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# Groups of Prime Power Order with Many Conjugacy Classes

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## 1. INTRODUCTION

Let $G$ be a finite group, and denote by $k(G)$ the number of conjugacy classes of $G$. Assume that $G$ contains two subgroups $H$ and $K$, such that $K \triangleleft H$, $H/K$ is abelian, and $|G:H|, |K|$ are bounded by some constant $C$. Then all classes of $H$ have size at most $C$, and therefore the classes of $G$ containing elements of $H$ have size at most $C^2$, so that we have

$$k(G) \geq \frac{|H|}{C^2} \geq \frac{|G|}{C^3}.$$

A converse of this observation is proved in [N]. Suppose that $k(G) \geq |G|/A$, for some finite group $G$ and some number $A > 0$. Then $G$ contains two normal subgroups $K \leq H$, with $H/K$ abelian, $|G:H| \leq A$, and $|K| \leq f(A)$, for a suitable function $f$ of $A$ (whose order of magnitude is $A^A$).

In this paper we restrict ourselves to $p$-groups, and we characterize those for which $A$ above is relatively small.

We remind the reader that an element $g$ of the $p$-group $G$ is said to have *breadth* $b_G(g)$ ($b(g)$ if no ambiguity is possible) if $p^{b_G(g)}$ is the size of the conjugacy class of $g$ in $G$. The breadth $b(G)$ of $G$ will be the maximum of the breadths of its elements. We have that $b(G) = 1$ if and only if $|G'| = p$ (see [K1] or the proof of Lemma 1). The $p$-groups of breadth 2 are studied in [K2]. They are shown to have nilpotency class at most 3, and commutator subgroup of order at most $p^3$.

Let $G_0 = G$ and, for $i > 0$, inductively let $G_i = [G, G_{i-1}]$ be the $i$th term of the lower central series of $G$. Our results here are

THEOREM 1. *Let $G$ be a finite $p$-group. If $G$ has class number $k(G) \geq 3|G|/p^3$ then at least one of the following occurs*:

(i) $|G_2| \leq p^2$,

(ii) $|G : Z(G)| \leq p^3$,

(iii) *$G$ has an abelian maximal subgroup $M$.*

THEOREM 2. *Let $G$ be a finite $p$-group. If $G$ has class number $k(G) \geq 5|G|/p^4$ then at least one of the following occurs*:

(i) $|G_2| \leq p^3$,

(ii) *there exists a minimal normal subgroup $N \triangleleft G$ contained in $G_2$ such that $Z(G/N)$ has index at most $p^3$ in $G/N$.*

(iii) $|G : Z(G)| \leq p^4$

(iv) *there exists a maximal subgroup $M \lessdot G$ with $b(M) \leq 1$.*

THEOREM 3. *If $p$ is an odd prime then there exists no $p$-group $G$ of order $|G| = p^n$ such that $3p^{n-3} \leq k(G) \leq p^{n-2} + 2p - 3$. If $p \geq 5$ then there exists no $p$-group $G$ such that $5p^{n-4} \leq k(G) \leq p^{n-3} + 3p - 4$.*

Note that each of the conditions (i) and (ii) in Theorem 1 implies $b(G) \leq 2$, while each of the conditions (i), (ii), and (iii) in Theorem 2 implies $b(G) \leq 3$. If $|G| = p^n$, the conditions on the maximal subgroup of $M$ in Theorems 1 and 2 imply that $M$ contains respectively at least $p^{n-2}$ and $p^{n-3}$ $G$-conjugacy classes. Thus Theorems 1 and 2 have converses respectively when $p$ is odd and when $p \geq 5$. Moreover a corollary of Theorem 1 is a characterization of $p$-groups of breadth 2 in terms of conditions (i) and (ii) of Theorem 1 when $p$ is odd. We have also a new proof of a result of Isaacs and Passman characterizing all finite groups having character degrees 1 and $p$. Our proof works only in the case of $p$-groups, but it turns out that this is the main case. In a similar way a corollary to Theorem 2 gives a characterization of $p$-groups of breadth 3 when $p \geq 5$.

COROLLARY 1. *Let $p$ be an odd prime and $G$ be a $p$-group. Then $b(G) = 2$ if and only if $|G_2| = p^2$ or $|G : Z(G)| = p^3$.*

COROLLARY 2 [I, Theorem 12.11]. *Let $G$ be a nonabelian finite $p$-group and let $p$ be a prime. Then $G$ has only irreducible characters of degree 1 and $p$ if and only if one of the following holds:*

(i) *there exists an abelian normal subgroup $A \lhd G$ of index $p$,*

(ii) $|G : Z(G)| = p^3$.

