p$ on this knot sequence are defined as follows: for $0 \leq k \leq p$ and $1 \leq i \leq (n + p) + p - k$,

$$N_{i,[p]}(x) := \begin{cases} 1, & \text{if } x \in [t_i, t_{i+1}), \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$N_{i,[p]}(x) := \frac{x - t_i}{t_{i+k} - t_i} N_{i,[k-1]}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} N_{i+[k-1]}(x), \quad k \geq 1,$$

where we assume that a fraction with zero denominator is zero. Then, we know that (see [5]) the functions $N_{i,[p]}, \ldots, N_{n+p,[p]}$ form a basis for the spline space

$$\{s \in C^{p-1}([0, 1]) : s|_{[\xi_i, \xi_{i+1}]} \in \mathbb{P}_p, \quad \forall i = 0, \ldots, n - 1\},$$

where $\mathbb{P}_p$ is the space of polynomials of degree less than or equal to $p$, and moreover

$$N_{i,[p]}(0) = N_{i,[p]}(1) = 0, \quad i = 2, \ldots, n + p - 1.$$

The Greville abscissae corresponding to the B-splines $N_{i,[p]}$, $i = 2, \ldots, n + p - 1$, are defined as

$$\xi_{i,[p]} := \frac{t_{i+1} + t_{i+2} + \cdots + t_{i+p}}{p}, \quad i = 2, \ldots, n + p - 1.$$
In higher dimensions \( d > 1 \), we consider tensor-product functions \( y_1 \otimes \cdots \otimes y_d \), which are constructed as
\[
(y_1 \otimes \cdots \otimes y_d)(x_1, \ldots, x_d) := y_1(x_1) \cdots y_d(x_d).
\]
Let \( p := (p_1, \ldots, p_d) \) and \( n := (n_1, \ldots, n_d) \in \mathbb{N}^d \), with \( p_i, n_i \geq 2 \) for all \( i = 1, \ldots, d \). The tensor-product B-splines \( N_{i,[p]} : \tilde{\Omega} \to \mathbb{R} \) and their Greville abscissae \( \xi_{i,[p]} \) are given by
\[
\begin{align*}
N_{i,[p]} &:= N_{i_1,[p_1]} \otimes N_{i_2,[p_2]} \otimes \cdots \otimes N_{i_d,[p_d]}, \\
\xi_{i,[p]} &:= (\xi_{i_1,[p_1]}, \xi_{i_2,[p_2]}, \ldots, \xi_{i_d,[p_d]}),
\end{align*}
\]
where \( 2 := (2, \ldots, 2) \) and \( j \leq i \leq k \) means that \( j_i \leq k_i \) for all \( r = 1, \ldots, d \).

In the framework of IgA based on B-splines, the functions \( \hat{\phi}_i \), \( i = 1, \ldots, N \), in (2.5) are chosen as the tensor-product B-splines in (2.11) and the collocation points \( \hat{\tau}_i \), \( i = 1, \ldots, N \), in (2.8) are chosen as the Greville abscissae in (2.12). In this setting, \( N = \prod_{k=1}^d (n_k + p_k - 2) \). Moreover, we adopt for the tensor-product B-splines (2.11) and for the associated Greville abscissae (2.12) the same ordering considered in [8] (see also [7, 10]), which is obtained by varying the multi-index \( i \) from \( 2 \) to \( n + p - 1 \) according to the following rule:
\[
\begin{bmatrix}
\cdots & [(i_1, i_2, \ldots, i_d)]_{i_1 = 2, \ldots, n_1 + p_1 - 1} & \cdots & [i_d = 2, \ldots, n_d + p_d - 1] & \cdots
\end{bmatrix}
\]
\begin{equation}
\tag{2.13}
\end{equation}

For instance, if \( d = 2 \), this ordering is
\[
(2), (3, 2), \ldots, (m_1, 2), (2, 3), (3, 3), \ldots, (m_1, 3), \ldots, (2, m_2), (3, m_2), \ldots, (m_1, m_2),
\]
with \( m_1 := n_1 + p_1 - 1 \) and \( m_2 := n_2 + p_2 - 1 \). The ordering (2.13) is followed when assembling the collocation matrix (2.4). This matrix can be compactly written as a linear combination of tensor-products of matrices arising in the univariate setting. We recall that, for every \( X \in \mathbb{C}^{m_1 \times m_1} \) and \( Y \in \mathbb{C}^{m_2 \times m_2} \), the (Kronecker) tensor-product \( X \otimes Y \) is the matrix in \( \mathbb{C}^{m_1 m_2 \times m_1 m_2} \) given by:
\[
X \otimes Y = \begin{bmatrix}
x_{11}Y & x_{12}Y & \cdots & x_{1m_1}Y \\
x_{21}Y & x_{22}Y & \cdots & x_{2m_1}Y \\
\vdots & \vdots & & \vdots \\
x_{m_11}Y & x_{m_12}Y & \cdots & x_{m_1 m_1}Y
\end{bmatrix}.
\]
\begin{equation}
\tag{2.14}
\end{equation}

Without loss of clarity, we use the same symbol \( \otimes \) for both tensor-products of matrices and of functions (see (2.10) and (2.14)).

When \( G \) is the identity map (and so \( \tilde{\Omega} = \tilde{\Omega} \)), the collocation matrix (2.4) resulting from the above choices of \( \hat{\phi}_i \) and \( \hat{\tau}_i \) is given by
\[
A_{[p]}^{n} := \sum_{r=1}^{d} n_r^2 D_{[n]}^{[p]}(\kappa_{r}) (M_{n_d}^{[p]} \otimes \cdots \otimes M_{n_{r+1}}^{[p]} \otimes K_{n_r}^{[p]} \otimes M_{n_{r+1}}^{[p]} \otimes \cdots \otimes M_{n_1}^{[p]})
\]
\[
- \sum_{r,s=1}^{d} n_r n_s D_{[n]}^{[p]}(\kappa_{rs} + \kappa_{sr}) (M_{n_d}^{[p]} \otimes \cdots \otimes M_{n_{r+1}}^{[p]} \otimes H_{n_s}^{[p]} \otimes M_{n_{r+1}}^{[p]} \otimes \cdots \otimes M_{n_1}^{[p]})
\]
\[
\otimes M_{n_{r+1}}^{[p]} \otimes H_{n_r}^{[p]} \otimes M_{n_{r+1}}^{[p]} \otimes \cdots \otimes M_{n_1}^{[p]})
\]
\[
+ \sum_{r=1}^{d} n_r D_{[n]}^{[p]}(\beta_{r}) (M_{n_d}^{[p]} \otimes \cdots \otimes M_{n_{r+1}}^{[p]} \otimes H_{n_r}^{[p]} \otimes M_{n_{r+1}}^{[p]} \otimes \cdots \otimes M_{n_1}^{[p]})
\]
\[
+ D_{[n]}^{[p]}(\gamma) (M_{n_d}^{[p]} \otimes \cdots \otimes M_{n_1}^{[p]}),
\]
\begin{equation}
\tag{2.15}
\end{equation}
where $D_n^p(a)$ denotes the $d$-level diagonal sampling matrix containing the samples of the function $a$ at the Greville abscissae, i.e.,

$$D_n^p(a) := \text{diag}_{i_d=1, \ldots, n_d, p_d=2} \left( \cdots \text{diag}_{i_1=1, \ldots, n_1, p_1=2} \left( a(\xi_{i_1+1, i_2+1, \ldots, i_d+1, [p_1, p_2, \ldots, p_d]) \right) \right) \cdots,$$

and the matrices $K_n^p$, $H_n^p$, $M_n^p$ are defined for all $p, n \geq 2$ by

$$n^2 K_n^p := \left[ -N_{j,1+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$
$$n H_n^p := \left[ N_{j,1+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$
$$M_n^p := \left[ N_{j,1+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2}.$$

We refer the reader to [8] for a derivation of (2.15) in the cases $d = 1, 2$, and to [9, Section 5.2] for a more detailed construction of the matrix $A_n^p$ in any dimension $d$. In particular, in the case $d = 1$ we have

$$A_n^p = n^2 D_n^p(\kappa) K_n^p + n D_n^p(\beta) H_n^p + D_n^p(\gamma) M_n^p,$$
with $D_n^p(a) = \text{diag}_{i_1=1, \ldots, n_1+p_1=2}(a(\xi_{j+1,[p]})).$ In the case $d = 2$ we have

$$A_{n_1,n_2}^{p_1,p_2} = n_1^2 D_{n_1,n_2}^{p_1,p_2}(\kappa_{11}) M_{n_2}^{p_2} \otimes K_{n_1}^{p_1} + n_2^2 D_{n_1,n_2}^{p_1,p_2}(\kappa_{21}) K_{n_2}^{p_2} \otimes M_{n_1}^{p_1}$$
$$-n_1 n_2 D_{n_1,n_2}^{p_1,p_2}(\kappa_{12} + \kappa_{21}) H_{n_2}^{p_2} \otimes H_{n_1}^{p_1} + n_1^2 D_{n_1,n_2}^{p_1,p_2}(\beta_1) M_{n_2}^{p_2} \otimes H_{n_1}^{p_1}$$
$$+ n_2^2 D_{n_1,n_2}^{p_1,p_2}(\beta_2) H_{n_2}^{p_2} \otimes M_{n_1}^{p_1},$$
with $D_{n_1,n_2}^{p_1,p_2}(a) = \text{diag}_{i_1=1, \ldots, n_1+p_1=2} \left( \text{diag}_{i_2=1, \ldots, n_2+p_2=2} \left( a(\xi_{i_1+1, i_2+1, [p_1, p_2]}) \right) \right).$

In the general case where $\Omega$ and $\mathbf{G}$ are nontrivial, let us consider, for any $u : \Omega \to \mathbb{R}$, the corresponding function

$$\hat{u} : \hat{\Omega} \to \mathbb{R}, \quad \hat{u}(\hat{x}) := u(x), \quad x = \mathbf{G}(\hat{x}).$$

In other words, $\hat{u} := u(\mathbf{G})$. Then, $u$ is the solution of (2.1) if and only if $\hat{u}$ satisfies the corresponding transformed problem

$$\begin{cases} -1(K_\mathbf{G} \circ H \hat{u})^T + \beta_\mathbf{G} \cdot \nabla \hat{u} + \gamma_\mathbf{G} \hat{u} = f_\mathbf{G}, & \text{in } \hat{\Omega}, \\ \hat{u} = 0, & \text{on } \partial \hat{\Omega}, \end{cases}$$

where $H \hat{u}$ is the Hessian of $\hat{u}$, $\gamma_\mathbf{G} := \gamma(\mathbf{G})$, $f_\mathbf{G} := f(\mathbf{G})$, and $K_\mathbf{G} := \left[ [K_{\mathbf{G}, i,j}]_{i,j=1}^{d} \right]$, $\beta_\mathbf{G} := \left[ [\beta_{\mathbf{G}, i,j}]_{i,j=1}^{d} \right]$ are the transformed coefficients of the PDE. The expression of $\beta_\mathbf{G}$ in terms of $K$, $\beta$, $\mathbf{G}$ is complicated and hence not reported here. For $K_\mathbf{G}$ we have

$$K_\mathbf{G} = (J_\mathbf{G})^{-1} K(\mathbf{G}) (J_\mathbf{G})^{-T},$$

where $J_\mathbf{G}$ is the Jacobian matrix of $\mathbf{G}$,

$$J_\mathbf{G} := \left[ \frac{\partial G_j}{\partial x_j} \right]_{i,j=1}^{d}.$$
In this case, the collocation matrix (2.4), with \( \varphi_i, \tau_i \) as in (2.7), (2.9) and \( \hat{\varphi}_i, \hat{\tau}_i \) as in (2.11)–(2.12), is given again by (2.15), in which \( \kappa_{rs}, \beta_r, \gamma \) are replaced by \( \kappa_{G,rs}, \beta_{G,r}, \gamma_G \), respectively. In the following, we will denote this matrix by \( A^{[p]}_{G,n} \).

