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Quantum Accelerator Modes: effects of atomic interactions and higher-order resonances.

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Outline.

1. **Quantum Resonance in a pulsed system:**
- Resonance and localization
- Translational invariance and Bloch Theorem
- Connection with cold atom experiments: Quantum Accelerator Modes (QAMs)

2. **Effects of atomic interactions on QAMs:**
- Theoretical model and “pseudo-classical” theory
- Model with interactions: Gross-Pitaevskii equation
- Dynamics of the wave packet: weak and strong nonlinearity
- Non-condensed particles

3. **QAMs near higher-order resonances:**
- Spinor dynamics at exact resonance
- Near-resonant motion in presence of gravity
- QAMs near the resonance $\tau=\pi$

Quantum Resonance and Localization in kicked dynamics.

$$\hat{H}(t') = \frac{\hat{p}^2}{2} + kV(\hat{\theta}) \sum_{t=-\infty}^{t=+\infty} \delta(t' - t\tau) \rightarrow \text{Floquet Operator} \quad \hat{U} = e^{-ikV(\hat{\theta})} \cdot e^{-i\tau \frac{\hat{n}^2}{2}}$$

$(\hbar = 1)$ $\quad \hat{n} = -i\partial_\theta$

kick

free evolution of duration τ

The Kicked Rotator

It is the most popular and simplest model of kicked dynamics.

$$V(\theta) = \cos\theta$$

QUANTUM RESONANCE

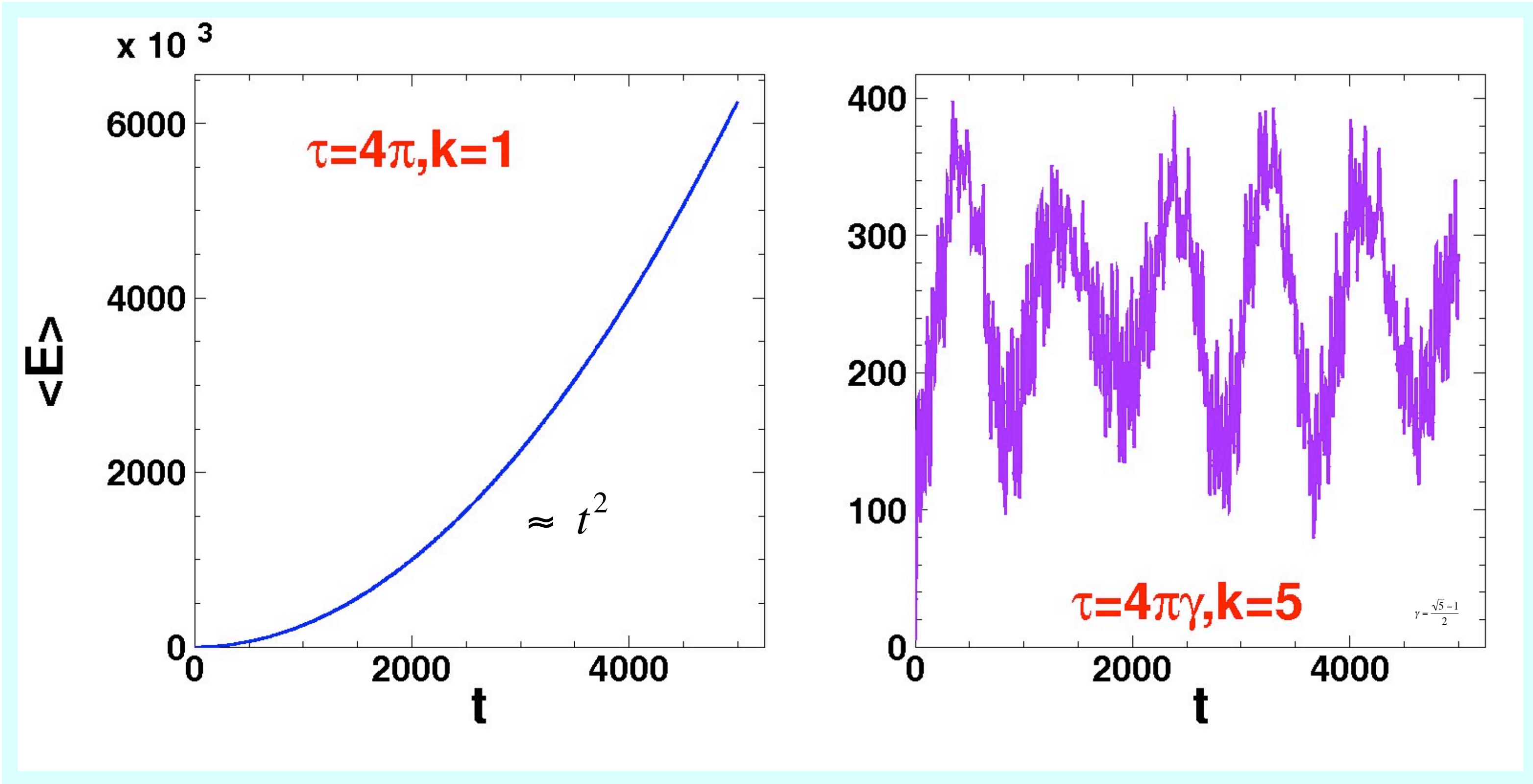
$\tau = 2\pi \frac{p}{q} \qquad p, q \in \mathbf{N}_0$

p, q : coprime integers

→ Principal Q.R. $\quad \tau = 2\pi p$

LOCALIZED REGIME

$\frac{\tau}{2\pi} \in \mathbf{R} - \mathbf{Q}$



Translational invariance and Bloch Theorem

Translational invariance in position

Periodicity in space $\hat{T}_\ell = e^{-i\ell\hat{n}}$ $\quad \hat{T}_\ell V(\theta) = V(\theta+\ell)\hat{T}_\ell \quad \psi(\theta+\ell) = e^{i\lambda}\psi(\theta) \quad \lambda \in \mathbf{R} \quad \lambda = \beta\cdot\ell$

$\left[\hat{T}_\ell, \hat{U}\right] = 0 \Rightarrow$ Conservation of eigenvalue $\lambda \Rightarrow$ Conservation of vector β in reciprocal space $\rightarrow$ First Brillouin Zone $0 \le \beta < \frac{2\pi}{\ell}$

$$V(\theta) = V(\theta+2\pi) \qquad \hat{T}_{2\pi} = e^{-i2\pi\hat{n}} \Rightarrow \left[\hat{T}_{2\pi}, \hat{U}\right] = 0 \Rightarrow \text{ Conservation of quasi-momentum } \beta$$

fractional part of momentum $\quad p = [p] + \{p\} = n + \beta = -i\frac{d}{d\theta} + \beta \qquad n \in \mathbf{Z} \quad 0 \le \beta < 1$

Bloch-Wannier fibration -- Kicked Particle Model:
The wavefunction can be written as a superposition of Bloch waves.
Particle dynamics is decomposed in a bundle of rotors with fixed quasi-momentum **β**, evolving independently.

$$\psi(x) = \int_0^1 d\beta e^{i\beta x}\varphi_\beta(x) \qquad \psi(x,t) = \hat{U}^t\psi(x) = \int_0^1 d\beta e^{i\beta x}\hat{U}_\beta^t\varphi_\beta(x) \qquad \hat{U}_\beta = e^{-ik\cos\hat{\theta}}\cdot e^{-i\frac{\tau}{2}(\hat{n}+\beta)^2}$$

$$\varphi_\beta(x) = \varphi_\beta(x+2\pi)$$

Translational invariance in momentum at Quantum Resonance

$$\tau = 2\pi\frac{p}{q}$$

Resonance condition in the Kicked Rotator Model : The Floquet operator U commutes with a non trivial subgroup $\mathbf{T}_l$ of the discrete unitary group of momentum translations, generated by multiplication by $e^{i\theta}$ in $L^2(\mathbf{t})$ (**t** denotes the 1-torus R/(2πZ))

$$\hat{\mathbf{T}}_\ell = e^{i\ell\hat{\theta}} \qquad e^{i\ell\hat{\theta}}\hat{\psi}(n) = \hat{\psi}(n-\ell)$$

order of the resonance = minimum index of the subgroup, the least positive integer such that

$$\left[\hat{U}, e^{i\ell\hat{\theta}}\right] = 0 \Rightarrow \text{ Conservation of Bloch phase } \xi \equiv \theta \left(\text{mod } \frac{2\pi}{\ell}\right)$$

General Quantum Resonance conditions

Kicked $\beta-$Rotor $\quad \hat{U}_\beta = e^{-ik\cos\hat\theta}\cdot e^{-i\frac{\tau}{2}(\hat n+\beta)^2}$

Resonant set of quasi-momenta:

$$\left[\hat U_\beta, e^{i\ell\hat\theta}\right]=0 \Rightarrow e^{-i\frac{\tau}{2}(n+\beta-\ell)^2} = e^{-i\frac{\tau}{2}(n+\beta)^2} \Rightarrow \frac{\tau}{2}\ell^2-\tau(n+\beta)\equiv 0 \qquad \text{mod } 2\pi$$

$$\tau = 2\pi\frac{p}{q} \qquad \ell = mq \qquad \beta_{res} = \frac{\nu}{mp}+\frac{mq}{2} \qquad m\in\mathbf{N}, \nu\in\mathbf{Z} \qquad \to \beta_r \in \beta_{res} \begin{cases} 0 & \text{q even} \\ 1/2 & \text{q odd}\end{cases}$$