COROLLARY 3. *Let $p \geq 5$ be a prime and $G$ be a $p$-group. Then $b(G) \leq 3$ if and only if one of the following conditions is satisfied:*

(i) $|G_2| \leq p^3$,

(ii) $|G : Z(G)| \leq p^4$,

(iii) $|G_2| > p^3$, $|G : Z(G)| > p^4$, *and there exists a unique minimal normal subgroup $N \lhd G$ contained in $G_2$ such that the index of $Z(G/N)$ in $G/N$ is at most $p^3$.*

When $p$ is odd Theorem 1 implies that $k(G) \geq p^{n-2}$ holds if and only if either $b(G) \leq 2$ or $G$ has a maximal abelian subgroup. Note that $p$-groups of order $p^n$ all of whose irreducible characters have degree 1 or $p$ admit more than $p^{n-2}$ distinct irreducible characters, and then have more than $p^{n-2}$ conjugacy classes. In fact by Corollary 2 they have breadth at most two, or they have an abelian maximal subgroup.

An example of a 2-group $G$ such that $k(G) \geq 2^{n-2}$, $b(G) \geq 2$, and $M_2 \neq 1$ for every maximal subgroup $M$, and an example of a 3-group $H$

such that $k(H) \geq 3^{n-3}$, $b(H) \geq 3$, and $b(M) > 1$ for every maximal subgroup $M$, can be found in Example 5.1 of [VLW], as the groups $G_{2,3}$ and $G_{3,4}$. We have the following general result which is useful in case $p = 2$ and $p = 3$.

PROPOSITION 1. *Let p be a prime and G be a group of order $p^n$. If $k(G) \geq 2p^{n-3}$ then G is metabelian. If $k(G) \geq 2p^{n-4} + p^{n-5}$ then $|G''| \leq p$.*

## 2. PROOFS

In the following we shall denote by $p$ a prime and by $G$ a $p$-group of order $p^n$. We split the proofs into several intermediate lemmas. We shall often use, without any further reference, the fact that the number of irreducible characters of degree $d$ of a finite nonabelian group $G$ is less than $|G|/d^2$. This comes immediately from the character degrees formula $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$.

*Burnside's Formula.* Let $G$ be a finite $p$-group and $M \lessdot G$ a maximal subgroup. If $s$ and $t$ are the numbers respectively of invariant and fused conjugacy classes of $M$ then

$$k(G) = ps + \frac{t}{p} = s\left(p - \frac{1}{p}\right) + \frac{k(M)}{p}.$$

*Proof.* See [B, p. 472]. ∎

LEMMA 1. *Let G be a finite p-group. Then*

(i) $b(G) \leq 1 \Leftrightarrow |G_2| \leq p \Leftrightarrow k(G) \geq p^{n-1} + p - 1$,

(ii) $b(G) \geq 2 \Leftrightarrow |G_2| \geq p^2 \Leftrightarrow k(G) \leq 2p^{n-2} - p^{n-4}$.

*Proof.* The argument of the proof is mainly due to Berkovitch. Let $|G_2| = p^x$. Note that $G$ has $p^{n-x}$ linear characters and less than $p^{n-2} - p^{n-x-2}$ irreducible characters of degree greater than or equal to $p$ so that

$$k(G) \leq p^{n-2} + p^{n-x} - p^{n-x-2}.$$

The case $G$ abelian is trivial so that we begin by assuming $b(G) = 1$. We have $p^{n-1} < k(G) \leq p^{n-2} + p^{n-x}$. This forces $x = 1$. If $Z$ is the centre of $G$, we have that $g \in G \setminus Z$ if and only if $b(g) = 1$. Putting $|Z| = p^z$, we have $z \geq 1$ and $k(G) = (|G| - |Z|)/p + |Z| = p^{n-1} + p^{z-1}(p-1) \geq p^{n-1} + p - 1$.

If $b(G) \geq 2$ then $x \geq 2$. From the last display it follows that $k(G) \leq 2p^{n-2} - p^{n-4}$. Since for all primes $p$ the inequality $2p^{n-2} - p^{n-4} < p^{n-1} + p - 1$ holds and since for all finite $p$-groups we have either $b(G) \leq 1$ or $b(G) \geq 2$, the proof is done. ∎

LEMMA 2. *Suppose that* $|G_2| \geq p^3$ *and* $|G : Z(G)| \geq p^4$. *Assume further that either* $k(G) \geq 3p^{n-3}$ *or* $p = 2$, *G has no element of breadth* 1, *and* $k(G) \geq 2^{n-2}$. *If no maximal subgroup* $M \lessdot G$ *is abelian, then* $|M_2| \geq p^2$ *for every maximal subgroup* $M \lessdot G$.

*Proof.* Assume by way of contradiction that there exists $M \lessdot G$ with $|M_2| = p$. Let $p^z = |Z(G)|$ and $p^x = |Z(M)|$. Then $k(M) = p^{n-2} + p^x - p^{x-1}$. Moreover $Z(G) \leq M$, otherwise $M_2 = G_2$. Hence $z \leq x$ and $z \leq n - 4$, $x \leq n - 3$. Let $C = C_G(G/M_2)$. We want to show that $C \leq M$. Indeed if we assume the contrary we have that the central subgroup $C/M_2 \leq G/M_2$ would be a supplement to the abelian subgroup $M/M_2$, hence $G/M_2$ would be abelian. In this case we have again the contradiction $G_2 = M_2$.

