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THE CARTER SUBGROUPS OF SOME CLASSICAL GROUPS

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ABSTRACT

It is shown that a Carter subgroup of a non-soluble unitary group can only be the normalizer of a Sylow 2-subgroup. This result, combined with previous results, implies that no finite simple group can be a minimal counterexample to the conjugacy conjecture of Carter subgroups in a finite group.

1. *Introduction*

A subgroup of a finite group is called a *Carter subgroup* if it is nilpotent and self-normalizing. By [**1**], any finite solvable group contains exactly one conjugacy class of Carter subgroups, and it is reasonable to conjecture that a finite group contains at most one conjugacy class of such subgroups. It has been shown in [**3**] that a smallest counterexample to this conjecture should be an almost-simple group, and a wide list of almost-simple groups that cannot be minimal counterexamples is given in [**15**]. This list includes, in particular, all finite simple groups, except for the unitary groups. So the main motivation for this paper is to fill this gap, although the determination of Carter subgroups of classical groups may be of some interest in itself. We prove the following theorem.

THEOREM 1.1. *Let q be a prime power, and assume that $H = \mathrm{Sp}_n(q)$, or $\mathrm{SO}_n^{\varepsilon}(q) \leqslant H \leqslant \mathrm{O}_n^{\varepsilon}(q)$ with q odd in this case, or $\mathrm{SU}_n(q) \leqslant H \leqslant \mathrm{U}_n(q)$. If H admits a Carter subgroup G, then either G is the normalizer of a Sylow 2-subgroup of H, or else one of the following statements holds:*

(1) *$H \in \{\mathrm{Sp}_2(3),\ \mathrm{SU}_2(9),\ 2.\mathrm{SU}_2(9)\}$ and G is the normalizer of a Sylow 3-subgroup of H;*

(2) *$H = \mathrm{U}_3(4)$ has order $2^3 \cdot 3^4$, and G has order $2 \cdot 3^2$.*

Moreover, when H is orthogonal, G is a 2-group, except possibly when $H = \mathrm{SO}_2^{\varepsilon}(q)$.

This result generalizes previous work of L. Di Martino, M. C. Tamburini and A. Zalesskii, who, in [**7**], determined the Carter subgroups of $\mathrm{Sp}_n(q)$, $\mathrm{O}_n^{\varepsilon}(q)$ with q odd, and $\mathrm{U}_n(q)$. Thus our generalization concerns the orthogonal and unitary groups, since $|\,\mathrm{O}_n^{\varepsilon}(q) : \mathrm{SO}_n^{\varepsilon}(q)| = 2$ and $|\,\mathrm{U}_n(q) : \mathrm{SU}_n(q)| = \sqrt{q}+1$. Our methods are in fact a refinement of those in [**7**].

From Theorem 1.1, combined with the results in [**15**], it follows that no finite simple group is a minimal counterexample to the conjecture of conjugacy of Carter subgroups. This in turn implies that if X is a minimal counterexample to the conjecture and p is any prime dividing the order of $X/\mathrm{Soc}(X)$, then p divides the order of every Carter subgroup of X.

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2. *Notations and preliminary results*

Let $\mathbb{F}_q$ be a finite field of characteristic r, and let τ be an automorphism of $\mathbb{F}_q$. For each $A=(\alpha_{ij})\in \mathrm{Mat}_n(q)$, we denote by A^τ the matrix (α_{ij}^τ), and by A^t we denote the transpose of A. We recall that a non-degenerate matrix J is called *reflexive* if it is symmetric or alternating or hermitian. More explicitly, this occurs if $J^t=J$, or if $J=A-A^t$ for some A, or if $J^t=J^\tau$, where τ is an automorphism of $\mathbb{F}_q$ of order 2. Let H be one of the classical groups listed in the introduction. It is well known (see, for example, [**4**]) that there exist τ of order less than or equal to 2 and a reflexive matrix J such that, for a suitable $d\in\mathbb{N}$, H coincides with

$$H_n^d(q,J,\tau)=\{X\in \mathrm{Mat}_n(q)\mid XJX^{\tau t}=J,\ \det(X)^d=1\}.$$

In the rest of the paper, for the sake of clarity, we will often denote $H_n^d(q,J,\tau)$ by $H_n^d(q)$ or H. It is worth noting that H is soluble (in particular, it has a unique conjugacy class of Carter subgroups) precisely when it is contained in one of the following groups: $\mathrm{O}_1(q)=C_2$, $\mathrm{U}_1(q)=C_{\sqrt{q}+1}$, $\mathrm{O}_2^\varepsilon(q)=D_{2(q\pm1)}$, $\mathrm{O}_3(3)$, $\mathrm{O}_4^+(3)$, $\mathrm{U}_2(4)$, $\mathrm{U}_2(9)$ and $\mathrm{U}_3(4)$.