For instance, let us consider the problem (2.1) in the one-dimensional case \( d = 1 \) with \( \Omega = (a,b) \):

\[
\begin{cases}
-k(x)u''(x) + \beta'(x)u'(x) + \gamma(x)u(x) = f(x), & a < x < b, \\
u(a) = 0, & u(b) = 0.
\end{cases}
\]

Given any geometry function \( G : [0,1] \to [a,b] \), the transformed problem reads as

\[
\begin{cases}
-k(G(\hat{x}))\hat{u}''(\hat{x}) + \left( \frac{\kappa(G(\hat{x}))G''(\hat{x})}{(G'(\hat{x}))^3} + \beta(G(\hat{x})) \right) \hat{u}'(\hat{x}) + \gamma(G(\hat{x}))\hat{u}(\hat{x}) = f(G(\hat{x})), & 0 < \hat{x} < 1, \\
\hat{u}(0) = 0, & \hat{u}(1) = 0,
\end{cases}
\]

and the resulting collocation matrix \( A^{[p]}_{G,n} \) is given by (2.16) in which \( \kappa, \beta, \gamma \) are replaced by

\[
\kappa_G := \frac{\kappa(G)}{(G')^3}, \quad \beta_G := \frac{\kappa(G)G''}{(G')^3} + \frac{\beta(G)}{G'} = \frac{\beta(G)}{G'}, \quad \gamma_G := \gamma(G),
\]

respectively.

2.2. Spectral distribution, symbol, and multilevel Toeplitz matrices

We now introduce the definition of the asymptotic spectral distribution and of the symbol for a given sequence of matrices. We denote by \( \mu_d \) the Lebesgue measure in \( \mathbb{R}^d \) and by \( C_c(\mathbb{C}) \) the space of continuous functions \( F : \mathbb{C} \to \mathbb{C} \) with compact support.

**Definition 2.1.** Let \( \{X_n\}_n \) be a sequence of matrices, with \( X_n \) of size \( m_n \) tending to infinity, and let \( f : D \to \mathbb{C} \) be a measurable function defined on the measurable set \( D \subset \mathbb{R}^d \) with \( 0 < \mu_d(D) < \infty \). We say that \( \{X_n\}_n \) is distributed like \( f \) in the sense of the eigenvalues and we write \( \{X_n\}_n \sim f \), if

\[
\lim_{n \to \infty} \frac{1}{m_n} \sum_{j=1}^{m_n} F(\lambda_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(f(x)) \, dx, \quad \forall F \in C_c(\mathbb{C}).
\]

In this case, \( f \) is called the (spectral) symbol of the sequence of matrices \( \{X_n\}_n \).

For the informal meaning behind the above definition we refer the reader to [7, Remark 3.1].

Multilevel Toeplitz matrices are certainly one of the most important examples of matrices possessing an asymptotic spectral distribution described by a (multivariate) symbol, in the sense of Definition 2.1. Given a \( d \)-variate function \( f : [-\pi,\pi]^d \to \mathbb{C} \) in \( L_1([-\pi,\pi]^d) \), we denote its Fourier coefficients by

\[
f_k := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\theta) e^{-i \mathbf{k} \cdot \theta} \, d\theta, \quad \mathbf{k} := (k_1, \ldots, k_d) \in \mathbb{Z}^d,
\]

where \( \mathbf{k} \cdot \theta := k_1 \theta_1 + \ldots + k_d \theta_d \). For any multi-index \( \mathbf{m} := (m_1, \ldots, m_d) \in \mathbb{N}^d \), we define the \( \mathbf{m} \)-th \( d \)-level Toeplitz matrix associated with \( f \) as the matrix of size \( m_1 \cdots m_d \) given by

\[
T_m(f) := [f_{i-j}]_{i,j=1}^m.
\]
Here, we used the standard multi-index notation (see [22, Sections 6 and 8] or [9, Section 1.1.1]), in which the multi-indices $i, j$ vary from $1$ to $m$ according to the lexicographic ordering

$$
\left[ \ldots \left[ (j_1, \ldots, j_d) \right]_{j_{d-1}=1, \ldots, m_d} \right]_{j_{d-2}=1, \ldots, m_{d-1}} \ldots \left[ (j_1, \ldots, j_d) \right]_{j_1=1, \ldots, m_1} \right].
$$

(2.22)

Note that this is not the ordering considered in (2.13). For instance, in the case $d = 2$, the lexicographic ordering is

$$(1, 1), (1, 2), \ldots, (1, m_2), (2, 1), (2, 2), \ldots, (2, m_2), \ldots, (m_1, 1), (m_1, 2), \ldots, (m_1, m_2).$$

In particular, the 1-level Toeplitz matrix, or Toeplitz matrix for short, is a square matrix whose entries are constant along each diagonal:

$$T_m(f) := \begin{bmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-(m-1)} \\ f_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ f_{m-1} & \cdots & \cdots & f_{-1} & f_0 \end{bmatrix} \in \mathbb{C}^{m \times m},$$

with $f_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i k \theta} d\theta$ and $k \in \mathbb{Z}$. The 2-level Toeplitz matrix is given by

$$T_{m_1, m_2}(f) := \begin{bmatrix} F_0 & F_{-1} & \cdots & \cdots & F_{-(m_1-1)} \\ F_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ F_{m_1-1} & \cdots & \cdots & F_{-1} & F_0 \end{bmatrix} \in \mathbb{C}^{m_1 m_2 \times m_1 m_2},$$

where, for every $k \in \mathbb{Z}$,

$$F_k := \begin{bmatrix} f_{k,0} & f_{k,-1} & \cdots & \cdots & f_{k,-(m_2-1)} \\ f_{k,1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ f_{k,m_2-1} & \cdots & \cdots & f_{k,-1} & f_{k,0} \end{bmatrix} \in \mathbb{C}^{m_2 \times m_2},$$

with $f_{k_1, k_2} := \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(\theta_1, \theta_2) e^{-i(k_1 \theta_1 + k_2 \theta_2)} d\theta_1 d\theta_2$ and $k_1, k_2 \in \mathbb{Z}$.

The function $f$ is called the generating function of the Toeplitz family $\{T_m(f)\}_{m \in \mathbb{N}^d}$. It is known that, if $f$ is real-valued, then all the matrices $T_m(f)$ are Hermitian. Moreover, under the assumption that $f$ is real-valued, the following distribution result (known as the Szegö-Tilli theorem) holds [21]:

$$
\lim_{m \to \infty} \frac{1}{m_1 \cdots m_d} \sum_{j=1}^{m_1 \cdots m_d} F(\lambda_j(T_m(f))) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} F(f(\theta)) d\theta,
$$

(2.23)

where for a multi-index $m$, the notation $m \to \infty$ means that $\min(m_1, \ldots, m_d) \to \infty$. Due to (2.23), $f$ is the symbol of the Toeplitz family $\{T_m(f)\}_{m \in \mathbb{N}^d}$ and we write $\{T_m(f)\}_{m \in \mathbb{N}^d} \sim \chi f$. Note that (2.23) amounts to saying that each sequence $\{T_m(n)(f)\}_n$ extracted from $\{T_m(f)\}_{m \in \mathbb{N}^d}$ and such that $m(n) \to \infty$ as $n \to \infty$ is distributed like $f$ in the sense of the eigenvalues, according to Definition 2.1.
2.3. Symbols for the IgA collocation matrices

In this section we recall from [8] the symbols of our (normalized) IgA collocation matrices. We start with the univariate setting and then we consider the general \(d\)-variate setting, with a particular focus on the case \(d = 2\).

2.3.1. Symbol of the univariate sequence \(\left\{ \frac{1}{n^2} A_{G,n}^{[p]} \right\}_n\)

Let \(\phi_{[p]} : \mathbb{R} \rightarrow \mathbb{R}\) be the cardinal B-spline of degree \(p\) over the uniform knot sequence \(\{0, 1, \ldots, p + 1\}\), which is defined recursively as follows [5]:

\[
\phi_{[p]}(t) := \begin{cases} 
1, & \text{if } t \in [0, 1), \\
0, & \text{elsewhere},
\end{cases}
\]

and

\[
\phi_{[p]}(t) := \frac{t}{p} \phi_{[p-1]}(t) + \frac{p + 1 - t}{p} \phi_{[p-1]}(t - 1), \quad p \geq 1.
\]

The cardinal B-spline \(\phi_{[p]}\) belongs piecewisely to \(\mathbb{P}_p\) and is globally of class \(C^{p-1}(\mathbb{R})\). We denote by \(\dot{\phi}_{[p]}\) and \(\ddot{\phi}_{[p]}\) the first and second derivative of \(\phi_{[p]}\), respectively, and we define the following three functions:

\[
h_p : [-\pi, \pi] \rightarrow \mathbb{R}, \quad h_p(\theta) := \phi_{[p]} \left( \frac{p + 1}{2} \right) + 2 \sum_{k=1}^{[p/2]} \phi_{[p]} \left( \frac{p + 1}{2} - k \right) \cos(k\theta), \quad p \geq 0, \tag{2.24}
\]

\[
g_p : [-\pi, \pi] \rightarrow \mathbb{R}, \quad g_p(\theta) := -2 \sum_{k=1}^{[p/2]} \dot{\phi}_{[p]} \left( \frac{p + 1}{2} - k \right) \sin(k\theta), \quad p \geq 2, \tag{2.25}
\]

\[
f_p : [-\pi, \pi] \rightarrow \mathbb{R}, \quad f_p(\theta) := -\ddot{\phi}_{[p]} \left( \frac{p + 1}{2} \right) - 2 \sum_{k=1}^{[p/2]} \ddot{\phi}_{[p]} \left( \frac{p + 1}{2} - k \right) \cos(k\theta), \quad p \geq 2, \tag{2.26}
\]

with the usual assumption that a sum is zero when the upper index is less than the lower one. Note that

\[
h_0(\theta) = h_1(\theta) = 1, \quad g_2(\theta) = g_3(\theta) = -\sin\theta, \quad f_2(\theta) = f_3(\theta) = 2 - 2\cos\theta.
\]

Remark 2.1. The functions (2.24) and (2.26) of odd degree \(p = 2q + 1, q \geq 1\), have already been analyzed and used in the IgA Galerkin context [7, 10]. Note that \(f_q\) (resp. \(h_q\)) in [7, 10] coincides with \(f_{2q+1}\) (resp. \(h_{2q+1}\)) here.