If $m \in M$ is such that $b_M(m) = b_G(m) = 1$ then $m \in C \setminus Z(G)$. Therefore if we denote by $s$ the number of $M$-classes which are $G$-invariant and by $s_i$ the number of those of breadth $i$ we have $s \leq s_0 + s_1 \leq p^z + p^{c-1} - p^{z-1}$, where $p^c = |C|$. Since $C/M_2$ is central in $G/M_2$ and since Lemma 1(i) yields $b(G/M_2) \geq 2$ we have that $|G : C| \geq p^3$ and $c \leq n - 3$. So, by Burnside's Formula,

$$k(G) \leq p^{c-1}(p - 1/p) + p^{n-3} + p^z(p - 1/p)(1 - 1/p) + p^{x-1}(1 - 1/p).$$

Observing that this is an increasing function of its arguments and using the inequalities on $c, x, z$, we get $k(G) < 3p^{n-3}$.

If $p = 2$ and $G$ has no element of breadth 1 then $Z(M) = Z(G) = C$ and $s_1 = 0$. Hence

$$k(G) = \left(2 - \frac{1}{2}\right)2^z + \frac{2^{n-2} + 2^z - 2^{z-1}}{2} = 2^{n-3} + 2^z(2 - 2^{-2}) < 2^{n-2}. \quad \blacksquare$$

LEMMA 3. *Let* $M \lessdot G$ *be a maximal subgroup of a finite p-group G and let A and B be two subgroups of G such that* $[A, B] \leq M_2$. *Then one of the following occurs*:

(i) $[G, B] \leq M_2$,

(ii) $A_2 \leq M_2$.

*Proof.* Without loss of generality we can assume that $M_2 = 1$ and $G_2 \neq 1$. If $A \leq M$ then (ii) holds. Thus, by the maximality of $M$, we can assume $G = MA$. If $B \leq M$ then we have the claim since $[G, B] = [MA, B] \leq [A, B] \leq 1$. If $B \nleq M$ then let $L = M \cap A$. Clearly $A = \langle x, L \rangle$

for some $x \in A \setminus M$ and $L$ is a maximal subgroup of $A$. Let $y \in B \setminus M$. Then $ym = x^i$ for some $m \in C_M(x) = Z(G)$ and $i \in \{1, \dots, p-1\}$. Therefore $A_2 = [x^i, L]L_2 = [ym, L]L_2 = [y, L]L_2 = 1$. ∎

The roles of $A$ and $B$ in the previous lemma are clearly symmetric. However, we have preferred to state it as above in view of the following:

COROLLARY 4. *Let $x \in G$ and $C = C_G(x)$. If $M$ is a maximal subgroup of $G$, then one of the following occurs*:

(i) $[G, x] \leq M_2$,

(ii) $C_2 \leq M_2$.

LEMMA 4. *Let $|G_2| \geq p^3$ and $M_2 = G_2$ for all maximal subgroups $M \lessdot G$. Then $k(G) < 2p^{n-3}$.*

*Proof.* If $\chi$ is an irreducible character of $G$ of degree $\chi(1) \leq p$, since $G$ is monomial, there exists a subgroup $M \leq G$ of index $i = |G : M| \leq p$ and a linear character $\lambda$ of $M$ such that $\chi = \lambda^G$. Then $\ker \chi \geq M_2 = G_2$ and $\chi$ is linear. Hence $G$, having no irreducible character of degree $p$, has at most $p^{n-3} + p^{n-4}$ irreducible characters. Thus $k(G) \leq p^{n-3} + p^{n-4} < 2p^{n-3}$. ∎

The fact that $G$ has no irreducible characters of degree $p$ is a particular case of an unpublished result of Mann [M].

*Proof of Theorem* 1. Let $G$ be a counterexample. Then $|G_2| \geq p^3$, $|G : Z(G)| \geq p^4$ and $b(M) \geq 1$ for all maximal subgroups $M \lessdot G$. Assume that there exists $x \in G$ such that $b_G(x) = 1$. Let $C = C_G(x) \lessdot G$ and $L = \langle [G, x] \rangle$, a normal subgroup of $G$. Any irreducible character $\chi$ of degree $p$ is induced by a linear character of a maximal subgroup $M$ such that $M_2 \leq \ker \chi$. By Corollary 4 we have $C_2 \leq \ker \chi$ or $[G, x] \leq \ker \chi$, so that $\chi$ is actually a character either of $G/C_2$ or of $G/L$. By Lemma 2, $|C_2| \geq p^2$ so that $G/C_2$ has less than $p^{n-4}$ irreducible character of degree $p$; the group $G/L$ has at most $p^{n-3}$ of them. Thus $G$ has at most $p^{n-3}$ linear characters, $p^{n-3} + p^{n-4}$ irreducible characters of degree $p$, and less then $p^{n-4}$ irreducible characters of degree greater than $p$. Thus $k(G) < 2(p^{n-3} + p^{n-4}) \leq 3p^{n-3}$ a contradiction. Hence we may assume that $G$ has no element of breadth 1. Choose a maximal subgroup $M \lessdot G$ such that $M_2 \neq G_2$ (Lemma 4). Let $s_0$, $s_1$, and $s_2$ be the numbers of invariant conjugacy classes of $M$ having respectively breadth 0, 1, and greater than or equal to 2. Note that any invariant class of breadth 0 is contained in $Z(G)$ and any invariant class of breadth 2 is contained in $C_G(G/M_2)$. Hence $s_0 \leq |Z(G)| \leq p^{n-4}$, $s_1 = 0$, and $s_2 < |C_G(G/M_2)|/p^2 \leq p^{n-4}$, since $b(G/M_2) \geq 1$. By Lemmas 1(ii) and 2 we have $k(M) \leq$