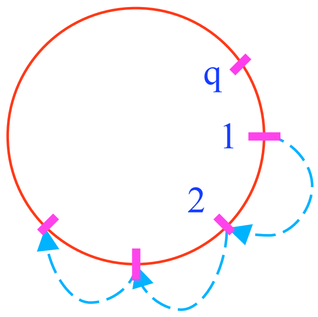
Principal Quantum Resonances : $q=1 \qquad \tau = 2\pi p$

Higher Order Quantum Resonances $\qquad \tau = 2\pi\frac{p}{q} \qquad p,q$ coprime

Basic formula at resonance

Using Poisson's summation formula $\sum_{l=-\infty}^{+\infty} e^{i2\pi\phi l} = \sum_{j=-\infty}^{+\infty}\delta(\phi-j)$

and the resonance conditions for a resonant quasi-momentum:



$$e^{-i\frac{\tau}{2}\left(\hat n+\beta_r\right)^2}\psi(\theta) = e^{-i\frac{\pi}{q}p\left(-i\partial_\theta+\beta_r\right)^2}\psi(\theta) = \sum_{j=0}^{q-1} G(p,q,j,\beta_r)\,\psi\left(\theta-2\pi\frac{j}{q}\right)$$

$$\tau = 2\pi p/q \qquad \beta=\beta_r\in\beta_{res} = \left\{\frac{r}{p}+\frac{q}{2}\right\}_{0\le r<p} \qquad G(p,q,j,\beta_r) = \frac{1}{q}\sum_{l=0}^{q-1} e^{-i\frac{\pi}{q}p\left(l+\beta_r\right)^2} e^{i\frac{2\pi}{q}lj}$$

For example

$$q=2 \qquad p=1 \qquad \beta_r = 0$$

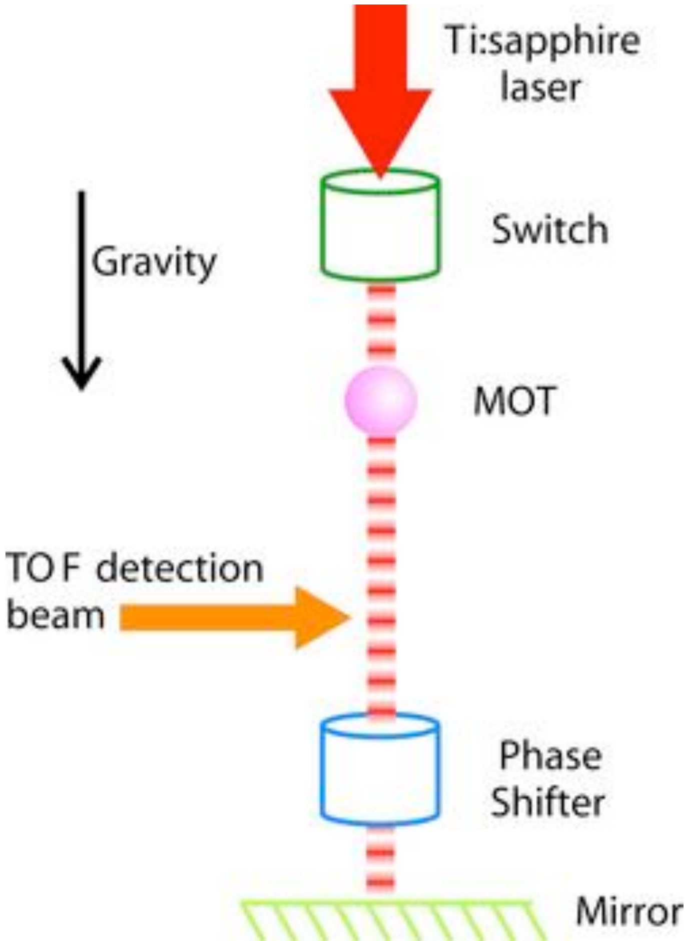
$$\to\to = \frac{1}{\sqrt 2}e^{i\frac{\pi}{4}}\left(\psi(\theta-\pi)-\psi(\theta)\right)$$

Quantum Accelerator Modes near principal resonances

The Quantum Optics experiment and the discovery of Quantum Accelerator Modes.

The experiment was realized at Oxford University by the **Atom Optics Group** in **1999**.
It belongs to the experimental line pioneered by the group of Mark Raizen (1995), in which new quantum optic techniques were used to laser-cool and manipulate gases of alkali atoms.
The paradigmatic abstract model of the Quantum Chaos – the **Kicked Rotator** – was experimentally realized.
The introduction of variants of this original model (such as the **gravity** in Oxford experiments) was able to produce new unexpected phenomena.

The **atom interferometer** was oriented **parallel** to the direction Earth's acceleration:



10^7 (Sodium) Cesium atoms — temperature $\approx 5\mu K$

$G \approx 1.4 \cdot 10^7\ \mathrm{m}^{-1}$ ($\lambda \approx 500$ nm) — $I_{\max} = \frac{2\hbar}{\delta T} K_{\max} \approx 5 \cdot 10^4\ \frac{mW}{cm^2}$

pulse period $30\ \mu s \leq T \leq 200\ \mu\, s$ — $\omega \neq$ any atomic transition

pulse duration $\delta T \leq 500$ n s

distance MOT - detector $\approx$ 50 cm

$$\hat{H}(t') = \frac{\hat{P}^2}{2m} - mg\hat{X} + K\cos(G\hat{X}) \sum_{t=-\infty}^{t=+\infty} \delta(t' - tT)$$

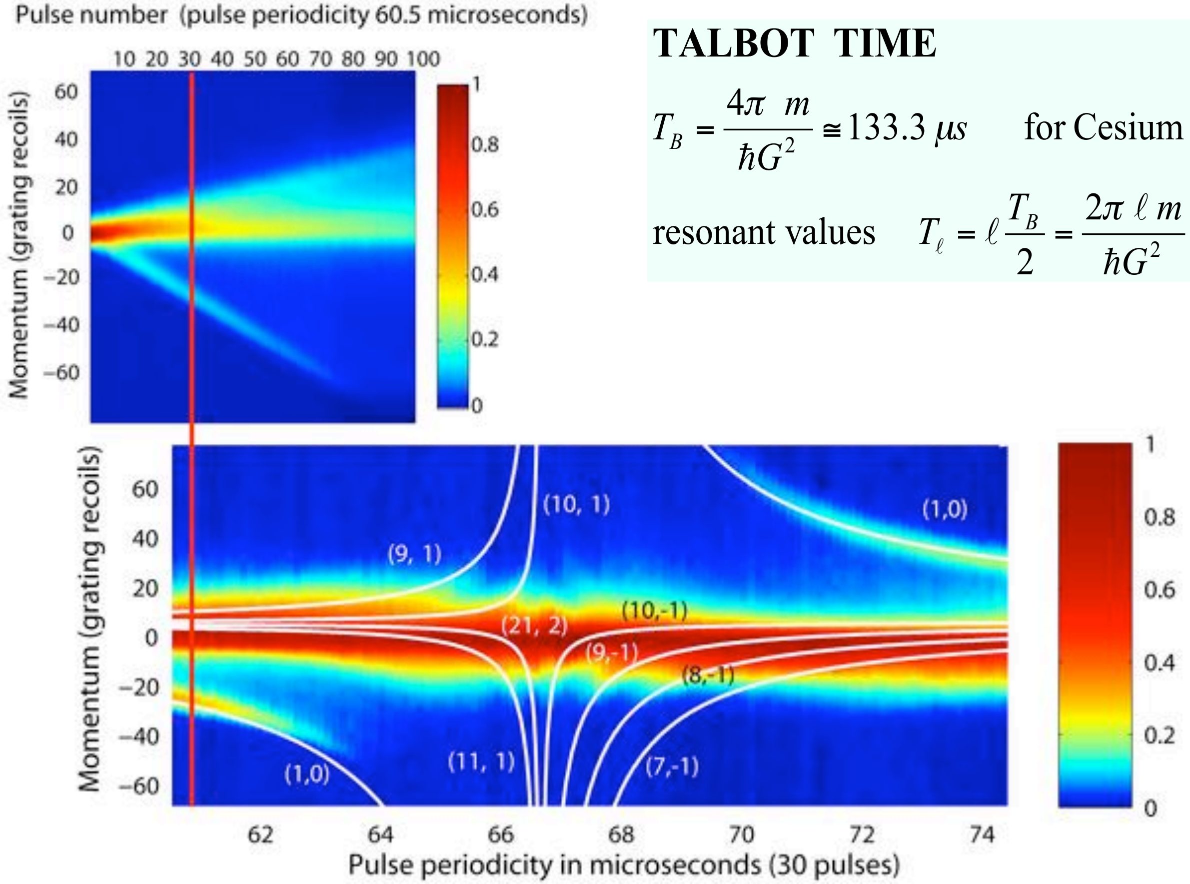
m = mass

$2\pi\, G^{-1}$ = spatial periodicity of the kicking potential

$\hbar$G = grating recoil = 2 photons of the driving wave

The Quantum Accelerator Modes. [1]

The Quantum Accelerator Modes (QAMs) are a manifestation of a novel type of **quantum ballistic transport** in momentum and manifest themselves as a **portion of the atoms accelerating coherently** away from the main cloud, in a manner strongly dependent on the effect of gravity.