Now, let us consider the sequence of matrices \(\{A_{G,n}^{[p]}\}_n\), coming from the IgA collocation approximation of (2.1) in the univariate case \(d=1\), with \(\Omega = (a, b)\) and with any geometry function \(G : [0, 1] \rightarrow [a, b]\). We recall from Section 2.1 that \(A_{G,n}^{[p]}\) is given by (2.16) with \(\kappa, \beta, \gamma\) replaced by \(\kappa_G, \beta_G, \gamma_G\), respectively (see (2.20)). Then, from [8], we know that the symbol of the sequence of normalized matrices \(\left\{ \frac{1}{n^2} A_{G,n}^{[p]} \right\}_n\)

\[
f_{G,p} := \kappa_G \otimes f_p : [0, 1] \times [-\pi, \pi] \rightarrow \mathbb{R},
\]

that is

\[
\left\{ \frac{1}{n^2} A_{G,n}^{[p]} \right\}_n \sim \kappa_G \otimes f_p.
\]

In particular, \(f_p : [-\pi, \pi] \rightarrow \mathbb{R}\) is the symbol of the sequence \(\{A_{G,n}^{[p]}\}_n\) obtained when \(\kappa = 1\) and \(G : [0, 1] \rightarrow [0, 1]\) is the identity map. Lemma 2.1 collects some properties of \(f_p\) (see [8]).
Lemma 2.1. The following properties hold for all $p \geq 2$ and $\theta \in [-\pi, \pi]$:

1. $\left(\frac{2}{\pi}\right)^{p-1} \leq h_{p-2}(\theta) \leq h_{p-2}(0) = 1$ and $h_{p-2}(\pi) \leq \frac{h_{p-2}(\pi)}{h_{p-2}(\frac{\pi}{2})} \leq 2^{\frac{p-1}{2}}$;

2. $f_p(\theta) = (2 - 2 \cos \theta)h_{p-2}(\theta)$ and $\frac{f_p(\pi)}{M_{f_p}} \leq \frac{f_p(\pi)}{f_p(\frac{\pi}{2})} \leq 2^{\frac{p-2}{2}}$, where $M_{f_p} := \max_{\theta \in [-\pi, \pi]} f_p(\theta)$.

In particular, $f_p$ has a unique zero of order two at $\theta = 0$ (like the function $2 - 2 \cos \theta$), but the value $f_p(\pi)/M_{f_p}$ converges to 0 exponentially as $p \to \infty$ (see Figure 1 and Table 1).

The zero of the symbol $f_p$ at $\theta = 0$ is interpreted by saying that the related IgA collocation matrices $\frac{1}{n^2}A_n^{[p]}$ are ill-conditioned in the low frequencies; see [7, Section 3.2.2] for the terminology of frequencies. On the other hand, the fact that the normalized symbol $f_p/M_{f_p}$ has a numerical zero at $\theta = \pi$ for large $p$ means that the matrices $\frac{1}{n^2}A_n^{[p]}$ are ill-conditioned (for large $p$) also in the high frequencies. The ill-conditioning in the low frequencies is expected, because it is a canonical feature of the symbol associated with the discretization matrices of second order differential problems like (2.1). However, the ill-conditioning in the high frequencies is not expected and is responsible for the deterioration in the convergence rate of the standard multigrid methods when $p$ increases.

A way to overcome this problem consists in adopting a multi-iterative strategy, as discussed in Section 4.

2.3.2. Symbol of the bivariate sequence $\{\frac{1}{n^2}A_n^{[p]}\}_{n \in \mathbb{N}}$ with $n = n\nu$

Let $p := (p_1, p_2)$ and $\nu := (\nu_1, \nu_2)$ such that $p_1, p_2 \geq 2$ and $\nu_1, \nu_2 \in \mathbb{Q}_+ := \{r \in \mathbb{Q} : r > 0\}$.

We consider the sequence of matrices $A_n^{[p]}$, coming from the IgA collocation approximation of (2.1) in the bivariate case $d = 2$, with a geometry map $G : [0, 1]^2 \to \Omega$ and with $n = n\nu$ (assuming...
that \( n \) varies in the set of natural numbers such that \( n \nu \in \mathbb{N}^2 \). It was proved in [8] that the symbol of the sequence of normalized collocation matrices \( \{ \frac{1}{n^2} A_n^{|p|} \}_n \) is \( f_{G,p}^{(\nu)} : [0,1]^2 \times [-\pi, \pi]^2 \to \mathbb{R} \),
\[
\begin{align*}
f_{G,p}^{(\nu)} := & (\nu_1)^2 \kappa_{G,1} \otimes h_{p_1} \otimes f_{p_1} + (\nu_2)^2 \kappa_{G,2} \otimes f_{p_2} \otimes h_{p_1} + \nu_1 \nu_2 (\kappa_{G,12} + \kappa_{G,21}) \otimes g_{p_2} \otimes g_{p_1} \\
= & [\nu_1 \nu_2 \kappa_G] [H_{p_1,p_2}] [\nu_1 \nu_2]^T,
\end{align*}
\]
with
\[
H_{p_1,p_2} := \begin{bmatrix} h_{p_2} \otimes f_{p_1} & g_{p_2} \otimes g_{p_1} \\ g_{p_2} \otimes g_{p_1} & f_{p_2} \otimes h_{p_1} \end{bmatrix}.
\]
Recall that the functions \( h_p, g_p, f_p \) are defined in (2.24)–(2.26), and \( K_G \) is given in (2.19). It turns out (see [8]) that \( H_{p_1,p_2} \) is symmetric positive semidefinite over \([-\pi, \pi]^2\). Moreover,
\[
f_p^{(\nu)} : [-\pi, \pi]^2 \to \mathbb{R}, \quad f_p^{(\nu)} := (\nu_1)^2 h_{p_2} \otimes f_{p_1} + (\nu_2)^2 f_{p_2} \otimes h_{p_1}
\]
is the symbol of the sequence \( \{ \frac{1}{n^2} A_n^{|p|} \}_n \) obtained when \( G : [0,1]^2 \to [0,1]^2 \) is the identity map and \( K \) is the identity matrix. Note that \( f_p^{(\nu)} \) is symmetric in each variable on \([-\pi, \pi]^2\), i.e.,
\[
f_p^{(\nu)}(\pm \theta_1, \pm \theta_2) = f_p^{(\nu)}(\theta_1, \theta_2).
\]
Some properties of the symbol \( f_p^{(\nu)} \) can be deduced from Lemma 2.1 and are given in Lemma 2.2.

**Lemma 2.2.** The following properties hold for all \( p := (p_1,p_2) \in \mathbb{N}^2 \) with \( p_1, p_2 \geq 2 \), for all \( \nu := (\nu_1, \nu_2) \in (\mathbb{Q}_+)^2 \) and for all \( \theta := (\theta_1, \theta_2) \in [-\pi, \pi]^2 \):

1. \[ \left( \frac{2}{\pi} \right)^{p_1+p_2} \min(\nu_1, \nu_2)^2 \sum_{k=1}^{2} (2 - 2 \cos \theta_k) \leq f_p^{(\nu)}(\theta) \leq \max(\nu_1, \nu_2)^2 \sum_{k=1}^{2} (2 - 2 \cos \theta_k); \]

2. \[ \frac{f_p^{(\nu)}(\pi, \theta_2)}{M_{f_p^{(\nu)}}} \leq f_p^{(\nu)}(\frac{\pi}{2}, \theta_2) \leq \frac{2^{-p_2}}{2^{-p_2}} \quad \text{and} \quad \frac{f_p^{(\nu)}(\theta_1, \pi)}{M_{f_p^{(\nu)}}} \leq f_p^{(\nu)}(\frac{\theta_1}{2}, \pi) \leq \frac{2^{-p_2}}{2^{-p_2}}, \quad \text{where} \quad M_{f_p^{(\nu)}} := \max_{\theta \in [-\pi, \pi]^2} f_p^{(\nu)}(\theta). \]

In particular, \( f_p^{(\nu)} \) has a unique zero of order two at \( \theta = 0 \) (like the function \( \sum_{k=1}^{2} (2 - 2 \cos \theta_k) \)), but the values \( f_p^{(\nu)}(\pi, \theta_2)/M_{f_p^{(\nu)}} \) and \( f_p^{(\nu)}(\theta_1, \pi)/M_{f_p^{(\nu)}} \) converge to 0 exponentially when \( p_2 \to \infty \) and \( p_1 \to \infty \), respectively.

As in the univariate case, the zero of the symbol \( f_p^{(\nu)} \) at \( \theta = 0 \) is expected and is interpreted as a source of ill-conditioning for the corresponding IgA collocation matrices \( \frac{1}{n^2} A_n^{|p|} \) in the low frequencies. On the other hand, when the spline degrees \( p_1 \) and \( p_2 \) are large, the normalized symbol \( f_p^{(\nu)}/M_{f_p^{(\nu)}} \) presents numerical zeros located at the '\( \pi \)-edge points'

\[ \{ \theta \in [0, \pi]^2 : \theta_1 = \pi \text{ or } \theta_2 = \pi \}. \]

These are interpreted as an ill-conditioning of \( \frac{1}{n^2} A_n^{|p|} \) in the high frequencies corresponding to such points. This second (non-canonical) source of ill-conditioning is responsible for the convergence deterioration of standard multigrid methods when \( p_1, p_2 \) increase. In Section 4 we will see how we can overcome this problem by adopting a multi-iterative multigrid method whose convergence rate does not deteriorate for increasing spline degrees.
2.3.3. Symbol of the $d$-variate sequence $\{\frac{1}{n^2}A_{\mathbf{n}}^{[p]}\}_{n}$ with $\mathbf{n} = n\mathbf{\nu}$