$2p^{n-3}$. Thus Burnside's Formula yields the contradiction

$$k(G) = s\left(p - \frac{1}{p}\right) + \frac{k(M)}{p} < (s_0 + s_1 + s_2)p + 2p^{n-4} \le 3p^{n-3}. \quad \blacksquare$$

*Proof of Corollary* 1. If $b(G) = 2$ then $k(G) > p^{n-2} \ge 3p^{n-3}$. By Theorem 1 we have only to consider the case when $G$ has a maximal subgroup $M$ which is abelian. Let $x \in G \setminus M$. Then $|G_2| = |[x, M]M_2| = |[x, m]| \le p^2$, since $b_G(x) = 2$.

To prove the converse note that $b(G) \ge k$ implies $|G : Z(G)| \ge p^{k+1}$, while $|G_2| \le p^k$ implies $b(G) \le k$ for every natural number $k$. Thus we are left to prove that if $|G_2| = p$ and $|G : Z(G)| \le p^3$ then $|G : Z(G)| \le p^2$. Indeed this is a consequence of the fact that $\Omega_1(Z(H)) \le H_2$ for a $p$-group $H$ of minimal order in the isoclinism class of $G$. Then $H$ is a generalized extraspecial $p$-group and the quotient $H/Z(H)$ must be elementary abelian of even rank (see [R, Exercise 8, p. 146]). Since $p^3 \ge |G : Z(G)| = |H : Z(H)|$ we have $|G : Z(G)| = p^2$. $\blacksquare$

LEMMA 5. *Assume that $|G : G_2| \le p^{n-4}$. If the number $n_1 = n_1(G)$ of irreducible characters of degree p of G satisfies the inequality $n_1 \le 3p^{n-4}$ then $k(G) < 5p^{n-4}$.*

*Proof.* Let $n_0, n_1, n_2$ be respectively the numbers of irreducible characters $\chi$ such that $\chi(1) = 1$, $\chi(1) = p$, and $\chi(1) \ge p^2$. We have $n_0 = |G : G_2| \le p^{n-4}$ and the equation $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$ implies $n_2 < p^{n-4}$. Hence $k(G) = n_0 + n_1 + n_2 < 5p^{n-4}$.

LEMMA 6. *Let $M \lessdot G$ be a maximal subgroup. Assume that $|M_2| \ge p^2$ and that the number s of G-invariant conjugacy classes of M satisfies the inequality $s \le 3p^{n-5}$. Then $k(G) < 5p^{n-4}$.*

*Proof.* By Lemma 1(ii), we have $k(M) \le 2p^{n-3}$. By Burnside's Formula $k(G) = s(p - 1/p) + k(M)/p < 5p^{n-4}$. $\blacksquare$

Let $B_i(G)$ be the subgroup of $G$ generated by the elements $g \in G$ having breadth $b_G(g) = i$.

LEMMA 7. *Assume that $k(G) \ge 5p^{n-4}$, that $|G : G_2| \le p^{n-4}$, and that the inequality $|G/N : Z(G/N)| \ge p^4$ holds for all minimal normal subgroups N contained in $G_2$. If $b(M) \ge 2$ for every maximal subgroup $M \lessdot G$ then*

(i) $B_1(G) = 1$

(ii) *if $g \in G$ and $b(g) = 2$ then $C_G(g)$ is abelian.*

*Proof.* Let $x \in G$ be an element of breadth $b_G(x) = 1$ and let $X = [x, G]$, a normal subgroup of $G$. If $C = C_G(x)$, then $C \lessdot G$ is maximal in