TALBOT TIME

$$T_B = \frac{4\pi\ m}{\hbar G^2} \cong 133.3\ \mu s \qquad \text{for Cesium atoms}$$

resonant values $\quad T_\ell = \ell \frac{T_B}{2} = \frac{2\pi\ \ell\ m}{\hbar G^2} \cong \ell \times 66.7\ \mu s \qquad \ell = 1,2,3,\ldots$

Connections with the experiments:

$$\tau_{res} = 2\pi\ \ell \rightarrow T_\ell = \ell \frac{T_B}{2}$$

$\ell = 1$

[1] R. M. Godun, M. B. d'Arcy, M. K. Oberthaler, G. S. Summy,and K. Burnett, Phys. Rev. A **62**, 013411 (2000);
M. B.d'Arcy, R. M. Godun, M. K. Oberthaler, G. S. Summy, K.Burnett, and S. A. Gardiner, Phys. Rev. E 64, 056233 (2001);
S. Schlunk, M. B. d'Arcy, S. A. Gardiner, D. Cassettari, R. M.Godun, and G. S. Summy, Phys. Rev. Lett. 90, 054101 (2003);
S. Schlunk, M. B. d'Arcy, S. A. Gardiner, and G. S. Summy, Phys. Rev. Lett. 90, 124102 (2003);
Z.-Y. Ma, M. B. d'Arcy, and S. A.Gardiner, Phys. Rev. Lett. **93**, 164101 (2004);
M. B. d'Arcy, G. S. Summy, S. Fishman and I. Guarneri, Physica Scripta **69**, C25 (2004).
A. Buchleitner, M. B. d'Arcy, S. Fishman, S. A. Gardiner, I. Guarneri, Z.-Y. Ma, L. Rebuzzini and G. S. Summy, Phys. Rev. Lett. **96**, 164101 (2006).

The theoretical model and the "pseudo-classical" theory. [2]

It is a variant of the well-known problem of the **kicked particle** on the line (the momentum is not a discrete).
The potential gravity energy breaks the **spatial periodicity** of the system.
The broken invariance respect to spatial translations (**of 2π**) is recovered passing to a **"temporal gauge"**, in which the momentum of the particle is measured relative to that of the free-fall (gauge transformation $e^{igxt'}$):

$$\hat{H}(t') = \frac{\hat{P}^2}{2m} - mg\hat{X} + K\cos(G\hat{X})\sum_{t=-\infty}^{t=+\infty}\delta(t'-tT) \longrightarrow \hat{H}_g(t') = \frac{1}{2}\left(\hat{p}+gt'\right)^2 + k\cos\hat{x}\sum_{t=-\infty}^{t=+\infty}\delta(t'-t\tau)$$

$k = \frac{K}{\hbar}, \quad \tau = \frac{\hbar T G^2}{m}, \quad \eta = \frac{mg'T}{\hbar G}, \quad g = \frac{\eta}{\tau}$ (rescaled units of $\hat{p},\hat{x}$ and m : $\hbar G$, G^{-1} and m)

Bloch-Wannier fibration:
Particle dynamics is decomposed in a bundle of rotors with fixed quasi-momentum **β**, evolving independently.

$$\varphi_\beta(x) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbf{Z}}\langle n+\beta|\psi\rangle e^{inx} \qquad \psi(x,t) = \hat{U}^t\psi(x) = \int_0^1 d\beta e^{i\beta x}\hat{U}_\beta^t\varphi_\beta(x)$$

QAMs arise in the vicinity of a **resonant** value of the period: $\tau = 2\pi\ell + \varepsilon \qquad \ell = 1,2,3,\ldots$

"Pseudo-classical" limit (ε→0): the role of the Planck constant is played by the detuning **ε** from the resonance.

$$\hat{U}_\beta(t) = e^{-ik\cos\hat{\theta}}e^{-i\frac{\tau}{2}\left(n+\beta+\eta t+\frac{\eta}{2}\right)^2} \xrightarrow[n=I/|\varepsilon|]{\tau=2\pi l+\varepsilon,\ k=\tilde{k}/|\varepsilon|} \hat{U}_\beta(t) = e^{-\frac{i}{|\varepsilon|}\tilde{k}\cos\hat{\theta}}\ e^{-\frac{i}{|\varepsilon|}\hat{H}_\beta(\hat{I},t)}$$

where

$$\hat{I} = |\varepsilon|\hat{n} = -i|\varepsilon|\frac{d}{d\theta} \qquad \hat{H}_\beta(\hat{I},t) = \frac{1}{2}\mathrm{sign}(\varepsilon)\hat{I}^2 + \hat{I}\left(\pi l + \tau\left(\beta + t\eta + \frac{\eta}{2}\right)\right).$$

[2] S. Fishman, I. Guarneri and L. Rebuzzini, Phys. Rev. Lett. **89**,084101 (2002); S. Fishman, I. Guarneri and L. Rebuzzini, J. Stat. Phys. **110**, 911 (2003).

"Pseudo-classical" map.

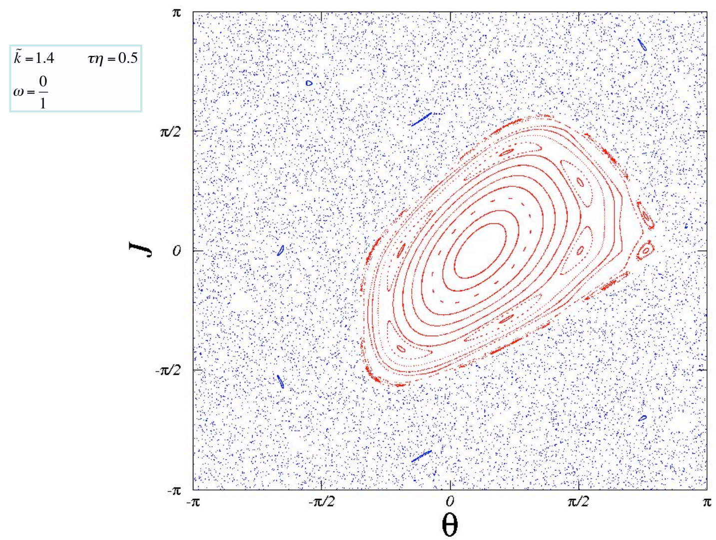
If $|\varepsilon|$ is assigned the role of the **Planck's constant**, it is the formal quantization of a classical system, described by the time-dependent Hamiltonian

$$H = H_\beta + \tilde{k}\cos\theta \sum_{t\in\mathbf{Z}} \delta(t'-t)$$

The explicit time-dependence is removed by a change in momentum variable:

$$I \to J = I \pm \pi\ell \pm \tau\left(\beta + \eta t + \frac{\eta}{2}\right)$$

$$\begin{cases} \theta_{t+1} = \theta_t \pm J_t & \text{mod } 2\pi \\ J_{t+1} = J_t + \tilde{k}\sin\theta_{t+1} \pm \tau\eta \end{cases}$$



$$\tilde{k} = k\,|\varepsilon| \qquad\qquad \eta = g\tau$$

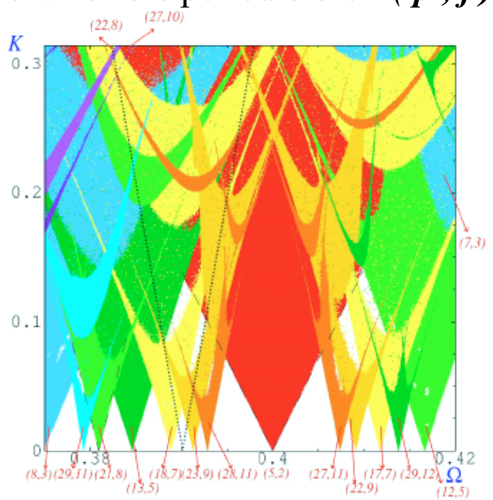
t = time measured in number of periods

The QAMs are shown to correspond to the **stable periodic orbits** of the **area-preserving** map.
To each stable periodic orbit ***(p , j)***, a **rational winding number** in the J variable, can be associated:

$$w = \frac{j}{p}$$

$j \in \mathbf{Z}$: jumping index (number of revolutions of J)

$p \in \mathbf{N}_0$: period of the orbit



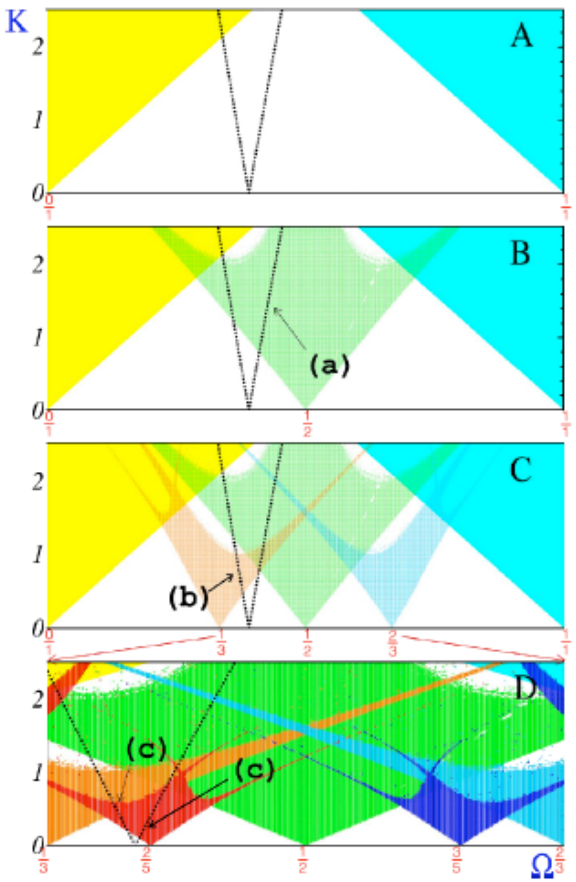
The **acceleration of the mode in momentum** is:

$$a_0 = \frac{2\pi}{|\varepsilon|}\frac{j}{p} - \frac{\tau\eta}{\varepsilon}$$

Arnol'd Tongues and arithmetic of the Accelerator Modes: Farey rule [3].