In the general $d$-variate setting, the situation is completely analogous to the bivariate setting. Let $\mathbf{p} := (p_1, \ldots, p_d)$ and $\mathbf{\nu} := (\nu_1, \ldots, \nu_d)$ such that $p_j \geq 2$ for all $j$ and each $\nu_j$ belongs to $\mathbb{Q}_+$. We consider the sequence of matrices $\{\frac{1}{n^2}A_{\mathbf{n}}^{[p]}\}_{n}$, coming from the IgA collocation approximation of (2.1) in the $d$-variate case, with an arbitrary $\mathbf{G} : \Omega \to \Omega$ and with $\mathbf{n} = n\mathbf{\nu}$. Then, the symbol of the sequence $\{\frac{1}{n^2}A_{\mathbf{n}}^{[p]}\}_{n}$ is $f^{(\nu)}_{\mathbf{G}, \mathbf{p}} : [0, 1]^d \times [-\pi, \pi]^d \to \mathbb{R}$,

$$f^{(\nu)}_{\mathbf{G}, \mathbf{p}} := [\nu_1 \cdots \nu_d][K_\mathbf{G} \circ H_{\mathbf{p}_1, \ldots, \mathbf{p}_d}][\nu_1 \cdots \nu_d]^T,$$

with

$$(H_{\mathbf{p}_1, \ldots, \mathbf{p}_d})_{rs} := \begin{cases} h_{p_d} \otimes \cdots \otimes h_{p_{r+1}} \otimes f_{p_r} \otimes h_{p_{r-1}} \otimes \cdots \otimes h_{p_1}, & \text{if } r = s, \\ h_{p_d} \otimes \cdots \otimes h_{p_{r+1}} \otimes g_{p_r} \otimes h_{p_{r-1}} \otimes \cdots \otimes h_{p_1}, & \text{if } r < s, \\ h_{p_d} \otimes \cdots \otimes h_{p_{r+1}} \otimes g_{p_r} \otimes h_{p_{r-1}} \otimes \cdots \otimes h_{p_1}, & \text{if } r > s. \end{cases}$$

The formal proof of this statement can be found in [9, Theorems 5.1–5.2]. In particular,

$$f^{(\nu)}_{\mathbf{p}} := \sum_{k=1}^d (\nu_k)^2 (h_{p_d} \otimes \cdots \otimes h_{p_{k+1}} \otimes f_{p_k} \otimes h_{p_{k-1}} \otimes \cdots \otimes h_{p_1})$$

is the symbol of the sequence $\{\frac{1}{n^2}A_{\mathbf{n}}^{[p]}\}_{n}$ obtained when $\mathbf{G} : [0, 1]^d \to [0, 1]^d$ is the identity map and $K$ is the identity matrix.

**Lemma 2.3.** The following properties hold for all $\mathbf{p} \in \mathbb{N}^d$ with every $p_j \geq 2$, for all $\mathbf{\nu} \in (\mathbb{Q}_+)^d$ and for all $\theta \in [-\pi, \pi]^d$:

1. \(\left(\frac{2}{\pi}\right)^{p_1 + \cdots + p_d + d - 2} \min(\nu_1, \ldots, \nu_d)^2 \sum_{k=1}^d (2 - 2 \cos \theta_k) \leq f^{(\nu)}_{\mathbf{p}}(\theta) \leq \max(\nu_1, \ldots, \nu_d)^2 \sum_{k=1}^d (2 - 2 \cos \theta_k);\)

2. \(f^{(\nu)}_{\mathbf{p}}(\theta_1, \ldots, \theta_{j-1}, \pi, \theta_{j+1}, \ldots, \theta_d) \leq f^{(\nu)}_{\mathbf{p}}(\theta_1, \ldots, \theta_{j-1}, \pi, \theta_{j+1}, \ldots, \theta_d) \leq 2^{\frac{s - p_{j-1} + 1}{d}}\) for all $j = 1, \ldots, d$, where $M^{(\nu)}_{f^{(\nu)}_{\mathbf{p}}} := \max_{\theta \in [-\pi, \pi]^d} f^{(\nu)}_{\mathbf{p}}(\theta)$.

In particular, $f^{(\nu)}_{\mathbf{p}}$ has a unique zero of order two at $\theta = \mathbf{0}$ (like the function $\sum_{k=1}^d (2 - 2 \cos \theta_k)$), but, for every $j = 1, \ldots, d$, the value $f^{(\nu)}_{\mathbf{p}}(\theta_1, \ldots, \theta_{j-1}, \pi, \theta_{j+1}, \ldots, \theta_d)/M^{(\nu)}_{f^{(\nu)}_{\mathbf{p}}}$ converges to 0 exponentially when $p_j \to \infty$.

The normalized symbol $\frac{f^{(\nu)}_{\mathbf{p}}}{M^{(\nu)}_{f^{(\nu)}_{\mathbf{p}}}}$ has an actual zero at $\theta = \mathbf{0}$ and, when the spline degrees $p_j$ are large, also infinitely many numerical zeros located at the $\pi$-edge points

$$\{\theta \in [0, \pi]^d : \exists j \in \{1, \ldots, d\} \text{ with } \theta_j = \pi\}. \quad (2.27)$$

The interpretation of these zeros as sources of ill-conditioning for the associated IgA collocation matrices $\frac{1}{n^2}A_{\mathbf{n}}^{[p]}$ is the same as in the bivariate case and, in particular, the non-canonical zeros (2.27) are responsible for the deterioration of the convergence rate of standard multigrid methods for large $p_j$. In Section 4 we will see how to bypass this problem.
3. Optimal and robust P-GMRES for the general IgA collocation matrices

When considering the preconditioned version of the GMRES method for solving a linear system $Au = b$, it is assumed to have a preconditioning matrix at hand. A matrix $M$ is a good GMRES preconditioner if $M^{-1}$ is a good approximation to $A^{-1}$ (from the spectral point of view) and if the product of $M^{-1}$ and an arbitrary vector is easy to compute. Such a preconditioner is optimal if the number of P-GMRES iterations to reach a preassigned accuracy is independent of the size of the matrix $A$, and in addition, it is robust if it is independent of (or only mildly dependent on) all the relevant parameters involved in the discretization.

The B-spline discretization matrix related to the Laplacian on the hypercube will be referred to as the Parametric Laplacian matrix (or PL-matrix); this is the matrix coming from the IgA collocation approximation of (2.1) in the case where $G$ is the identity map (so $Ω = (0, 1)^d$), $K = I$ is the identity matrix, and $β = 0$, $γ = 0$. In this section we show that under mild assumptions the PL-matrix is a good GMRES preconditioner for the general IgA collocation matrices $A^{[p]}_{G,n}$ approximating the full elliptic problem (2.1) with arbitrary $K$, $β$, $γ$, $G$. This can be explained by means of the theory of Generalized Locally Toeplitz (GLT) sequences [18], which is a generalization of the standard Fourier Analysis to nonconstant coefficient differential operators, as discussed in [19].

3.1. The PL-matrix: a good choice as preconditioner for GMRES

Let us illustrate in the bivariate case $d = 2$ why the PL-matrix performs well as GMRES preconditioner, without entering into all the theoretical details. For simplicity of notation, we denote by $A^{[p]}_{G,n}$ the bivariate PL-matrix as given by (2.17) for $κ_{11} = κ_{22} = 1$ and $κ_{12} = κ_{21} = β_1 = β_2 = γ = 0$. From the analysis in [8], we know that both $\{\frac{1}{π^2} A^{[p]}_{G,n}\}_n$ and $\{\frac{1}{π^2} A^{[p]}_{G,n}\}_n (n = nν)$ are GLT sequences, with corresponding symbols

$$f^{(ν)}_{P,G} = [ν_1 \ ν_2] [K_G \circ H_{p_1,p_2}] [ν_1 \ ν_2]^T,$$

and

$$f^{(ν)}_P = [ν_1 \ ν_2] [I \circ H_{p_1,p_2}] [ν_1 \ ν_2]^T,$$

respectively. Since the GLT class is an algebra and since $f^{(ν)}_P$ vanishes only at $θ = 0$ (this implies that $\{\frac{1}{π^2} A^{[p]}_{G,n}\}_n$ is sparsely vanishing according to the terminology in [19]), it follows that $\{(\frac{1}{π^2} A^{[p]}_{G,n})^{-1} \frac{1}{π^2} A^{[p]}_{G,n}\}_n = \{A^{[p]}_{G,n}^{-1} A^{[p]}_{G,n}\}_n$ is still a GLT sequence with symbol $(f^{(ν)}_P)^{-1} f^{(ν)}_{G,P}$.

$$\{(A^{[p]}_{G,n})^{-1} A^{[p]}_{G,n}\}_n \sim λ (f^{(ν)}_P)^{-1} f^{(ν)}_{G,P} = [ν_1 \ ν_2] [K_G \circ H_{p_1,p_2}] [ν_1 \ ν_2]^T, \quad (3.1)$$

Now, suppose that there exist two positive constants $c, C$ such that

$$cI \leq K_G(\hat{x}) \leq CI, \quad ∀\hat{x} ∈ \hat{Ω}, \quad (3.2)$$

where the notation $X \geq Y$ means that $X - Y$ is Hermitian positive semidefinite. The condition in (3.2) is equivalent to

$$\min_{x ∈ \Omega} λ_{min}(K_G(x)) ≥ c > 0, \quad \max_{x ∈ \Omega} λ_{max}(K_G(x)) ≤ C < ∞. \quad (3.3)$$

Note that (3.2) is usually satisfied in practice. For instance, it is satisfied if

1. $cK I \leq K(x) ≤ CK I$ for some positive constants $c_K, C_K$ and for all $x ∈ Ω$. 


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2. \( c_G I \leq (J_G(\hat{x}))^T J_G(\hat{x}) \leq C_G I \) for some positive constants \( c_G, C_G \) and for all \( \hat{x} \in \hat{\Omega} \); in this case we can take \( c = c_K / C_G \) and \( C = C_K / c_G \). Under the assumption (3.2), a linear algebra argument [9, Lemma 1.6] gives the following relations:

\[
C I \circ H_{p_1,p_2} \leq K_G \circ H_{p_1,p_2} \leq C I \circ H_{p_1,p_2}.
\]

This implies that the ‘preconditioned symbol’ in (3.1) satisfies

\[
c \leq (f_p^{(\nu)})^{-1} f_G^{(\nu)} \leq C,
\]

i.e., it is uniformly bounded from above and below by the positive constants \( C \) and \( c \), respectively, which of course depend on \( K, G \), but not on \( n, p \). This explains why the PL-matrix \( A_{G,n}^{[p]} \) is expected to be an optimal and robust GMRES preconditioner for \( A_{G,n}^{[p]} \). In particular, the P-GMRES convergence rate should be independent of \( n \) and \( p \). This reduces the fast solution of linear systems associated with the general IgA collocation matrix \( A_{G,n}^{[p]} \) to the fast solution of linear systems related to the PL-matrix \( A_{G,n}^{[p]} \).

As we will see in Section 4, a fast solver is available for linear systems (4.2) related to the PL-matrix. The solver is of multi-iterative type, combining a standard multigrid strategy and a certain P-GMRES employed as a smoother at the finest level. The first method is effective for approximating the solution especially in the space of low frequencies, where a source of ill-conditioning exists, due to the fact that the symbol vanishes at \( \theta = 0 \). The second method is equipped with a specific preconditioner for dampening the high frequency error components, or, equivalently, for approximating the solution in the high frequencies, where another (unexpected) source of ill-conditioning shows up when the spline degrees \( p_j \) are large, due to the presence of the numerical zeros of the symbol at the \( \pi \)-edge points (2.27). The combination of these two methods — in the spirit of a multi-iterative strategy — leads to a solver whose convergence speed is optimal and robust, i.e., independent of the matrix-size and substantially independent of the other relevant parameters, like the approximation parameters \( p \) and the dimensionality \( d \).