$G$. Let $\mathscr{L}$ be the set of maximal subgroups $M \lessdot G$ such that $C_2 \leq M_2$ and $\mathscr{M}$ the set of maximal subgroups $M \lessdot G$ such that $X \leq M_2$. Then, by Corollary 4, any maximal subgroup of $G$ belongs to one of $\mathscr{L}$ or $\mathscr{M}$. Let $\chi$ be an irreducible character of degree $p$ of $G$. Then $\chi$ is induced by a linear character $\lambda$ of some maximal subgroup $M$. If $M \in \mathscr{L}$ then $C_2 \leq M_2 \leq \ker \chi$ and $\chi_C$ splits as a sum of linear characters of $C$. Hence $\chi$ is induced by some linear character of $C$. Thus, either $\chi$ is induced by a linear character of $C$ or $X \leq M_2 \leq \ker \chi$. Note that $C$ is maximal in $G$, so that $|C_2| \geq p^2$. Therefore there are at most $|C : C_2|/p \leq p^{n-4}$ irreducible characters of degree $p$ induced by $C$. If $|X| \geq p^2$ then the number of characters of degree $p$ of $G/X$ is at most $|G : X|/p^2 \leq p^{n-4}$. In this case we find $n_1 \leq 2p^{n-4}$ and from Lemma 5 we have $k(G) < 5p^{n-4}$. If $|X| = p$, then none of the conditions (i), (ii), or (iii) of Theorem 1 is satisfied by the group $G/X$, hence $k(G/X) < 3p^{n-4}$. Since any linear character of $G$ contains $X$ in its kernel, we have that the sum of the number of linear characters of $G$ and of the number of irreducible characters of degree $p$ containing $X$ in their kernel is less then $3p^{n-4}$. Therefore $G$ has at most $p^{n-4}$ irreducible characters of degree greater than or equal to $p^2$, $p^{n-4}$ irreducible characters of degree $p$ containing $C_2$ in their kernel, and less then $3p^{n-4}$ irreducible characters containing $X$ in their kernel. Since these are all the irreducible characters of $G$ we have again $k(G) < 5p^{n-4}$. Thus $G$ has no element of breadth 1.

Now assume that there exists an element $g \in G$ of breadth $b_G(g) = 2$. Let $A = C_G(g)$. By way of contradiction, assume that $|A_2| \geq p$. As above let $\mathscr{L}$ be the set of maximal subgroups $M \lessdot G$ such that $A_2 \leq M_2$ and $\mathscr{M}$ the set of maximal subgroups $M \lessdot G$ such that $[G, g] \leq M_2$. By Corollary 4 any maximal subgroup of $G$ is contained in one of $\mathscr{L}$ or $\mathscr{M}$. Let $\chi$ be as above. Then $\chi$ is induced by some linear character $\lambda$ of some maximal subgroup $M \lessdot G$. If $M$ is in $\mathscr{L}$, Then $\chi_A$ is a sum of linear characters. The irreducible characters of degree $p$ induced from the maximal subgroups in $\mathscr{L}$ contain the normal closure $L$ of $A_2$ in their kernel. Thus they are irreducible characters of degree $p$ of $G/L$. If $|L| \geq p^2$, there are less than $p^{n-4}$ such characters. If $|L| = p$ then, as above, we find $k(G/L) < 3p^{n-4}$. In particular the sum of the numbers of irreducible characters of degree $p$ induced by maximal subgroups in $\mathscr{L}$ and of the linear characters of $G$ is less than $3p^{n-4}$. Now we turn our attention to the irreducible characters of degree $p$ induced by the maximal subgroups in $\mathscr{M}$. Let $X = [G, g]$. Since $b_G(g) = 2$, then $|X| \geq p^2$. Any irreducible character of degree $p$ of $G$ induced from $M \in \mathscr{M}$ factors through $G/X$, and hence there are less then $p^{n-4}$ of them. Adding up we find that the number of irreducible characters $\chi$ of $G$ such that $\chi(1) \leq p$ is at most $4p^{n-4}$. Since there are less than $p^{n-4}$ irreducible characters of degree greater than $p$, we have $k(G) < 5p^{n-4}$, a contradiction. It follows that $A_2 = 1$. ∎

LEMMA 8. *Assume that* $|G:G_2| \leq p^{n-4}$. *If* $M_2 = G_2$ *for all maximal subgroups* $M \lessdot G$ *then* $k(G) < 2p^{n-4}$.

*Proof.* The argument is the same as the one used in the proof of Lemma 4. ∎

LEMMA 9. *Let* $\chi \in \operatorname{Irr}(G)$ *and* $A \lhd G$. *If* $\chi$ *splits in linear constituents over* $A$, *then there exists a subgroup* $L$ *of* $G$ *containing* $A$ *and a linear character* $\lambda$ *of* $L$, *such that* $\chi = \lambda^G$.