The accumulation point of the sequence of winding number is $\Omega^* = 2\pi\ell^2 g \qquad \ell = 1,2,3,\ldots$

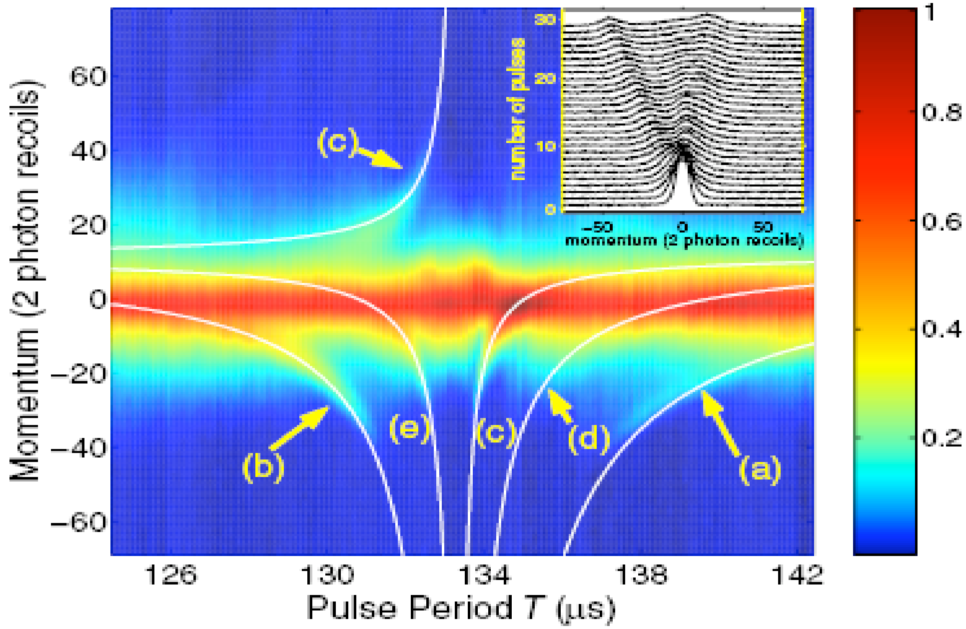
The sequence in constructed with the Farey Rule: $\frac{j}{p} = \frac{j_1}{p_1} \oplus \frac{j_2}{p_2} = \frac{j_1 + j_2}{p_1 + p_2}$ with $\frac{j_1}{p_1} \le \frac{j}{p} \le \frac{j_2}{p_2}$



FAREY SERIE

$\Omega^* = 0.3902 \quad (\ell = 2)$

$\left(\frac{0}{1},\frac{1}{1}\right)$	$\frac{0}{1}$	
$\left(\frac{0}{1},\frac{1}{2}\right)$	$\frac{1}{2}$	a
$\left(\frac{1}{2},\frac{1}{3}\right)$	$\frac{1}{3}$	b
$\left(\frac{1}{3},\frac{2}{5}\right)$	$\frac{2}{5}$	c
$\left(\frac{2}{5},\frac{3}{8}\right)$	$\frac{3}{8}$	d
$\left(\frac{2}{5},\frac{5}{13}\right)$	$\frac{5}{13}$	e
$\left(\frac{2}{5},\frac{7}{18}\right)$	$\frac{7}{18}$	
$\left(\frac{2}{5},\frac{9}{23}\right)$	$\frac{9}{23}$	



[3] I.Guarneri, L.Rebuzzini and S.Fishman, Nonlinearity **19**, 1141 (2006).

A. Buchleitner, M. B. d'Arcy, S. Fishman, S. A. Gardiner, I. Guarneri, Z.-Y. Ma, L. Rebuzzini and G. S. Summy, Phys. Rev. Lett. **96**, 164101 (2006).

Effects of atomic interactions on Quantum Accelerator Modes

The model with atomic interactions: the Nonlinear Schrödinger equation (Gross-Pitaevskii).

In a mean-field approach, the interactions in atomics gases can be well reproduced by a single nonlinear Schrödinger equation for the **condensate wave function**, i.e. the **Gross-Pitaevskii equation**.

The **effective potential** consists in an additional **nonlinear** term, proportional to the **square modulus** of the wave function. The condition of validity are:

1. **Low density** ($\rho<10^{15}$ atoms/cm^3) : only elastic binary s-wave collisions are relevant.
2. **Ultracold temperature** ($T \ll T_c$; $T \leq 1\mu K$)
3. The **thermalized fraction** is negligible (regime of almost pure condensate).

The **1-dimensional nonlinear Schrödinger equation** for the ensemble of interacting atoms in the temporal gauge, expressed in rescaled units, is:

$$i\dot{\psi}(x,t') = \left[\frac{1}{2}\left(p+gt'\right)^2 + k\cos\hat{x}\sum_{t=-\infty}^{t=+\infty}\delta(t'-t\tau) + u\,|\psi(x,t')|^2\right]\psi(x,t')$$

The **effective 1-dimensional coupling constant** for atoms confined transversally by an atom waveguide or a cigar-shaped atomic trap:

$$u' = 2N\frac{\hbar^2 a_0'}{m a_\perp'} \quad \rightarrow \quad u = 2N\frac{a_0}{a_\perp}$$

a_0' : 3-dim scattering length

$a_\perp'$: radial extension $\qquad a_\perp' = \sqrt{\hbar/m\omega_\perp} \gg a_0'$

Connection with the experiments [4] (^{87}Rb atom condensate with repulsive interaction)**:**

$N = 50000 \quad \lambda = 390\,\text{nm} \quad a_0' = 110 a_B = 110\cdot 0.53\cdot 10^{-10}\,\text{m} \quad a_\perp' \approx \text{waistbeam} \approx 12\mu\text{m}$

$N \approx 10^5 \cdot u \Rightarrow \qquad u \approx 0.5 \rightarrow N = 50000$

Numerical integration method: time-splitting spectral method [5]

[4] G. Behinaein, V. Ramareddy, P. Ahmadi and G. S. Summy, Phys. Rev. Lett. **97**, 244101 (2006)
[5] R. Artuso and L. Rebuzzini, Phys.Rev.E **66**,017203 (2002);
R. Artuso and L. Rebuzzini, Phys.Rev.E **68**, 036221 (2004);
L. Rebuzzini, S. Wimberger and R. Artuso, Phys.Rev.E **71**, 036220 (2005).

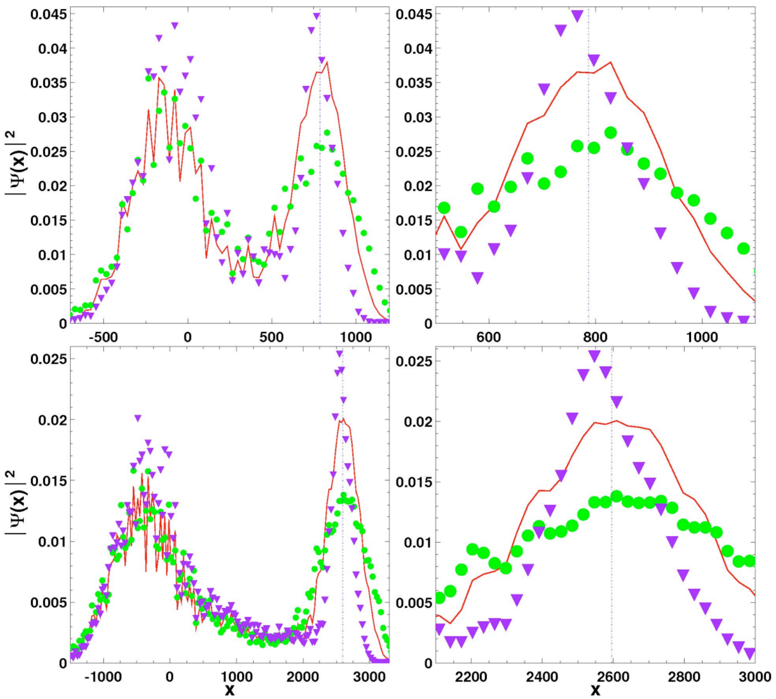
Sign of the coupling constant u:

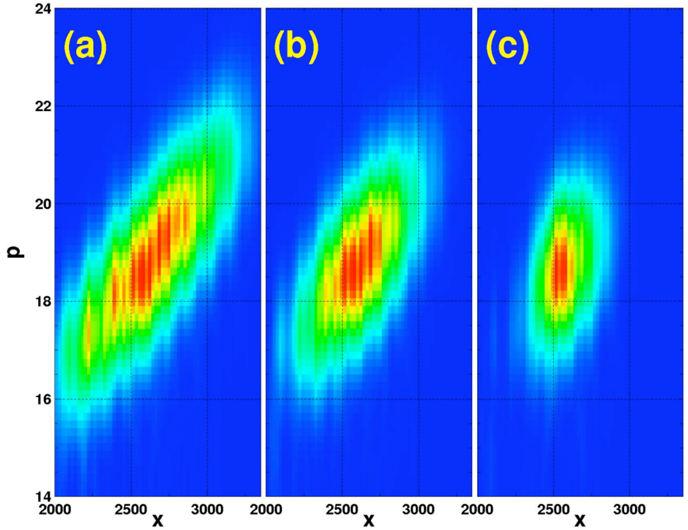
u<0 **focusing nonlinearity (attractive interactions)**

u>0 **defocusing nonlinearity (repulsive interactions)**

u= - 1.25   u=0   u=1.25

WEAK ATOMIC INTERACTIONS.