In the following examples we provide numerical evidence of the optimality of the PL-matrix as a GMRES preconditioner for the general IgA collocation matrices \( A_{G,n}^{[p]} \). In all the examples, we use the MATLAB gmres function without restarting and with a tolerance of \( 10^{-6} \). The method has been started with \( u^{(0)} = 0 \) and stopped at the first vector whose relative residual in 2-norm is less than \( 10^{-6} \):

\[
||A_{G,n}^{[p]} u^{(c)} - b|| \leq 10^{-6} ||b||.
\]

For notational convenience, the physical variables \( x \) are denoted by \((x,y)\) and \((x,y,z)\) in the 2D and 3D setting, respectively. A similar notation is used for the parametric variables \( \hat{x} \).

Example 1. Consider the problem (2.1) in the case \( d = 2 \), defined on the unit square

\[
\Omega = (0,1)^2, \quad G(\hat{x}, \hat{y}) = (\hat{x}, \hat{y}),
\]

and with

\[
K(x,y) = \begin{bmatrix}
(2 + \cos x)(1 + y) & \cos(x + y) \sin(x + y) \\
\cos(x + y) \sin(x + y) & (2 + \sin y)(1 + x)
\end{bmatrix},
\]

\[
\beta(x,y) = \begin{bmatrix}
11 + \sin x + y \sin x - 2 \cos^2(x + y) \\
-9 - \cos y - x \cos y - 2 \cos^2(x + y)
\end{bmatrix},
\]

\[
\gamma(x,y) = f(x,y) = 1.
\]
Table 2: Example 1: number of GMRES iterations without preconditioning (I) and with the PL-matrix as precon-
ditioner (P), for solving $A_{G,n,n}^{[p,p]} u = b$ up to a precision of $10^{-6}$, varying the fineness parameter $n$ and the spline
degree $p$.

We solved the linear system resulting from the IgA collocation approximation of this problem by
means of the GMRES method, first applied without preconditioning and then with the PL-matrix
as preconditioner. The results are collected in Table 2. We observe that the P-GMRES has an
optimal and robust convergence rate, completely independent of $n$ and $p$. This is in contrast with
the behavior of the simple GMRES, whose convergence rate worsens with respect to both $n$ and $p$
and, in particular, grows linearly with $n$ (the system size is $(n+p-2)^2 \sim n^2$). From Table 2 we
can conclude that the PL-matrix is an optimal and robust GMRES preconditioner for the general
IgA collocation matrix, whereas the simple GMRES is not.

Example 2. Consider the problem (2.1) in the case $d = 2$, defined on a quarter of an annulus:

$\Omega = \{(x,y) \in \mathbb{R}^2 : r^2 < x^2 + y^2 < R^2, \ x > 0, \ y > 0\}, \quad r = 1, \quad R = 4,$

with

$G(\hat{x}, \hat{y}) = (x, y), \quad \left\{ \begin{array}{l} x = [r + \hat{x}(R-r)] \cos(\frac{\pi}{2} \hat{y}), \\
y = [r + \hat{x}(R-r)] \sin(\frac{\pi}{2} \hat{y}) \end{array} \right.$

Note that the map $G$ provides an exact representation of the domain $\Omega$, but is not expressed in
terms of tensor-product B-splines. In this sense, our analysis is general, since we are not restricted
to use a B-spline approximation of the domain (following the isoparametric approach), but we
may use any exact representation of the domain. Moreover, we take

$K(x, y) = \begin{bmatrix} (2 + \cos x)(1 + y) & \cos(x + y) \sin(x + y) \\
\cos(x + y) \sin(x + y) & (2 + \sin y)(1 + x) \end{bmatrix},$

$\beta(x, y) = \begin{bmatrix} -5y \\
5x \end{bmatrix},$

$\gamma(x, y) = xy,$

and $f(x, y)$ computed from the exact solution

$u(x, y) = (x^2 + y^2 - 1)(x^2 + y^2 - 16) \sin x \sin y.$
Table 3: Example 2: number of GMRES iterations without preconditioning (I) and with the PL-matrix as precon-
ditioner (P), for solving $A_{G,n,n} \mathbf{u} = \mathbf{b}$ up to a precision of $10^{-6}$, varying the fineness parameter $n$ and the spline
degree $p$.

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<tr>
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<td>286 20</td>
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<td>316 19</td>
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To solve the corresponding B-spline collocation linear system, the GMRES method was applied
first without preconditioning and then with the PL-matrix as preconditioner. The results are
collected in Table 3, and they clearly indicate that the PL-matrix is an optimal and robust GMRES
preconditioner for the IgA collocation matrix $A_{G,n,n}$.

When increasing $p$, but keeping $n$ fixed, the number of P-GMRES iterations to reach the pre-
assigned accuracy $10^{-6}$ is slowly increasing for moderate $n$, whereas it seems practically constant
for large $n$: the observed convergence rate is about $10^{-6/19} \approx 0.483$. In this example, we also
computed the best constants for which the relations (3.2)–(3.3) are satisfied, i.e.,

$$
c := \min_{(\hat{x},\hat{y}) \in \hat{\Omega}} \lambda_{\min}(K_{G}(\hat{x}, \hat{y})) \approx 0.111, \quad C := \max_{(\hat{x},\hat{y}) \in \hat{\Omega}} \lambda_{\max}(K_{G}(\hat{x}, \hat{y})) \approx 2.436. \quad (3.6)
$$

Let us assume that $(A_{G,n,n}^{[p,p]})^{-1}A_{G,n,n}^{[p,p]}$ is ‘almost’ symmetric positive definite with its spectrum
contained in $[c, C]$. Note that this makes sense because the corresponding symbol $(f_{p,p}^{(1,1)})^{-1}f_{G,p,p}^{(1,1)}$
is nonnegative with range in $[c, C]$. Then, the classical GMRES convergence analysis based on
the values (3.6) provides an upper bound of 0.648 for the asymptotic convergence rate; see [14,
Proposition 6.32] and recall the classical estimate for the quantity $\epsilon_m$ appearing in the proposition,
which in our case becomes

$$
\epsilon_m \leq 2 \left( \frac{\sqrt{C/c} - 1}{\sqrt{C/c} + 1} \right)^m.
$$

Luckily, the observed convergence rate 0.483 is even better. Thus, the presence of eigenvalues
with small imaginary part, the existence of outliers, and the fact that the matrix of the eigenvectors
of $(A_{G,n,n}^{[p,p]})^{-1}A_{G,n,n}^{[p,p]}$ is not exactly unitary do not seem to negatively influence the observed
convergence rate.

We conclude the numerical example by showing in Figure 2 the error $|u(x, y) - \tilde{u}(x, y)|/\|u\|_{\infty}$, where $\tilde{u}(x, y)$ is the computed solution, for $n = 30$ and for different values of $p$. The 2-norm of
the relative error is also given in the figure.
Example 3. Consider the problem (2.1) in the case $d = 3$, defined on the unit cube

$$\Omega = (0, 1)^3, \quad G(\hat{x}, \hat{y}, \hat{z}) = (\hat{x}, \hat{y}, \hat{z}),$$

and

$$K(x, y, z) = \begin{bmatrix}
ed^{xyz} & \frac{xy}{2} & \frac{xz}{2} \\
\frac{xy}{2} & e^{x+y+z} & \frac{yz}{2} \\
\frac{xz}{2} & \frac{yz}{2} & xyz + 3
\end{bmatrix}, \quad \beta(x, y, z) = \begin{bmatrix}5xy + z \\
-10yz + x \\
5xz + y
\end{bmatrix},$$

$$\gamma(x, y, z) = \frac{x^2y - y^3}{1 + z}, \quad f(x, y, z) = 1.$$  

We solved the corresponding B-spline collocation linear system by means of the GMRES method with the PL-matrix as preconditioner. The results are collected in Table 4, and once again, they
show that the PL-matrix is an optimal and robust GMRES preconditioner for the general IgA collocation matrix.

### 3.2. Preconditioning in the presence of a singularity in the geometry map

The geometry map $G$ is singular if the determinant of its Jacobian matrix $J_G$ is zero at one or more points, the so-called singular points. We expect that the PL-matrix will not perform optimally as GMRES preconditioner when the geometry map is singular, because in this case the related preconditioned symbol (3.4) has a pole. Similarly, if $G$ is nearly singular, then the preconditioned symbol (3.4) has a numerical pole, and we expect that the PL-matrix is theoretically optimal, but the number of P-GMRES iterations for reaching a preassigned accuracy is too high for practical purposes. The reason of these predictions relies on the structure of the matrix $K_G$, which incorporates in its expression (2.19) the factor $(J_G)^{-1}$ and its transpose $(J_G)^{-T}$. In particular, when $G$ is singular and $n = n_\nu$, one expects that the largest eigenvalue of $\frac{1}{n^2} A_{G,n}^{[p]}$ (i.e., the eigenvalue of $\frac{1}{n^2} A_{G,n}^{[p]}$ with maximum modulus) is asymptotically unbounded and its absolute value grows like $n^{2\alpha}$, where $\alpha$ is the highest order of the zeros of the determinant $\text{det}(J_G)$.

As a consequence, the same growth is observed for the largest eigenvalue of the preconditioned matrix $\left( \frac{1}{n^2} A_{G,n}^{[p]} \right)^{-1} \frac{1}{n^2} A_{G,n}^{[p]} = (A_{G,n}^{[p]})^{-1} A_{G,n}^{[p]}$, because the preconditioner $\frac{1}{n^2} A_{G,n}^{[p]}$ (the normalized PL-matrix) has a bounded spectrum. The numerical experiments in Table 6 provide a confirmation of these forecasts in the case $\alpha = 1$. Note that the unboundedness of the largest eigenvalue of $\frac{1}{n^2} A_{G,n}^{[p]}$ can also be understood through the fact that the right inequalities in (3.2)–(3.3), as well as the left inequality in item 2 after (3.3), are all violated, and consequently the symbol $f_{G,p}^{(\nu)}$ is unbounded.

These considerations have a practical counterpart, because they suggest a clever and computationally very cheap variation in the preconditioning strategy. In practice, it suffices to ‘insert’ in the PL-matrix a global term with the following two properties:

1. its largest eigenvalue grows in the same way as the largest eigenvalue of $A_{G,n}^{[p]}$;
2. it acts in the subspace generated by the eigenvectors related to the exploding eigenvalues.

Following the GLT approach, a smart ‘insertion’ can be achieved by using diagonal matrices that perform a symmetric pre-scaling of the coefficient matrix $A_{G,n}^{[p]}$, as already proposed in the context of Finite Differences [17, 20].

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3 This could be proved by following the technique shown in [20].

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Table 4: Example 3: number of P-GMRES iterations for solving $A_{G,n,n,n}^{[p,p,p]} u = b$ up to a precision of $10^{-6}$, varying the fineness parameter $n$ and the spline degree $p$. 


Table 5: Example 4: the weights \( w_{i,1}, w_{i,5} \) and the control points \( P_{i,1} := (P_{i,1}^x, P_{i,1}^y), P_{i,5} := (P_{i,5}^x, P_{i,5}^y) \) used in the NURBS geometry map depending on the parameter \( \omega \).