*Proof.* Note that the splitting condition is equivalent to $A_2 \leq \ker \chi$, so, without loss of generality, we may assume that $A$ be abelian. Then the claim is Exercise 6.11 of [I]. ∎

LEMMA 10. *Let* $A$ *be a normal subgroup of* $G$ *of index* $|G:A| = p^2$. *If* $|M_2| \geq p^3$ *for all maximal subgroups* $M \lessdot G$ *containing* $A$, *then there exist at most* $p^{n-4} + p^{n-5}$ *irreducible characters* $\chi$ *of* $G$ *of degree* $p$ *whose restriction* $\chi_A$ *is a sum of linear characters.*

*Proof.* Let $S$ be the set of irreducible characters $\chi$ of $G$ of degree $p$ such that the restriction $\chi_A$ is a sum of linear characters. Let $\chi \in S$. By Lemma 9 there exists a maximal subgroup $M \lessdot G$ containing $A$ and a linear character $\lambda$ of $M$ such that $\chi = \lambda^G$. Then $|S| \leq (1/p)|T|$, where

$$T = \{(M, \lambda) | A \leq M \lessdot G \text{ and } \lambda \text{ is a linear character of } M\}.$$

Hence $|S| \leq (1/p)(p+1)(p^{n-1}/p^3) = p^{n-4} + p^{n-5}$. ∎

COROLLARY 5. *Suppose that*:

(i) $k(G) \geq 5p^{n-4}$,

(ii) $|G:G_2| \leq p^{n-4}$, *and*

(iii) $|M_2| \geq p^3$ *for every maximal subgroup* $M \lessdot G$.

*Then* $G$ *has no abelian subgroups of index* $p^2$.

*Proof.* Assume by way of contradiction that there exists an abelian subgroup $A$ of index $p^2$. Let $M$ be a maximal subgroup of $G$ containing $A$. By (iii) we have $b(M) \geq 2$. If there would exist an abelian subgroup $B \neq A$ of $M$ such that $|M:B| = p$, then $AB = M$ and $Z(M) \leq A \cap B$ would have index at most $p^2$ in $M$, contrary to $b(M) \geq 2$. Therefore $A$ is a characteristic subgroup of $M$ and so it is a normal abelian subgroup of $G$. Our claim then follows from Lemmas 5 and 10. ∎

*Proof of Corollary* 2. Assume that $G$ has only irreducible characters of degree 1 and $p$. Let $|G:G_2| = p^k$. Then $G$ has $p^k$ linear characters and $(p^n - p^k)/p^2 = p^{n-2} - p^{k-2}$ irreducible characters of degree $p$. Hence $k(G) = p^{n-2} + p^{k-2}(p^2 - 1)$. Assume that there exists an element $x \in G$

of breadth $b(x) = 1$. Let $C = C_G(x)$. Then $C$ is a maximal subgroup of $G$ and so we may assume $C_2 \neq 1$. By Corollary 4, all irreducible characters of degree $p$ are characters either of $G/C_2$ or of $G/[\langle x \rangle, G]$. Since $C_2$ and $[\langle x \rangle, G]$ are subgroups of $G_2$, each of this factors has at most $p^{n-3} - p^{k-2}$ irreducible characters of degree $p$. This implies that $k(G) \leq 2p^{n-3} + p^{k-2}(p^2 - 2)$ which contradicts the previous calculation of $k(G)$. Thus $G$ has no element of breadth 1. A similar argument shows that every element $y \in G$ of breadth $b(y) = 2$ has an abelian centralizer $C_G(y)$. Let $|Z(G)| = p^z$. Then $p^{n-2} + p^{k-2}(p^2 - 1) = k(G) \leq p^z + (p^n - p^z)/p^2 = p^{n-2} + p^{z-2}(p^2 - 1)$. Thus we have $z \geq k$. Now we split the proof in two cases depending on $p$ being odd or even.

If $p$ is an odd prime then $k(G) = p^{n-2} + p^{k-2}(p^2 - 1) > 3p^{n-3}$. If $G$ has no abelian subgroup of index $p$, then, by Theorem 1 we have that $k \geq n - 3$. Since $z \geq k$, the centre of $G$ has index at most $p^3$.

If $p = 2$ then $k(G) > 2^{n-2}$. Assume by contradiction that $z \leq n - 4$ and that $G$ has no abelian maximal subgroups. Since $k \leq z$, by Lemma 2, we have that the group $G$ has no maximal subgroup $M$ such that $|M_2| \leq 2$. If $G$ has an element $y$ of breadth $b(y) = 2$ then $A = C_G(y)$ is abelian. Arguing as in Corollary 5 and Lemma 10 we see that $A$ is normal in $F$ so that $G$ has less than $\frac{3}{2}2^{n-3}$ irreducible characters of degree $p$ and $k(G) < \frac{3}{2}2^{n-3} + 2^k = 2^{n-3} + 2^{n-4} + 2^k \leq 2^{n-3} + 2^{n-4} + 2^{n-4} = 2^{n-2}$, a contradiction. Thus $G$ has no element of breadth 2. Let $M$ be a maximal subgroup of $G$. Arguing as in Lemma 4 we can assume that $G_2 \neq M_2$. Since $G$ has no element of breadth 1 and 2 we have that $M$ has no element of breadth 1, no invariant class of breadth 2, and $Z(M) = Z(G)$. Thus $k(M) \leq 2^{n-3} + 2^z - 2^{z-2}$. Let $C = C_G(G/M_2)$. Then $|C| \leq 2^{n-2}$ and by Burnside's Formula

$$
\begin{aligned}
k(G) &\leq (2 - 1/2)(|C|/8 + 2^z) + \tfrac{1}{2}(2^{n-3} + 2^z - 2^{z-2}) \\
&\leq 2^{n-3} + 2^{n-4} - 2^{n-5} - 2^{n-6} + 2^{n-4} - 2^{n-7} + 2^{n-5} \\
&< 2^{n-3} + 2^{n-4} + 2^{n-4} = 2^{n-2},
\end{aligned}
$$

again a contradiction.