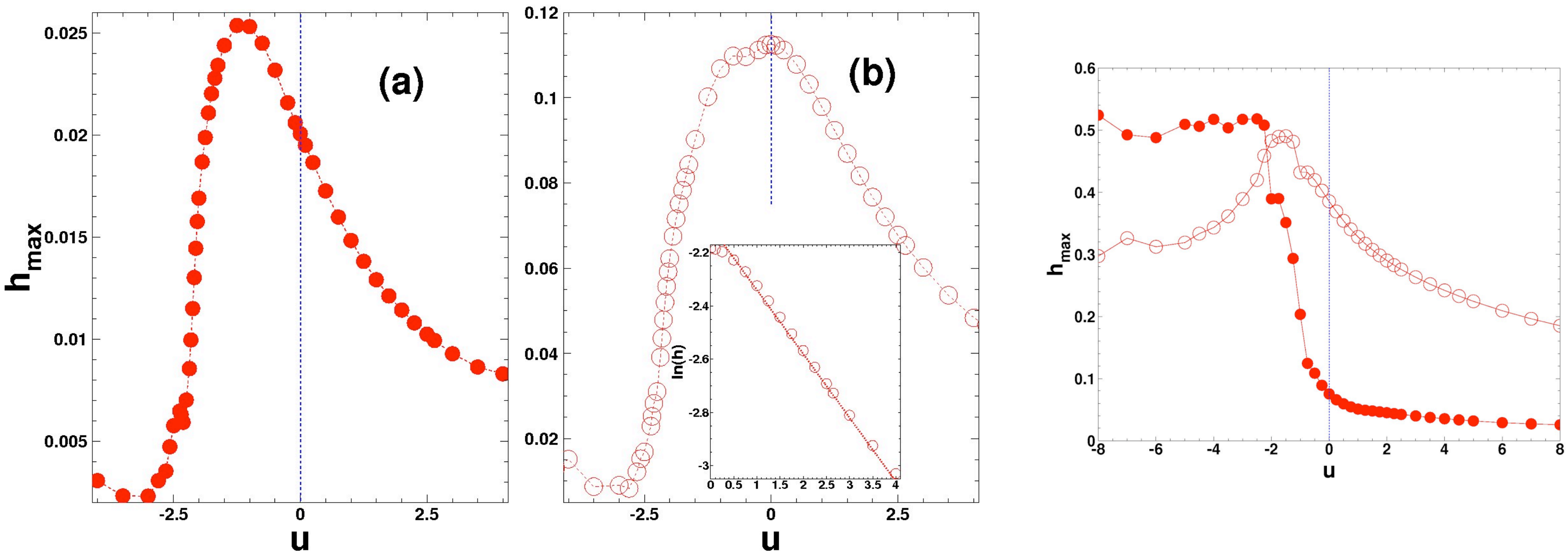




The spreading of the wave-function is wider (narrower) and the density probability is lower (higher) in presence of **repulsive** (**attractive**) interactions compared to the **linear** case.

initial state: a symmetric Gaussian wave packet centered in the stable fixed point of the "pseudo-classical" map of the correspondent linear system, i.e. ($x_0 \simeq 0.3027$; $p_0 = 0$), with a null jumping index. The value of the chosen parameters are $k = 1.4$, $\varepsilon = -1$, $\tau = 2\pi + \varepsilon \simeq 5.2832$, $\tau\eta \simeq 0.4173$, $g = \eta/\tau \simeq 0.0149$.

STRONG ATOMIC INTERACTIONS.



Repulsive interactions:

•The spreading of the distribution and the peak damping depend **monotonically** on the nonlinear strength.
•Exponential decrease of height of the mode

Attractive interactions:

•**Enhancement** of the mode only for small nonlinearity
•**Suppression** of the mode for larger nonlinearity
•Strong focusing nonlinearity opposes to the separation of the wave packet into two part (see **exact resonant case**)
•Behavior observed for a variety of **other parameter** choice
•Dependence on **initial state**.

Non-condensed particles.

The Gross-Pitaevskii equation is a **zero-temperature** equation and does not take into account **thermal excitations**.
A standard technique to estimate the growth of the number of thermal particles is provided by the **formalism** developed by **Castin** and **Dum** in a second quantization context:

•It is restriced to the case of positive scattering length, i.e. for repulsive particle-particle interactions as in the case of ^{87}Rb and ^{23}Na atom condensate.

•It is a particle conserving formalism: $\hat{N} \cong \hat{N}_0 + \delta\hat{N}$ $\qquad \delta\hat{N} << \hat{N}_0$

•Perturbation theory with the square root of non-condensed fraction as the expansion parameter: $\sqrt{\dfrac{\delta\hat{N}}{\hat{N}}} \approx \dfrac{1}{\sqrt{\hat{N}}}$

•Splitting of the field operator into a condensate mode and a remainder, which accounts for non-condensed particles:

$$\hat{\psi}(x,t) = \varphi_0(x,t)\hat{a}(t) + \delta\hat{\psi}(x,t) \xrightarrow{\text{lowest order}} \approx \hat{a}(t)\left[\varphi_0(x,t) + \frac{1}{\sqrt{N}}\sum_k \hat{b}_k u_k(x,t) + \hat{b}_k^+ v_k^*(x,t)\right]$$

$\hat{a}$: annihilation operator of a particle in the condensate mode $\varphi_0(x)$ $\qquad \hat{a}^+\hat{a} = N_0 \approx N$

$\hat{b}_k, \hat{b}_k^+$: creation and annihilation operators of excitations above the condensate

$(u_k(x,t), v_k(x,t))$ **MODAL FUNCTIONS**

•Couple of functions that represent the time-dependent coefficients of the decomposition, in terms of annihilation and creation operators, of the time evolution of the field operator describing the thermal excitation above the condensate.

•They describe the spatial dependence of the thermal excitation.

•They evolve by equations identical to the ones that describe the evolution of a perturbation orthogonal to the condensate wave function,which evolves according to a linearized Gross-Pitaevskii equation.

$$i\hbar\frac{d}{dt}\begin{pmatrix}|u_k(t)\rangle \\ |v_k(t)\rangle\end{pmatrix} = \hat{L}(t)\begin{pmatrix}|u_k(t)\rangle \\ |v_k(t)\rangle\end{pmatrix} \qquad \hat{L}(t) = \begin{pmatrix}\hat{H}_{GP} + u\hat{Q}|\varphi(x,t)|^2\hat{Q}^* & u\hat{Q}\varphi(x,t)^2\hat{Q}^* \\ -u\hat{Q}^*\varphi^*(x,t)^2\hat{Q} & -\hat{H}_{GP} - u\hat{Q}|\varphi(x,t)|^2\hat{Q}^*\end{pmatrix}$$

$$\hat{H}_{GP} = \hat{H}(t) + u|\varphi(x,t)|^2 \qquad \hat{Q}(t) = 1 - |\varphi(x,t)\rangle\langle\varphi(x,t)|$$

where Q and Q^* are projectors orthogonally to the state of the condensate and its complex conjugate.

To the lowest order in the perturbation parameter and in the limit of vanishing temperatures, the **number of non-condensed particles** is:

$$\left\langle \delta\hat{N} \right\rangle = \left\langle \delta\hat{\psi}^{+}\delta\hat{\psi} \right\rangle \xrightarrow{\text{lowest order estimate}} \approx \sum_k \left\langle v_k(t) \middle| v_k(t) \right\rangle = \sum_k \int dx \left| v_k(x,t) \right|^2$$

The **transition to instability** of the condensate is marked by a transition from **polynomial** to **exponential** growth in number of non-condensed atoms beyond a critical strength of interactions.