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<th>( i )</th>
<th>( w_{i,1} )</th>
<th>( w_{i,5} )</th>
<th>( w_{i,1}P_{i,1}^x )</th>
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<td>3</td>
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<td>( \frac{5 + \sqrt{2}}{12 + \frac{24}{24}} )</td>
<td>( \frac{1 + \frac{7\sqrt{2}}{12}}{3 + \frac{24}{24}} )</td>
<td>( \frac{1 + \frac{7\sqrt{2}}{12}}{3 + \frac{24}{24}} )</td>
<td>( \frac{1 + \frac{4\omega}{3} + \frac{7\sqrt{2}}{3}}{3 + \frac{6}{6}} )</td>
<td>( \frac{4 + \omega + \frac{7\sqrt{2}}{3}}{3 + \frac{6}{6}} )</td>
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<td>4</td>
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<td>( \frac{5 + \sqrt{2}}{12 + \frac{24}{24}} )</td>
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<td>( \frac{1 + \frac{7\sqrt{2}}{12}}{3 + \frac{24}{24}} )</td>
<td>( \frac{4 + \omega + \frac{7\sqrt{2}}{3}}{3 + \frac{6}{6}} )</td>
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In accordance with the previous reasoning, if \( G \) has a singularity of order 1 (i.e., the determinant of \( J_G \) has a simple zero), then we expect that the largest eigenvalue of \( \frac{1}{n^2} A_{G,n}^{[p]} \) tends to infinity as \( n^2 \). In such a case, the suggested preconditioner for the general IgA collocation matrix \( A_{G,n}^{[p]} \) takes the form

\[
DA_{n}^{[p]}D,
\]

where \( A_{n}^{[p]} \) is the PL-matrix and \( D^2 \) is the diagonal matrix whose diagonal entries coincide with those of \( A_{G,n}^{[p]} \). Whenever a diagonal entry of \( A_{G,n}^{[p]} \) becomes numerically zero or negative,\(^4\) the corresponding entry of \( D \) is defined as \( \tau \), with \( \tau \) a positive value such that \( \tau^2 \) is equal to the smallest positive diagonal entry of \( A_{G,n}^{[p]} \).

We want to point out that the PL-matrix plays again a crucial role in the construction of the preconditioner (3.7). In particular, linear systems related to the matrix (3.7) can be efficiently solved once we have a fast solver for linear systems related to the PL-matrix, like, for instance, the one presented in Section 4. Indeed, the linear system

\[
DA_{n}^{[p]}Dv = b
\]

is equivalent to

\[
A_{n}^{[p]}v = D^{-1}b, \quad u = D^{-1}v,
\]

and the inverse of a diagonal matrix is easy to compute.

In the next example we address a problem where the geometry plays a significant role, and we also consider a singular map.

**Example 4.** Consider the problem (2.1) in the case \( d = 2 \) with \( K = I, \beta = 0, \gamma = 0 \) and \( f = 1 \). In order to test the influence of the geometry on the P-GMRES solver, we take a sequence of geometry maps \( G_\omega \), represented in terms of \( C^3 \) NURBS of bi-degree \( p_1 = p_2 = p = 4 \) and defined

\(^4\) A diagonal entry of \( A_{G,n}^{[p]} \) should not be negative in practice, but the B-spline collocation theory developed so far using Greville abscissae does not exclude this and even does not guarantee the invertibility of \( A_{G,n}^{[p]} \) in general.
Figure 3: Example 4: image and some isoparametric lines of the geometry map $G_\omega$ for different values of $\omega$.

Table 6: Example 4: asymptotic behavior of $\lambda_{\max}(\frac{1}{n^2}A_{G_1,n,n}^{[p,p]})$ and $\lambda_{\max}((A_{n,n}^{[p,p]})^{-1}A_{n,n}^{[p,p]})$ for different values of the fineness parameter $n$ and fixed spline degree $p = 4$, where $\lambda_{\max}$ denotes the largest eigenvalue and $A_{n,n}^{[p,p]}$ is the PL-matrix.
B-splines of the same degrees. To solve the corresponding linear system, the GMRES method was applied first without preconditioning, then with the PL-matrix as preconditioner, and finally with the matrix (3.7) as preconditioner. The results are collected in Table 7. In the case of the annulus domain ($\omega = 0$), we observe that both preconditioners perform equally well and the P-GMRES has an optimal convergence rate independent of $n$, whereas this is clearly not true for the simple (non-preconditioned) GMRES. In the case of the intermediate domain ($\omega = 3/4$), we see that both GMRES preconditioners produce again an optimal convergence rate, but now the preconditioner (3.7) performs better. This can be explained by the fact that the mapping of the intermediate domain presents already some similarities with the singular mapping. In the case of the square domain with a hole ($\omega = 1$), we clearly notice the effect of the singularity in the geometry map: the PL-matrix is not an optimal GMRES preconditioner anymore, although the related P-GMRES is still better than the simple GMRES. On the other hand, the preconditioner (3.7) remains optimal.

### 4. Optimal and robust multi-iterative multigrid solver for the PL-matrix

Let us consider the linear system

$$A_n^{[p]} u = b,$$

(4.1)

coming from the IgA collocation approximation of the $d$-dimensional problem (2.1) with $n := (n_1, \ldots, n_d)$, $p := (p_1, \ldots, p_d)$, in the case where $K = I$, $\beta = 0$, $\gamma = 0$, $G$ is the identity map on the parametric domain $\hat{\Omega} = [0, 1]^d$, and $f = 1$. Hence, the matrix in (4.1) is the PL-matrix. In this section we present optimal and robust two-grid and multigrid methods to solve the linear system (4.1). To be precise, we address the equivalent (normalized) system

$$\left( \frac{1}{n^2} A_n^{[p]} \right) u = \frac{1}{n^2} b.$$

(4.2)

The used machinery is very similar to the one used in [7] in the IgA Galerkin context; so we refer the reader to [7] for a detailed description of most of the tools.

#### 4.1. Two-grid

We start with explaining our two-grid method (TGM) for solving the linear system (4.2). For this we need two ingredients, which will be detailed afterwards: an iterative method (the smoother) $S$ for the solution of the linear system (4.2), and a full-rank matrix (the projector) $P \in \mathbb{R}^{l \times m}$, with $l \leq m$ and $m$ the length of the vector $u$. 

<table>
<thead>
<tr>
<th>$n \times n$</th>
<th>$\omega = 0$</th>
<th>$\omega = 3/4$</th>
<th>$\omega = 1$</th>
</tr>
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<tr>
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<td>I P DPD</td>
<td>I P DPD</td>
<td></td>
</tr>
<tr>
<td>20 $\times$ 20</td>
<td>63 15 17</td>
<td>75 48 30</td>
<td>129 116 69</td>
</tr>
<tr>
<td>30 $\times$ 30</td>
<td>95 16 17</td>
<td>103 51 32</td>
<td>210 164 62</td>
</tr>
<tr>
<td>40 $\times$ 40</td>
<td>127 16 15</td>
<td>137 53 32</td>
<td>283 212 62</td>
</tr>
<tr>
<td>50 $\times$ 50</td>
<td>158 16 15</td>
<td>169 51 32</td>
<td>407 259 60</td>
</tr>
<tr>
<td>60 $\times$ 60</td>
<td>191 16 15</td>
<td>201 50 32</td>
<td>519 294 57</td>
</tr>
</tbody>
</table>

Table 7: Example 4: number of GMRES iterations without preconditioning (I), with the PL-matrix as precondi-
tioner (P) and with the matrix (3.7) as preconditioner (DPD), for solving $A_n^{[p]} u = b$ up to a precision of $10^{-6}$, keeping the spline degree $p = 4$ fixed, and varying the fineness parameter $n$ and the geometry parameter $\omega = 0$, $\omega = 3/4$, $\omega = 1$. 

```Mathematica

def g(x):
    return 1/(1 + x**2)

dt = 0.01
x = np.linspace(-5, 5, 100)
y = g(x)
plt.plot(x, y, label='g(x)')
plt.legend()
plt.show()
```
Algorithm 4.1 (TGM). Given an approximation \( \mathbf{u}^{(k)} \) to the solution \( \mathbf{u} = A^{-1}\mathbf{b} \), the new approximation \( \mathbf{u}^{(k+1)} \) is obtained by applying a coarse-grid correction and \( \sigma \) steps of post-smoothing as follows:

1. compute the residual: \( \mathbf{r} = \mathbf{b} - A\mathbf{u}^{(k)} \);
2. project the residual: \( \mathbf{r}^{(c)} = P\mathbf{r} \);
3. compute the correction: \( \mathbf{e}^{(c)} = (PA^TP)^{-1}\mathbf{r}^{(c)} \);
4. extend the correction: \( \mathbf{e} = P^T\mathbf{e}^{(c)} \);
5. correct the initial approximation: \( \tilde{\mathbf{u}}^{(k)} = \mathbf{u}^{(k)} + \mathbf{e} \);
6. apply \( \sigma \) post-smoothing iterations: \( \mathbf{u}^{(k+\sigma)} = S^\sigma(\tilde{\mathbf{u}}^{(k)}) \).

Note that Algorithm 4.1 only considers post-smoothing for the sake of simplicity, but it is clear that one can add a convergent pre-smoother as well to improve the convergence rate of the TGM.

The specific two-grid method that we will consider for the solution of (4.2) is denoted by \( T((P\text{-GMRES})_{[p],n}) \) and is obtained from Algorithm 4.1 by choosing \( A = \frac{1}{m}A^{[p]}_{n} \), \( S = P\text{-GMRES} \), \( P = P^{[p]}_{n} \) and \( \sigma = s[p] \). More precisely, it is formed by the following ingredients:

1. a canonical coarse-grid correction, with the standard full-weighting projector

   \[
   P^{[p]}_{n} := P_{n_d+p_d-2,...,n_1+p_1-2},
   \]

   where for any \( m \in \mathbb{N}^d \) with odd components \( m_1, \ldots, m_d \geq 3 \), the matrix \( P_m \) is given by

   \[
   P_m := P_{m_1} \otimes \cdots \otimes P_{m_d},
   \]

   and

   \[
   P_m := \frac{1}{2} \begin{bmatrix}
   1 & 2 & 1 \\
   1 & 2 & 1 \\
   & \ddots & \\
   1 & 2 & 1
   \end{bmatrix} \in \mathbb{R}^{m-1 \times m}.
   \]

   According to Algorithm 4.1, we follow the Galerkin approach, so that the prolongation operator is just the transpose of the projector (4.3) and the coarse-grid correction matrix is

   \[
   I - (P^{[p]}_{n})^T \left( P^{[p]}_{n} A^{[p]}_{n} (P^{[p]}_{n})^T \right)^{-1} P^{[p]}_{n} A^{[p]}_{n};
   \]