Conversely, if one of (i) or (ii) holds, then Ito's Theorem and Corollary 2.30 in [I] imply that $G$ has only irreducible characters of degree 1 and $p$. ∎

LEMMA 11. *Assume that* $k(G) \geq 5p^{n-4}$, *that* $|G : G_2| \leq p^{n-4}$, *that* $|G : Z(G)| \geq p^5$, *and that* $|C_G(G/N)| \leq p^{n-4}$ *for all minimal normal subgroups* $N$ *contained in* $G_2$. *Suppose further that* $b(M) \geq 2$ *for all maximal subgroups* $M \lessdot G$. *Then* $G$ *has no maximal subgroup* $M$ *such that* $|M_2| = p^2$.

*Proof.* Suppose that there exists a maximal subgroup $M$ of $G$ with $|M_2| = p^2$. By Lemma 7(i), there are no invariant classes of breadth 1. Any invariant class of breadth 0 is contained in $Z(G)$ and any invariant class of breadth 2 is contained in $C_G(G/M_2)$. Since $|G_2/M_2| \geq p^2$, we have $b(G/M_2) \geq 2$, so that $|C_G(G/M_2)| \leq p^{n-3}$. Thus the total number $s$ of invariant classes of $M$ satisfies the inequality $s < |Z(G)| + 0 + |C_G(G/M_2)|/p^2 \leq 2p^{n-5}$. Then a contradiction arises from Lemma 6. ∎

*Proof of Theorem* 2. Let $G$ be a counterexample. By Lemma 11 we have $|M_2| \geq p^3$ for every maximal subgroup $M \lessdot G$. By Lemma 7(i), $G$ has no element of breadth 1. By Lemma 7(ii), Lemma 11, and Corollary 5, $G$ has no element of breadth 2. Let $M$ be a maximal subgroup of $G$ such that $M_2 \neq G_2$ (Lemma 8). Any invariant class of $M$ of breadth 0 is contained in $Z(G)$ and hence there are at most $p^{n-5}$ invariant classes of $M$ of breadth 0. There are no invariant classes of breadth 1 or 2 and the number of invariant classes of breadth greater than or equal to 3 is bounded by $|C_G(G/M_2|/p^3 \leq p^{n-5}$, where the last inequality holds since the group $G/M_2$ is not abelian. Thus $M$ has at most $2p^{n-5}$ invariant classes. Then Lemma 6 yields a contradiction. ∎

COROLLARY 6. *Assume that $G$ satisfies the condition* (ii) *of Theorem* 2, *that $|G : Z(G)| \geq p^5$, and that $|G_2| \geq p^3$. Then the subgroup $N$ in condition* (ii) *of Theorem* 2 *is unique*.

*Proof.* By contradiction suppose that $N_1$ and $N_2$ are two distinct minimal normal subgroups of $G$ such that $|G/N_i : Z(G/N_i)| \leq p^3$, for $i = 1, 2$. Put $C_i = C_G(G/N_i)$ and let $H = N_1N_2$. If $g \in C_1 \cap C_2$, then $[g, G] \leq N_1 \cap N_2 = 1$, so $C_1 \cap C_2 = Z(G)$. Note that $C_1C_2/H \leq Z(G/H)$ and so we have $|G/H : Z(G/H)| \leq |G : C_1C_2| = |G : C_1|\,|G : C_2|/|G : C_1 \cap C_2| \leq p^6/|G : Z(G)| \leq p$. Therefore $G/H$ is abelian and $|G'| \leq |H| = p^2$, a contradiction. ∎

*Proof of Corollary* 3. We can assume $b(G) = 3$ and $p \geq 5$. Then $|G_2| \geq p^3$, $|G : Z(G)| \geq p^2$, and $k(G) > |G|/p^3 \geq 5|G|/p^4$. By Theorem 2, Corollary 1, and Corollary 6 we have only to consider the case when there exists a maximal subgroup $M \lessdot G$ such that $|M_2| \leq p$. If $M$ is abelian then an argument similar to that used in Corollary 1 shows that $|G_2| = p^3$. Thus assume that $|M_2| = p$. Let $x \in G \setminus M$. Then $G_2 = [x, M]M_2$. Use the bar notation to denote the reduction modulo $M_2$ and let $A_x = \{[x, m] \| m \in M\}$. If $M_2 \subseteq A_x$ then $b_{\overline{G}}(\bar{x}) = 2$ and $\overline{G}_2 = \overline{A}_x$ so that $|G_2| = p \cdot |\overline{G}_2| = p^3$. Thus we are left with the case $M_2 \not\subseteq A_y$ for every $y \in G \setminus M$. Let $x \in G \setminus M$. We show that in this case $C_M(x) \leq Z(M)$. Assume that there exists $m \in C_M(x) \setminus Z(M)$. Choose an element $m' \in M$ such that $[m, m'] \neq 1$ and let $y = xm'$. Then $y \in G \setminus M$ and $[y, m] = [xm', m] =$