Example: kicked rotor for dynamically localized states

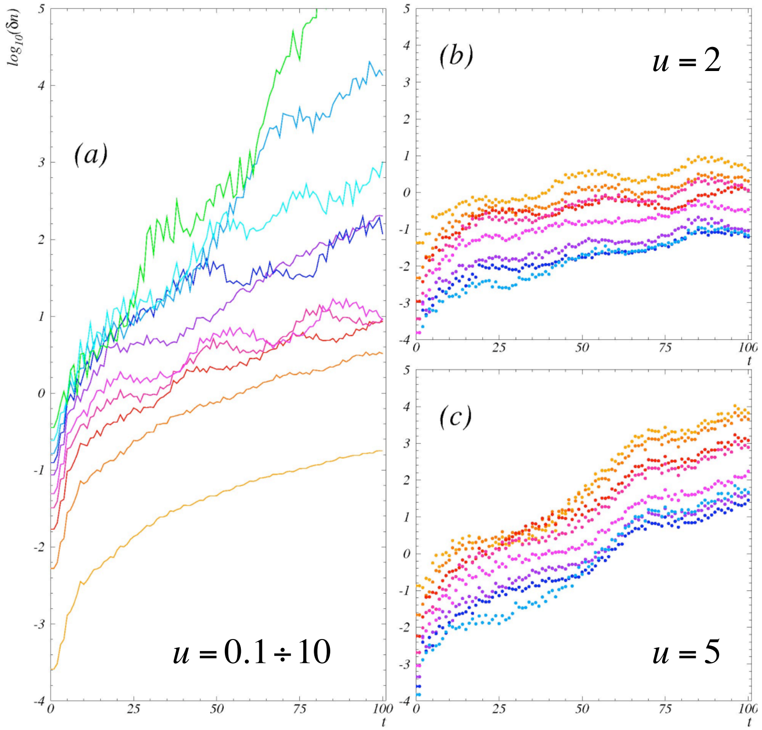
The **initial condition of the modal functions** are the **eigenvectors** of the operator *L(t=0)*.

$$\begin{pmatrix} u_k(\theta,0) \\ v_k(\theta,0) \end{pmatrix} = \frac{1}{2}\frac{e^{ik\theta}}{\sqrt{2\pi}} \begin{pmatrix} \frac{1}{C_k} + C_k \\ \frac{1}{C_k} - C_k \end{pmatrix}$$

$$C_k = \left(\frac{k^2 + 4\mu}{k^2} \right)^{1/4}$$

$$\mu = \frac{u}{2\pi} = u\,|\psi(\theta,0)|^2$$

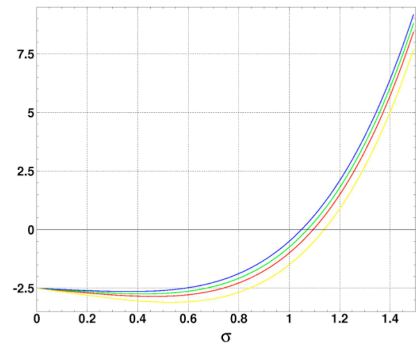
The **evolution** is determined by a TSS method and it is performed parallel to numerical integration of the GP equation for the condensate.



The **initial state of the system** is assumed to be the fundamental state of Gross-Pitaevskii equation with an harmonic trapping potential, which is analytically determined by a variational method. The condition for a stationary energy functional is imposed on a trial function in the form of a gaussian wave packet with the standard deviation as a variational parameter:

$$E\left[\psi_0,\psi_0^*\right]=\int dx\left\{\frac{1}{2}|\nabla\psi_0|^2+V(x)|\psi_0|^2+\frac{u}{2}|\psi_0|^4\right\} \qquad V(x)=\frac{1}{2}\omega_0 x^2$$

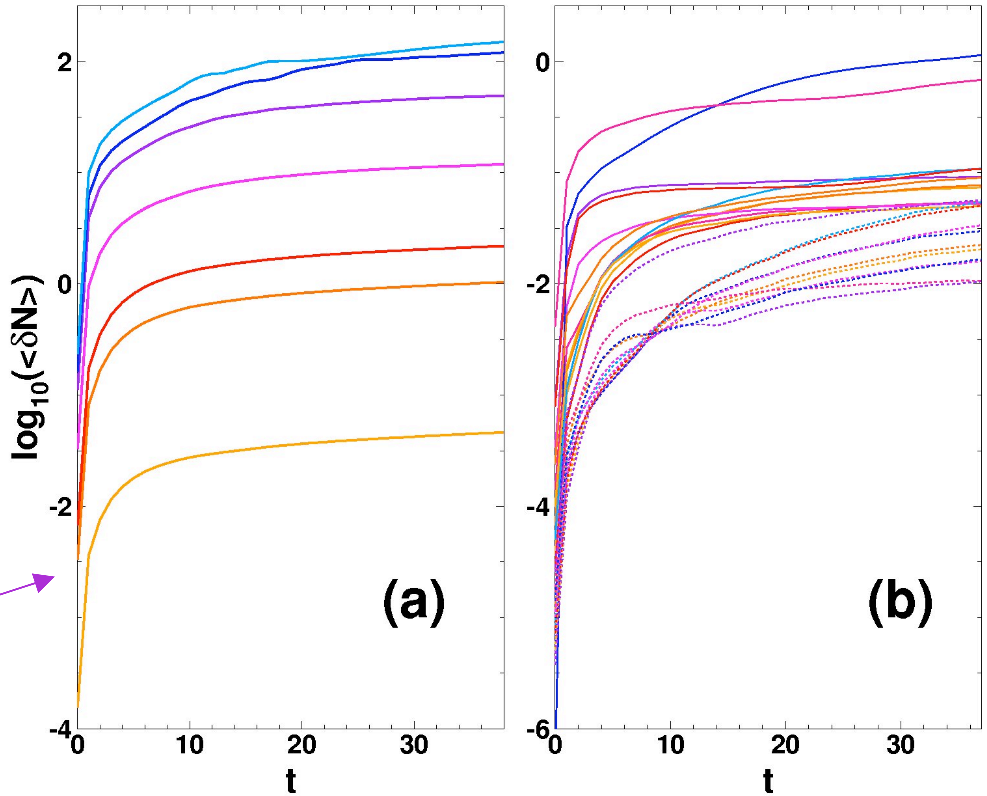
$$\psi_0(x)=\frac{1}{\sqrt{\sigma\sqrt{\pi}}}e^{-\frac{1}{2\sigma^2}(x-x_0)^2} \qquad \sqrt{2\pi}\,\omega_0^2\sigma^4-u\sigma-\sqrt{2\pi}=0$$



The results are consistent with a **polynomial growth** of non-condensed particles, namely in within considered parameter region and time scale, no exponentially instability takes place.

The sum is truncated to 12 terms, assuming the contribution of the others is vanishing (fig. part b).

$u = 0.1 \div 10$



Quantum Accelerator Modes near higher-order resonances

Higher-Order Resonances: Spinor dynamics

$\tau = 2\pi \frac{p}{q}$

$$\left[\hat{U}, e^{iq\theta}\right] = 0 \Rightarrow \text{ Conservation of Bloch phase } \xi \equiv \theta \bmod (2\pi / q)$$

The **state of the rotor** may be described by a ***q*-spinor**, specified by q complex functions of ξ varying in the Brillouin zone $\mathbf{B_q}$.

$$\psi(\theta) \in L^2(\mathbf{T}) \quad \to \quad \to \quad \begin{pmatrix} \psi_0(\xi) \\ \psi_1(\xi) \\ \cdots \\ \psi_{q-1}(\xi) \end{pmatrix} \qquad \xi \in B_q \equiv \left[0, \frac{2\pi}{q}\right] \qquad j = 1, \ldots q-1$$

Introducing the unitary map $\mathbf{b} : L^2(T) \to L^2(B_q) \otimes \mathbf{C}^q$

the **spinor components** are defined: $\langle \xi, j | \mathbf{b}\psi \rangle \equiv \psi_j(\xi) = \psi\left(\xi + 2\pi \frac{j}{q}\right)$

The **Resonant Floquet operator** admits a direct integral decomposition in fibers $\hat{\mathbf{B}}_{p,q,k,\beta_r}$

$$\hat{U}^{\text{res}}_{2\pi p/q,k,\beta_r} = e^{-ikV(\theta)} \cdot e^{-i\frac{\pi}{q}p(\hat{m}+\beta_r)^2} \to b\hat{U}^{\text{res}} b^{-1} = \hat{B}_{p,q,k,\beta_r} = \int_{B_q}^{\oplus} d\xi \, \hat{\mathbf{B}}_{p.q.k.\beta_r}(\xi),$$

$\hat{\mathbf{B}} : B_q \to U(q)$ (Unitary Matrices of rank q)

- **Representation** in which the free **rotation** is **diagonal**:
  - the spinors and fibers have periodicity 2π/q
  - the Brillouin zone B_q has been turned into the torus $\mathbf{R}/(2\pi\mathbf{Z}/q)$

- **Re-parametrization** of the torus by an angle $\vartheta = q\xi \in \mathbf{T} = [0, 2\pi[$

- Switch to a unitary decomposition: $b: \; L^2(\mathbf{T}) \to L^2(\mathbf{T}) \otimes \mathbf{C}^q$

  spinor components $\to$ $\langle \vartheta, j | \mathbf{b}\psi \rangle \equiv \psi_j(\vartheta) = \sum_{l \in \mathbf{Z}} \hat{\psi}(j + lq) \, e^{il\vartheta}$

  $\hat{\psi} \in \ell^2(\mathbf{Z})$

Floquet operator at exact resonance.