2. \( s[p] \) post-smoothing iterations by the P-GMRES method with preconditioner

   \[
   T_{n_d+p_d-2,...,n_1+p_1-2}(h_{p_d-2} \otimes \cdots \otimes h_{p_1-2}) = T_{n_d+p_d-2}(h_{p_d-2}) \otimes \cdots \otimes T_{n_1+p_1-2}(h_{p_1-2}).
   \]

   We want to emphasize that the preconditioner (4.6) is easy to construct and is effectively solvable. To see this, we recall from \([8]\) that the \( d \)-level Toeplitz matrix (4.6) is a small-rank perturbation of the B-spline mass matrix \( M^{[p_d-2]}_{n_d+2} \otimes \cdots \otimes M^{[p_1-2]}_{n_1+2} \) related to the fineness parameters \( n + 2 \) and to the spline degrees \( p - 2 \), where \( 2 := (2, \ldots, 2) \). In particular, the Toeplitz matrix \( T_{n_d+p_d-2}(h_{p_d-2}) \) is a small-rank perturbation of the 1D B-spline mass matrix \( M^{[p_d-2]}_{n_d+2} \) and, moreover, the generic central row of \( T_{n_d+p_d-2}(h_{p_d-2}) \) coincides with the generic central row of \( M^{[p_d-2]}_{n_d+2} \). In addition, we know from \([8]\) that the central entries of \( M^{[p]}_{n} \) can be easily computed as evaluations of the cardinal B-spline \( \phi^{[p]} \), i.e.,

   \[
   (M^{[p]}_{n})_{ij} = \phi^{[p]} \left( \frac{p + 1}{2} + i - j \right), \quad i, j = p, \ldots, n - 1.
   \]
By using (4.7), one can show that
\[
(T_{n_k+p_k-2}(h_{p_k-2}))_{ij} = \phi_{[p_k-2]} \left( \frac{p_k - 1}{2} + i - j \right), \quad i, j = 1, \ldots, n_k + p_k - 2,
\]
and it follows that \(T_{n_k+p_k-2}(h_{p_k-2})\) has a banded structure, thanks to the local support of the cardinal B-spline \(\phi_{[p_k-2]}\), given by \([0, p_k - 1]\). Thus, the matrix (4.6) is a tensor-product of banded Toeplitz matrices whose entries are nothing else than evaluations of cardinal B-splines.

In addition, the proposed preconditioner (4.6) is effectively solvable. Due to the tensor-product structure and the bandedness of \(T_n\), \(k = 1, \ldots, d\), the computational cost for solving a linear system with coefficient matrix (4.6) is linear in the matrix size \(\prod_{k=1}^d (n_k + p_k - 2)\). Let us illustrate this in the 2D case. By the properties of Toeplitz matrices and tensor-products,
\[
(T_{n_2+p_2-2,n_1+p_1-2}(h_{p_2-2} \otimes h_{p_1-2}))^{-1} = (T_{n_2+p_2-2}(h_{p_2-2}))^{-1} \otimes (T_{n_1+p_1-2}(h_{p_1-2}))^{-1}.
\]
Let \(c := \text{vec}(C) \in \mathbb{R}^{(n_1+p_1-2)(n_2+p_2-2)}\) be the vector obtained by stacking the columns of the matrix \(C \in \mathbb{R}^{(n_1+p_1-2) \times (n_2+p_2-2)}\), where vec denotes the stacking operator. Then, the linear system
\[
T_{n_2+p_2-2,n_1+p_1-2}(h_{p_2-2} \otimes h_{p_1-2})\mathbf{v} = \mathbf{c}
\]
can be solved by
\[
\mathbf{v} = \text{vec} \left( (T_{n_1+p_1-2}(h_{p_1-2}))^{-1} C (T_{n_2+p_2-2}(h_{p_2-2}))^{-T} \right).
\]
This requires to solve \((n_2+p_2-2)\) linear systems with the banded Toeplitz matrix \(T_{n_1+p_1-2}(h_{p_1-2})\), plus \((n_1 + p_1 - 2)\) linear systems with the matrix \((T_{n_2+p_2-2}(h_{p_2-2}))^T\); see [11, Lemma 4.3.1]. Of course, this trick does not apply to the original system (4.2), which consists of sums of tensor-product matrices.

In Tables 8 and 9 we solved the (normalized) system (4.2) for \(d = 1, 2\), using the two-grid method \(TG((P-GMRES)^{[p]} , P_n^{[P]})\). The two-grid procedure has been started with \(u^{(0)} = 0\) and stopped at the first vector \(u^{(c)}\) whose relative residual in 2-norm is less than \(10^{-6}\); see (3.5).

Let us give a motivation for the optimal performance of our two-grid method through the symbol. We first consider the case \(d = 1\) and then we generalize the argument to the case \(d \geq 2\). We will not provide all the necessary details, since they were already described in [7, Section 5] in the (analogous) context of Galerkin IgA. For a better understanding of the following discussion, the reader is recommended to first read [7].

For \(d = 1\), the symbol of \(\{ \frac{1}{\pi^2} A_n^{[p]} \}_n\) is
\[
f_p(\theta) = (2 - 2 \cos \theta) h_{p-2}(\theta).
\]
As already pointed out, see Section 2.3.1, the symbol \(f_p\) has a unique zero at \(\theta = 0\), implying that the low frequency subspace is ill-conditioned for \(\frac{1}{\pi^2} A_n^{[p]}\). However, this ill-conditioning in low frequencies is canonical when dealing with matrices coming from the approximation of PDEs, and, in fact, it causes no problems for any standard two-grid or multigrid procedure which employs the usual full-weighting projector \(P_n^{[P]}\). Indeed, \(P_n^{[P]}\) is designed to be highly contractive in low frequencies and hence any classical two-grid or multigrid method using such a projector combined with any standard smoother (e.g. Gauss-Seidel) will have a convergence rate independent of the matrix size. However, when \(p\) is large, a numerical zero of the (normalized) symbol \(f_p/M_{f_p}\) occurs at \(\theta = \pi\) (see Lemma 2.1, Figure 1 and Table 1). Therefore, for large \(p\), also the high frequency subspace is ill-conditioned for \(\frac{1}{\pi^2} A_n^{[p]}\), and this non-canonical ill-conditioning in high frequencies is completely ignored by the full-weighting projector \(P_n^{[P]}\). This is the reason why classical two-grid
Table 8: Number of iterations $c_n^{[2]}$ needed by the two-grid $TG((P-GMRES)^{[p]}_{h_{n}})$ for solving $(1/n^2)A_{n}^{[p]}u = b/n^2$ up to a precision of $10^{-6}$. The parameter $s^{[p]}$ is specified between brackets $[\cdot]$.

<table>
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<tr>
<th>$n$</th>
<th>$c_n^{[2]}$</th>
<th>$c_n^{[4]}$</th>
<th>$c_n^{[6]}$</th>
<th>$c_n^{[8]}$</th>
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<td>81</td>
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<td>6</td>
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<td>4</td>
</tr>
<tr>
<td>161</td>
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<td>2560</td>
<td>9</td>
<td>7</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 9: Number of iterations $c_{n,n}^{[2,2]}$ needed by the two-grid $TG((P-GMRES)^{[p,p]}_{h_{n,n}})$ for solving $(1/n^2)A_{n,n}^{[p,p]}u = b/n^2$ up to a precision of $10^{-6}$. The parameter $s^{[p,p]}$ is specified between brackets $[\cdot]$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$c_{n,n}^{[2,2]}$</th>
<th>$c_{n,n}^{[4,4]}$</th>
<th>$c_{n,n}^{[6,6]}$</th>
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<table>
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<tr>
<td>100</td>
<td>8</td>
<td>7</td>
<td>7</td>
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</tr>
</tbody>
</table>

and multigrid procedures with full-weighting projector and standard Gauss-Seidel smoother have a convergence rate that, despite being independent of the matrix size, worsens with $p$.

Instead of the standard Gauss-Seidel smoother, we have chosen as smoother the P-GMRES method with preconditioner $T_{n+p-2}(h_{p-2})$, as given by (4.6) for $d = 1$. This choice is made in order to gain a $p$-independent convergence rate. Actually, we see from Table 8 that the resulting two-grid method is quite successful, since its convergence rate is independent of both $n$ and $p$. This success was not unexpected. Indeed, the idea of using the preconditioner $T_{n+p-2}(h_{p-2})$ follows from the observation that $[h_{p-2}(\theta)]^{-1}f_{p}(\theta) = 2 - 2\cos \theta$ is $p$-independent, which means that the symbol $h_{p-2}$ of the Toeplitz preconditioner $T_{n+p-2}(h_{p-2})$ ‘erases’ the numerical zero of the symbol $f_{p}$ of $\frac{1}{n}A_{n}^{[p]}$ at $\theta = \pi$; see [7, Section 5.5] for a more detailed explanation. Therefore, we expect that
- the P-GMRES alone for $\frac{1}{n^2} A_n^{[p]}$ has a convergence rate substantially independent of $p$ but worsening with $n$;

- the standard two-grid and multigrid procedures with full-weighting projector and classical smoothers (like Gauss-Seidel) have a convergence rate independent of $n$ but worsening with $p$;

- the combination of this two methods in a unique two-grid or multigrid procedure has a convergence rate independent of both $n$ and $p$, according to the multi-iterative idea introduced in [16] and summarized in [7, Sections 5.1–5.3].

In the case $d = 2$, the symbol of $\{ \frac{1}{n^2} A_n^{[p_1, p_2]} \}_n$, with $n_1 = \nu_1 n$ and $n_2 = \nu_2 n$, is

$$f^{(\nu_1, \nu_2)}_{p_1, p_2}(\theta_1, \theta_2) = (\nu_1)^2 h_{p_2}(\theta_1) f_{p_1}(\theta_2) + (\nu_2)^2 f_{p_2}(\theta_1) h_{p_1}(\theta_2)$$

$$= h_{p_2}(\theta_1) h_{p_1}(\theta_2) \left[ (\nu_1)^2 h_{p_2}(\theta_1) - 2 \cos \theta_2 \right] + (\nu_2)^2 \left[ 2 - 2 \cos \theta_1 \right] h_{p_1}(\theta_2)$$

$$= (h_{p_2} \otimes h_{p_1})(\theta_1, \theta_2) \left[ (\nu_1)^2 w_{p_2}(\theta_1) - 2 \cos \theta_2 \right] + (\nu_2)^2 \left[ 2 - 2 \cos \theta_1 \right] w_{p_1}(\theta_2),$$

where $w_{p}(\theta) := h_{p}(\theta)/h_{p-2}(\theta)$, and for general $d \geq 2$ the symbol of $\{ \frac{1}{n^2} A_n^{[p]} \}_n$, with $n = \nu n = (\nu_1 n, \ldots, \nu_d n)$, is

$$f^{(\nu)}_{p}(\theta) = \sum_{k=1}^{d} (\nu_k)^2 (h_{p_{d-k}} \otimes \cdots \otimes h_{p_{k+1}} \otimes f_{p_k} \otimes h_{p_{k-1}} \otimes \cdots \otimes h_{p_1})(\theta)$$

$$= (h_{p_{d-2}} \otimes \cdots \otimes h_{p_{1}})(\theta) \cdot \sum_{k=1}^{d} (\nu_k)^2 w_{p_{d-k}}(\theta_1) \cdots w_{p_{k+1}}(\theta_{d-k})(2 - 2 \cos \theta_{d-k+1}) w_{p_{k-1}}(\theta_{d-k+2}) \cdots w_{p_1}(\theta_{d}).$$