$[m', m] \in M_2 \setminus 1$. Hence $M_2 = \{[y, m^i] \| i = 0, 1, \ldots, p-1\} \subseteq A_y$, a contradiction. Since then $Z(G) = Z(M) \cap C_M(x) = C_M(x)$ and $C_G(x) = \langle x, C_M(x) \rangle$, we have $|G : C_G(x)| \leq p^3$ and $|G : Z(G)| \leq p^4$.

The converse is trivial. ∎

Let $p \geq 5$ be a prime. Then the group

$$G = \Big\langle a, b | a^p, b^p, [b, a, a, b], [b, a, b, a], [b, a, a, a], [b, a, b, b]^{-1}, [b, a, a, a, a], [b, a, a, a, b] \Big\rangle$$

is an example of $p$-group of breadth 3 satisfying the condition (iii) of Corollary 3.

LEMMA 12. *If $h \leq n-2$ is a natural number and $b(G) \leq h$ then $k(G) \geq p^{n-h} + h(p-1)$.*

*Proof.* Let $1 = Z_0 < Z_1 < \cdots < Z_n = G$ be a principal series for $G$. Then for all $g$ in $Z_{i+1} \setminus Z_i$ we have $b(g) \leq \min(i, h)$. Hence $G$ has at least $p + (p^2 - p)/p + \cdots + (p^h - p^{h-1})p^{h-1} + (p^n - p^h)/p^h = p^{n-h} + h(p-1)$ conjugacy classes. ∎

PROPOSITION 2. *If either $b(G) \leq 2$ or $G$ has a maximal subgroup which is abelian, then $k(G) \geq p^{n+2} + 2p - 2$.*

*Proof.* By Lemma 12 we only need to consider the case when there exists a maximal subgroup $M \lessdot G$ such that $M_2 = 1$. Since $M$ has at least $p$ $G$-invariant conjugacy classes, by Burnside formula we have

$$k(G) = (p - 1/p)s + k(M)/p \geq p^2 - 1 + p^{n-2} > p^{n-2} + 2p - 2. \blacksquare$$

PROPOSITION 3. *If either $b(G) \leq 3$ of $G$ has a maximal subgroup $M$ with $b(M) \leq 1$, then $k(G) \geq p^{n-3} + 3p - 3$.*

*Proof.* As above we only need to consider the case when there exists a maximal subgroup $M \lessdot G$ such that $b(M) \leq 1$. In the same way by Burnside formula and Lemma 1(ii), we have $k(G) = (p - 1/p)s + k(M)/p \geq p^2 - 1 + p^{n-3} + 1 - 1/p > p^{n-3} + 3p - 3$. ∎

As an immediate corollary to Theorems 1 and 2 and to the previous propositions we have Theorem 3. In case $p = 2, 3$, for which the inequalities of Theorems 1 and 2 are too weak, Proposition 1 supplies further information.

*Proof of Proposition* 1. Suppose $k(G) \geq 2p^{n-3}$. If $|G'| \leq p^3$, it is well known that $G'$ is abelian. So let $|G'| > p^3$. Then $G$ has at most $|G|/p^4$ linear characters. Note that the intersection $K$ of the kernels of the irreducible characters of $G$ of degree $p$ is trivial, otherwise the number of nonlinear characters would be less than $|G|/p^3 + |G|/p^4$, and then total number of the irreducible characters would be less than $2p^{n-3}$. Since $G'' \leq K$ by Theorem 5.12 of [I], we have that $G$ is metabelian.

Now assume that $k(G) \geq 2p^{n-4} + p^{n-5}$. As above if $|G_2| \leq p^4$, let $Z$ be a minimal normal subgroup contained in $G''$. Then $|G_2/Z| \leq p^3$ is an abelian subgroup of $G/Z$; this implies $G'' \leq Z$. Hence $|G''| \leq p$. If $|G'| \geq p^5$ then let $K$ as above and assume by contradiction that $|K| \geq p^2$. Let $|G : G_2| = p^x$. The irreducible characters of $G$ of degree $p$ are characters of $G/K$ and so their number is bounded by $p^{n-2}/p^2$. Thus $k(G) < 2p^{n-4} + p^x \leq 2p^{n-4} + p^{n-5}$, a contradiction. The claim follows again from Theorem 5.12 of [I]. ∎

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