$$\left|\psi(t+1)\right\rangle = \hat{U}^{res}\left|\psi(t)\right\rangle \qquad \hat{U}^{res} = \hat{K}\cdot\hat{R} = e^{-ikV(\hat{\theta})}\cdot e^{-i\frac{\tau}{2}\hat{p}^2} = e^{-ik\cos\hat{\theta}}\cdot e^{-i\pi\frac{p}{q}\hat{p}^2} \qquad \hat{p} = \hat{m} + \beta_r \qquad \beta_r \in \beta_{res} \text{ fixed}$$

Evolution of the **spinor components**:

$$\psi_j(\vartheta, t+1) = \sum_{l\in\mathbf{Z}} e^{il\vartheta}\hat{\psi}_{j+lq}(t+1) \qquad \sum_{m\in\mathbf{Z}} \longrightarrow \sum_{j=0}^{q-1}\sum_{l\in\mathbf{Z}}$$

$$\hat{\psi}_{j+lq}(t+1) = \sum_{j_1=0}^{q-1}\sum_{l_1=-\infty}^{+\infty}\hat{K}_{j+lq,\,j_1+l_1q}\cdot e^{-i\pi\frac{p}{q}(j_1+\beta_r)^2}\hat{\psi}_{j_1+l_1q}(t) = \sum_{j_1=0}^{q-1}\hat{K}_{j-j_1}\cdot e^{-i\pi\frac{p}{q}(j_1+\beta_r)^2}\psi_{j_1}(\vartheta,t)$$

$$\left\langle j\middle|\hat{K}\middle|j_1\right\rangle \equiv \hat{K}_{j-j_1} = \frac{e^{-i(j-j_1)\frac{\vartheta}{q}}}{q}\sum_{j_2=0}^{q-1} e^{i\frac{2\pi}{q}j_2(j-j_1)}\, e^{-ik\cos\left(\frac{\vartheta}{q}-\frac{2\pi}{q}j_2\right)}$$

Fibers of the Floquet operator

the free rotation is diagonal, the “kick” mixes the spinor components:

$$\hat{\mathbf{B}}_{p,q,k,\beta_r} = e^{-i\hat{\mathbf{V}}'_{k,q}(\vartheta/q)}\cdot e^{-i\hat{\mathbf{G}}_{p,q,\beta_r}}$$

$$\hat{\mathbf{G}}_{p,q,\beta_r} = \pi\frac{p}{q}\left(\hat{\mathbf{S}} + \beta_r\hat{\mathbf{I}}\right) \qquad \hat{\mathbf{V}}'_{k,q}(\vartheta/q) = \frac{1}{2}\left\{\sum_{j=1}^{q-1}\left(\left|j\right\rangle\left\langle j+1\right| + \left|j+1\right\rangle\left\langle j\right|\right) + \left(\left|1\right\rangle\left\langle q\right|e^{i\vartheta} + \left|q\right\rangle\left\langle 1\right|e^{-i\vartheta}\right)\right\}$$

where $\{|j\rangle\}_{1\le j\le q}$: **canonical basis,** $\hat{\mathbf{I}} = \sum_{j=1}^{q}|j\rangle\langle j|$: **identity operator** and $\hat{\mathbf{S}} = \sum_{j=1}^{q}(j-1)|j\rangle\langle j|$: **spin operator** in $\mathbf{C}^q$

Recipe to translate the momentum-dependent part of the evolution in spinor form:

$$\hat{m} = -i\partial_\theta \longrightarrow b(\hat{m})b^{-1} = -iq\partial_\vartheta \otimes \hat{\mathbf{I}} + \hat{I}\otimes\hat{\mathbf{S}}$$

eigenvalues $m \longrightarrow ql + j$ $\quad j = 0,\dots,q-1, \quad m,l\in\mathbf{Z}, \quad \hat{I}$: identity in $L^2(\mathbf{T})$

Near resonance, with gravity: ε-classical approximation.

- τ is slightly incommensurate with *2πp/q*
- β may not have a rational - resonant value
- Presence of gravity, described by the term Φ_t

→ The evolution operator is not decomposable, that is a direct integral of fibers

Floquet operator:
(apart from an irrelevant phase factor)

$$\hat{U}_{\tau,k,\beta,\eta} = e^{-ik\cos(\hat{\theta})} \cdot e^{-i\frac{\tau}{2}(\hat{p}+\omega_t)^2} \qquad p = m + \beta, \qquad \omega_t = \beta + \frac{\eta}{2} + \eta t \qquad \eta = g\tau$$

$$\tau = 2\pi\frac{p}{q} + \varepsilon \longrightarrow \hat{U}_{\tau,k,\beta,\eta} = \hat{U}^{KR}_{2\pi p/q,k,\beta_r} \cdot e^{-i\frac{\varepsilon}{2}(\hat{m}+\beta_r)^2} \cdot e^{-i\tau\phi_t\hat{m}} \qquad \Phi_t = \beta - \beta_r + \frac{\eta}{2} + \eta t$$

The evolution operator in spinor representation is given by:

$$\hat{m} = -i\partial_\theta \longrightarrow b(\hat{m})\hat{b}^{-1} = -iq\partial_\vartheta \otimes \hat{\mathbf{I}} + \hat{I} \otimes \hat{\mathbf{S}}$$

$$\hat{\mathbf{U}}_{\tau,k,\beta,\eta} = b\hat{U}_{\tau,k,\beta,\eta}b^{-1} = \hat{\boldsymbol{B}}^{\boldsymbol{res}}{}_{p,q,k,\beta_r} \cdot \hat{\boldsymbol{A}}_{p,q,\varepsilon,\beta,\eta}(t) = \hat{\boldsymbol{B}}^{\boldsymbol{res}}{}_{p,q,k,\beta_r} \cdot e^{-i\frac{\varepsilon}{2}\left(-iq\partial_\vartheta \otimes \hat{\mathbf{I}} + \beta_r\hat{\mathbf{I}} + \hat{I}\otimes\hat{\mathbf{S}}\right)^2} e^{-i\tau\Phi_t\left(-iq\partial_\vartheta \otimes \hat{\mathbf{I}} + \hat{I}\otimes\hat{\mathbf{S}}\right)}$$

RESONANT REPRESENTATION:

the dynamics is transformed to the representation where the resonant fibers are diagonal

$$\hat{\mathbf{D}}^{res}_{p,q,k,\beta_r} = \hat{\Theta}^{+}\hat{\mathbf{B}}^{\boldsymbol{res}}{}_{p,q,k,\beta_r}\hat{\Theta} = e^{-i\hat{\mathbf{H}}^{res}(\vartheta)} \qquad \left\langle j\middle|\hat{\Theta}(\vartheta)\middle|l\right\rangle = \left\langle l \,\middle|\, \varphi^{(n)}(\vartheta)\right\rangle$$

where $\varphi^{(n)}(\vartheta)$ are the eigenvectors of the resonant fiber, $\hat{\mathbf{H}}^{res}(\vartheta)$ are hermitian matrices

The spinor components evolve independently at exact resonance.
The detuning from exact resonance and gravity mix the components.
In the limits of **small detunings** (ε→0), **small gravity accelerations** (g→0) and **small kicking strength** (*k*→*0*), mixing of the spinor components is assumed to be negligible and orbital motion to be decoupled by the spin motion.

Analogy with Born Oppenheimer approximation:
separation of the nuclear motion to the electron one based on the discrepancy between their masses and consequently their velocities and level spectra.

Electrons ⟶ **spin motion** (variable *j=1,…q)*

Nuclei ⟶ **orbital motion** (variables $(\vartheta, -iq\partial_\vartheta)$)

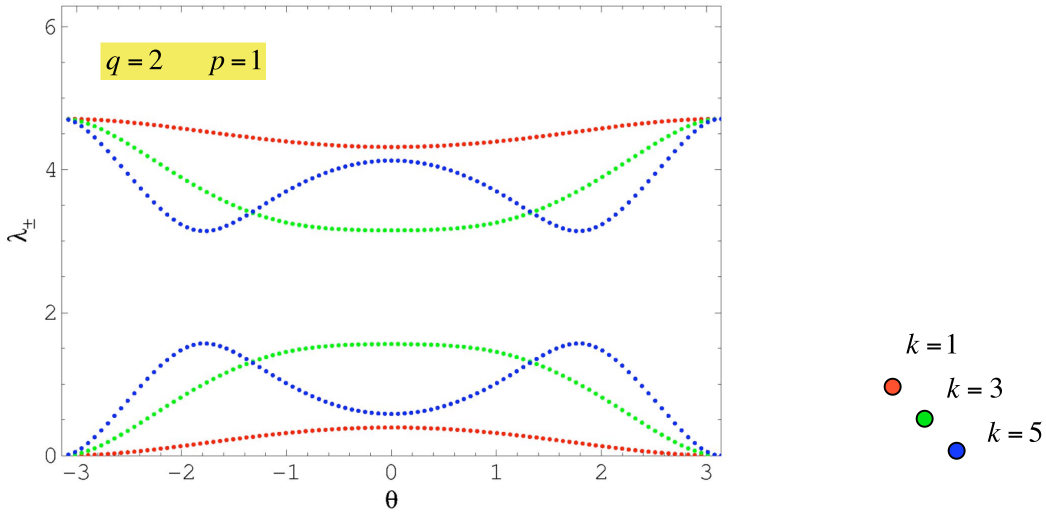
The nuclear (orbital) motion is assumed not to induce transitions between electronic (spin) states.
The eigenvalue of the electronic problem (depending parametrically on the nuclear position) appears in the Schrödinger equation for the nuclei as an effective potential energy.