We see from Figure 4 that the function $w_p$ is $p$-independent, in the sense that it is uniformly bounded from above and below by two positive constants independent of $p$. Actually, it seems that $w_p$ converges uniformly to some function with range in $[0.4, 1]$. Therefore, the idea of using
the P-GMRES with preconditioner (4.6) as smoother has the same motivation as for the case $d = 1$: the preconditioned symbol
\[
\left[ (h_{p_{d-2}} \otimes \cdots \otimes h_{p_{1-2}}) (\theta) \right]^{-1} p^{(\nu)} (\theta)
\]
\[
= \sum_{k=1}^{d} \left( \nu_k \right)^2 w_{p_{d}} (\theta_1) \cdots w_{p_{d-k}} (\theta_{d-k}) (2 - 2 \cos \theta_{d-k+1}) w_{p_{d-k-2}} (\theta_{d-k+2}) \cdots w_{p_{2}} (\theta_d)
\]
is $p$-independent. This implies that the numerical zeros of $f_p^{(\nu)}$ at the $\pi$-edge points (2.27) are completely 'erased' by the symbol $h_{p_{d-2}} \otimes \cdots \otimes h_{p_{1-2}}$ of the Toeplitz preconditioner (4.6), and this motivates why such a preconditioner is effective. We refer the reader again to [7, Section 5.5] for a more detailed explanation.

### 4.2. Multigrid: V-cycle and W-cycle

In practice, the coarser linear system of the TGM could be too large to be solved directly. Hence, the third step in Algorithm 1 is usually replaced by one recursive call, obtaining a multigrid V-cycle algorithm, or by two recursive calls, obtaining a multigrid W-cycle algorithm. In the following, we focus on the V-cycle and W-cycle multigrid algorithms that we will use for solving the linear system (4.2). They are formed by:

1. standard coarse-grid corrections at each level, which use the full-weighting projector (4.4), with properly adjusted size, as restriction operator, and the transpose of the projector as prolongation operator;
2. $s[p]$ post-smoothing iterations by the P-GMRES with preconditioner (4.6) at the finest level, and a single standard Gauss-Seidel post-smoothing iteration at all the other levels.

Let us now describe in more details our V-cycle and W-cycle algorithms in the case where $\mathbf{n} = (n, \ldots, n)$ and $\mathbf{p} = (p, \ldots, p)$, because this is the choice of $\mathbf{n}$, $\mathbf{p}$ considered in the numerical experiments. We denote by index 0 the finest level and by index $\ell[\mathbf{p}] := \log_2 (n + p - 1) - 1$ the coarsest level. Let $A_{n,i}^{[\mathbf{p}]}$ be the matrix at level $i$, whose size is $(m_{n,i})^d$, $0 \leq i \leq \ell[\mathbf{p}]$, with $m_{n,i}^{[\mathbf{p}]} := \frac{n+p-1}{2^i} - 1$. In this notation, we have $A_{n,0}^{[\mathbf{p}]} := \frac{1}{\sin^2 \theta_1} A_n^{[\mathbf{p}]}$ and
\[
A_{n,i+1}^{[\mathbf{p}]} := P_{n,i}^{[\mathbf{p}]} A_{n,i}^{[\mathbf{p}]} (P_{n,i}^{[\mathbf{p}]})^T, \quad i = 0, \ldots, \ell[\mathbf{p}] - 1,
\]
where
\[
P_{n,i}^{[\mathbf{p}]} := P_{m_{n,i}^{[\mathbf{p}]}} \otimes \cdots \otimes P_{m_{n,i}^{[\mathbf{p}]}}, \quad i = 0, \ldots, \ell[\mathbf{p}] - 1,
\]
is the full-weighting projector at level $i$, and $P_m$ is defined in (4.5). Regarding the smoother, at each coarse level $i \geq 1$ we used one single post-smoothing iteration with the standard Gauss-Seidel method; at the finest level $i = 0$ we used $s^{[\mathbf{p}]}$ post-smoothing iterations by the P-GMRES with preconditioner (4.6). At each level $i$, we first performed a coarse-grid correction, with one recursive call in the V-cycle and two recursive calls in the W-cycle, and then we applied one post-smoothing iteration by the Gauss-Seidel method if $i \geq 1$, or $s^{[\mathbf{p}]}$ post-smoothing iterations by the P-GMRES with preconditioner (4.6) if $i = 0$.

We observe that these V-cycle and W-cycle essentially coincide with those considered in [7, Section 9] for the IgA Galerkin PL-matrix, with the only difference that now, at the finest level, we use a P-GMRES smoother instead of a PCG smoother, because of the non-symmetry of the IgA collocation PL-matrix.

In Tables 10, 11, 12 we solved the (normalized) system (4.2) for $d = 1, 2, 3$, using the V-cycle and W-cycle multigrid procedure described above. We see that the number of V-cycle and W-cycle iterations for reaching the preassigned accuracy $10^{-6}$ is substantially independent of all the
Table 10: Number of iterations $c_{n,n}^{[p]}$ needed for solving $(1/n^2)A_{n,n}^{[p]}u = b/n^2$ up to a precision of $10^{-6}$, when using the multigrid cycle with $s^{[p]}$ post-smoothing steps by the P-GMRES at the finest level and 1 post-smoothing step by standard Gauss-Seidel at the coarse levels. The parameter $s^{[p]}$ is specified between brackets [·]. The methods have been started with $u^{(0)} = 0$ and stopped at the first term $u^{(c)}$ satisfying the relative criterion (3.5). For each pair $(p,n)$, the first entry corresponding to $c_{n,n}^{[p]}$ refers to the V-cycle, the second entry to the W-cycle.

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Table 11: Number of iterations $c_{n,n}^{[p,p]}$ needed for solving $(1/n^2)A_{n,n}^{[p,p]}u = b/n^2$ up to a precision of $10^{-6}$, when using the multigrid cycle with $s^{[p,p]}$ post-smoothing steps by the P-GMRES at the finest level and 1 post-smoothing step by standard Gauss-Seidel at the coarse levels. The parameter $s^{[p,p]}$ is specified between brackets [·]. The methods have been started with $u^{(0)} = 0$ and stopped at the first term $u^{(c)}$ satisfying the relative criterion (3.5). For each pair $(p,n)$, the first entry corresponding to $c_{n,n}^{[p,p]}$ refers to the V-cycle, the second entry to the W-cycle.

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Table 11: Number of iterations $c_{n,n}^{[p,p]}$ needed for solving $(1/n^2)A_{n,n}^{[p,p]}u = b/n^2$ up to a precision of $10^{-6}$, when using the multigrid cycle with $s^{[p,p]}$ post-smoothing steps by the P-GMRES at the finest level and 1 post-smoothing step by standard Gauss-Seidel at the coarse levels. The parameter $s^{[p,p]}$ is specified between brackets [·]. The methods have been started with $u^{(0)} = 0$ and stopped at the first term $u^{(c)}$ satisfying the relative criterion (3.5). For each pair $(p,n)$, the first entry corresponding to $c_{n,n}^{[p,p]}$ refers to the V-cycle, the second entry to the W-cycle.

relevant parameters: $p, n, d$. In particular, the convergence rate of the W-cycle is practically constant (the number of iterations is always around 8). The only unpleasant fact is that the number of P-GMRES post-smoothing steps $s^{[p,...,p]}$ needed for keeping a fixed number of W-cycle iterations around 8 slightly increases when $p$ and $d$ increase. However, we should also say that, if we decrease $s^{[p,...,p]}$ a little bit, the number of iterations does not increase so much. For instance,
Table 12: Number of iterations $c_{[p,n,n]}^{[p,p,p]}$ needed for solving $(1/n^2)A_{n,n,n}^{[p,p,p]} u = b/n^2$ up to a precision of $10^{-6}$, when using the multigrid cycle with $s^{[p,p,p]}$ post-smoothing steps by the P-GMRES at the finest level and 1 post-smoothing step by standard Gauss-Seidel at the coarse levels. The parameter $s^{[p,p,p]}$ is specified between brackets [ ]. The methods have been started with $u^{(0)} = 0$ and stopped at the first term $u^{(c)}$ satisfying the relative criterion (3.5). For each pair $(p,n)$, the first entry corresponding to $c_{n,n,n}^{[p,p,p]}$ refers to the V-cycle, the second entry to the W-cycle.

if in Table 12 we chose $s^{[9,9,9]} = 9$ (instead of $s^{[9,9,9]} = 12$), then the resulting number of W-cycle iterations $c_{n,n,n}^{[9,9,9]}$ for $n = 56$ would be 12 (instead of 8).

5. Conclusion and perspectives

By following the multi-iterative approach [16] and by using the knowledge of the symbol [8], we have designed a fast multi-iterative strategy for the large linear systems arising from the IgA collocation approximation based on B-splines of classical $d$-dimensional full elliptic problems. The strategy consists of two steps:

1. an external solver, which is the P-GMRES method considered in Section 3, whose preconditioner is the PL-matrix or the matrix (3.7) based on the PL-matrix;
2. an internal multi-iterative multigrid solver, i.e. the V-cycle or W-cycle considered in Section 4, for the fast solution of a linear system associated with the PL-matrix (or, more precisely, its normalized version $\frac{1}{n^2}A_{n_1,...,n_d}$). This uses the standard full-weighting projector (4.4) at each level, a few post-smoothing iterations by the P-GMRES method with preconditioner $\bigotimes_{j=1}^d T_{n_d,...,n_1} h_{p_j}$ at the finest level, and a single post-smoothing iteration by the standard Gauss-Seidel method at all other levels.

The main features of this multi-iterative solver are the following:

1. it has an optimal global cost, i.e., the overall number of operations is proportional to the number of degrees of freedom;
2. it is robust, i.e., its convergence speed is substantially independent or only mildly dependent on all the relevant parameters, such as the coefficients of the PDE, the dimensionality $d$, the geometric map $G$ describing the physical domain, the matrix size (related to the fineness parameters), and the spline degrees (associated with the IgA approximation order).

A completely analogous two-step strategy has proved to be also very effective in the IgA Galerkin context [7]. This can be explained through the similarities of the spectral distributions of the involved (normalized) matrices in the Galerkin and collocation cases. Indeed, in the simplified setting where $K$ is the identity matrix and $\Omega = (0,1)^d$, we know that the same symbol is shared in both cases; see [8] and Remark 2.1. Moreover, there is a well rooted conjecture that both symbols are also similar in the fully general elliptic setting.
A further direction of research would be to consider several theoretical issues related to the rigorous proofs of the optimal and robust convergence of the proposed solver, which is quite nontrivial, given the rich structure of the considered algorithms and the non-symmetry of the involved matrices.

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