Case *q*=2 ($\tau_{res}=\pi$)

The fibers of the Floquet operator can be approximated by:

$$\hat{\mathbf{U}}_{\tau,k,\beta,\eta} = \hat{\mathbf{B}}^{res}{}_{p,q,k,\beta_r} \cdot \hat{\mathbf{A}}_{p,q,\varepsilon,\beta,\eta}(t) \approx e^{-i\hat{\mathbf{H}}^{\mathbf{res}}(\vartheta)} \cdot e^{-i\frac{\varepsilon}{2}\left(-i2\hat{\mathbf{M}}+\beta_r\hat{\mathbf{I}}+\hat{\mathbf{S}}\right)^2} e^{-i\tau\Phi_t\left(-i2\hat{\mathbf{M}}+\hat{\mathbf{S}}\right)}$$

$$\hat{\mathbf{H}}^{res} = \begin{pmatrix} \lambda_+(\vartheta) & 0 \\ 0 & \lambda_-(\vartheta) \end{pmatrix} \qquad \hat{\mathbf{M}} = \begin{pmatrix} -i\partial_\vartheta & 0 \\ 0 & -i\partial_\vartheta \end{pmatrix}$$



$$\lambda_\pm(\vartheta) = -\frac{\pi}{4} \mp \omega(\vartheta)$$

$$\omega(\vartheta) = \arccos\left(\frac{\cos v(\vartheta)}{\sqrt{2}}\right)$$

$$v(\vartheta) = k\cos(\vartheta/2)$$

Map for q=2, p=1 $(\tau = \pi + \varepsilon)$

Introducing the "pseudo-classical" momentum variable $I = -i\,|\varepsilon|\,\partial_\vartheta$
the Floquet operators for the decoupled spinor components and the correspondent "pseudo-classical" Hamiltonian can be approximated by:

$$U_j \approx e^{-\frac{i}{|\varepsilon|} g_j(\vartheta)} e^{-\frac{i}{|\varepsilon|} f_j(I)} \quad \rightarrow \quad h_j(\vartheta, I, t') = g_j(\vartheta) \sum_t \delta(t' - t) + f_j(I) \qquad j = 0,1$$

$$g_j(\vartheta) = |\varepsilon|\,\lambda_\pm(\vartheta) \qquad f_j(I) = \pm\frac{1}{2} q^2 I^2 + \tau\phi_t q I + \varepsilon q I (j + \beta_r) + f_0(\beta_r, \tau, \eta, j, t)$$

The dynamics can be reduced to the 2-torus:

$$\begin{cases} \vartheta_{t+1} = \vartheta_t \pm J_t \\ J_{t+1} = J_t \mp q\tilde{k} \sin\left(\frac{\vartheta_{t+1}}{q}\right) \dfrac{\sin\left(k\cos\left(\frac{\vartheta_{t+1}}{q}\right)\right)}{\sqrt{1 + \sin^2\left(k\cos\left(\frac{\vartheta_{t+1}}{q}\right)\right)}} \pm q\tau\eta \end{cases} \qquad \text{mod}(2\pi)$$

$$\tilde{k} = |\varepsilon|\,k, \quad \tau\eta, \quad k$$

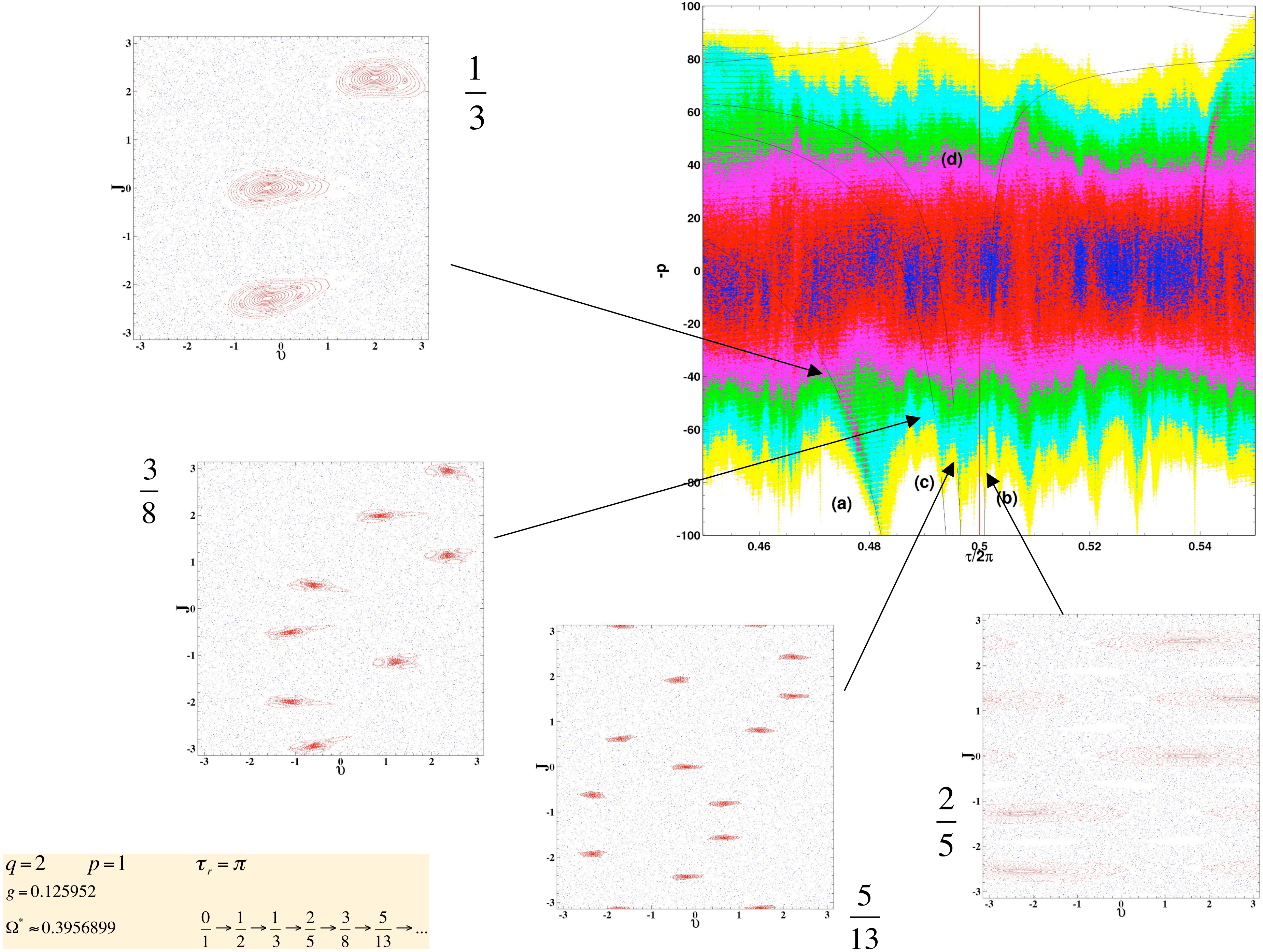
Periodic orbit of winding number $\omega = \dfrac{r}{s}$

acceleration of the QAMs

$$J_s - J_0 = 2\pi r \rightarrow a = \frac{1}{|\varepsilon|}\left(\frac{2\pi r}{qs} \mp \tau\eta\right)$$

accumulation point of winding numbers

$$\left|\frac{r}{s} \mp \frac{\tau\eta}{2\pi}\right| \leq \tilde{k}\mathrm{p} \rightarrow \tau = 2\pi\frac{p}{q} + \varepsilon \Rightarrow \frac{r}{s} \rightarrow \Omega^* = 2\pi g \frac{p^2}{q}$$



1/3
3/8
5/13
2/5
(a)
(b)
(c)
(d)
-p
τ/2π
J
ϑ
$q=2 \quad p=1 \quad \tau_r=\pi$
$g=0.125952$
$\Omega^* \approx 0.3956899$
$\frac{0}{1}\to\frac{1}{2}\to\frac{1}{3}\to\frac{2}{5}\to\frac{3}{8}\to\frac{5}{13}\to\ldots$

Bibliography

Effects of atomic interactions on Quantum Accelerator Modes
Laura Rebuzzini, Roberto Artuso,Shmuel Fishman, Italo Guarneri, Phys. Rev. A **76**, 031603, (2007).

Arnol'd Tongues and Quantum Accelerator Modes
Italo Guarneri, Laura Rebuzzini and Shmuel Fishman, Nonlinearity **19**,1141-1164, (2006).

Quantum Accelerator Modes from the Farey Tree
A.Buchleitner, M.B.d'Arcy, S.Fishman, S.A.Gardiner, I.Guarneri, Z.-Y.Ma, L.Rebuzzini, and G.S.Summy, Phys. Rev. Lett. **96**,164101 (2006)

Decay of Quantum Accelerator Modes
Michael Sheinman, Shmuel Fishman, Italo Guarneri, Laura Rebuzzini, Phys. Rev. A **73**, 052110, (2006).

Bose-Einstein condensate in the atomic gases: simple theoretical results
Y. Castin, Les Houches School.

Y. Castin and R. Dum, Phys.Rev.A **57**, n.3, 3008 (1998)

G. Behinaein, V. Ramareddy, P. Ahmadi and G. S. Summy, Phys. Rev. Lett. **97**, 244101 (2006)