For any subgroup G of $\mathrm{GL}_n(q)$, we write $\mathbb{F}_qG$ for the subalgebra of $\mathrm{Mat}_n(q)$ consisting of all linear combinations of elements of G with coefficients in $\mathbb{F}_q$. Given a partition $n=n_1+\ldots+n_k$, we identify $\mathrm{Mat}_{n_1}(q)\oplus\ldots\oplus\mathrm{Mat}_{n_k}(q)$ with the obvious subalgebra of $\mathrm{Mat}_n(q)$, namely with:

$$\{\text{block-diag}(A_1,\ldots,A_k)\mid A_i\in\mathrm{Mat}_{n_i}(q),\ i\leqslant k\}.$$

LEMMA 2.1. *Let G be a Carter subgroup of $H_n^d(q,J,\tau)$. Assume that $n=n_1+\ldots+n_k$, $\mathbb{F}_qG\leqslant \mathrm{Mat}_{n_1}(q)\oplus\ldots\oplus\mathrm{Mat}_{n_k}(q)$ and $J=\text{block-diag}(J_1,\ldots,J_k)$ with $J_i\in\mathrm{Mat}_{n_i}(q)$. For each $i\leqslant k$, denote by G_i the projection of G into $\mathrm{Mat}_{n_i}(q)$, and set $d_i=|\{\det(x)\mid x\in G_i\}|$. Then the following statements hold.*

(1) Whenever $n_i=n_\ell$, we may assume that $J_i=J_\ell$, except possibly when $H_n^d(q,J,\tau)$ is orthogonal.

(2) G_i is a Carter subgroup of $H_{n_i}^{d_i}(q,J_i,\tau)$.

(3) Let $k>1$, and assume that G_1 is the normalizer of a Sylow p_1-subgroup P_1 of $H_{n_1}^{d_1}(q)$ (for some prime p_1), and either that G_2 is the normalizer of a Sylow p_2-subgroup P_2 of $H_{n_2}^{d_2}(q)$ (p_2 a prime), or that $H_{n_2}^{q-1}(q)$ is soluble. Then $d_1=d_2=2$ in the orthogonal case, and $d_1=d_2=\sqrt{q}+1$ in the unitary case.

Proof. (1) This claim is obvious, since $H_{n_i}^{d_i}(q,J_i,\tau)$ is symplectic, unitary or orthogonal, according to whether $H_n^d(q,J,\tau)$ is.

(2) Clearly, each G_i is a subgroup of $H_{n_i}^{d_i}(q)$. Moreover, from the fact that G is nilpotent and self-normalizing, we obtain:

$$G=(G_1\times\ldots\times G_k)\cap H_n^d(q).$$

We want to show that G_i is self-normalizing in $H_{n_i}^{d_i}(q)$. To this purpose, setting $i=1$ for simplicity, we let $h_1\in N_{H_{n_1}^{d_1}(q)}(G_1)$. By the definition of d_1 given in the statement, there exists $x_1\in G_1$ such that $\det(x_1)=\det(h_1)$. Let $y\in G_2\times\ldots\times G_k$ be such that

$$g=\begin{pmatrix}x_1 & 0\\ 0 & y\end{pmatrix}\in G.$$

It follows that the matrix

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & y \end{pmatrix}$$

has the same determinant of g. Thus $h \in H_n^d(q)$, since it is an isometry with respect to J. Clearly, h normalizes $G_1 \times \ldots \times G_k$, and hence normalizes G. It follows that $h \in G$, and we conclude that $h_1 \in G_1$.

(3) $H_{n_1}^{d_1}(q)$ is a normal subgroup of $H_{n_1}^{q-1}(q)$. So we see that $P_1 = \widehat{P_1} \cap H_{n_1}^{d_1}(q)$, where $\widehat{P_1}$ is a Sylow p_1-subgroup of $H_{n_1}^{q-1}(q)$. We recall that the determinant map takes the normalizer of a Sylow p_1-subgroup of $H_{n_1}^{q-1}(q)$ onto the normalizer of a Sylow p_1-subgroup of its image in $\mathbb{F}_q^*$. Thus there exists

$$x_1 \in N_{H_{n_1}^{q-1}(q)}(\widehat{P_1})$$

such that $|\det(x_1)|$ is 2 in the orthogonal case and $\sqrt{q}+1$ in the unitary case. From $P_1 = \widehat{P_1} \cap H_{n_1}^{d_1}(q)$, it follows that $P_1^{x_1} = P_1$, and hence

$$\left(N_{H_{n_1}^{d_1}(q)}(P_1)\right)^{x_1} = N_{H_{n_1}^{d_1}(q)}(P_1);$$

that is, $G_1^{x_1} = G_1$. If G_2 is the normalizer of P_2, by the same argument there exists $x_2 \in H_{n_2}^{q-1}(q)$ such that $\det(x_1) = \det(x_2)$ and $G_2^{x_2} = G_2$. Thus the matrix block-diag$(x_1, x_2^{-1}, I_{n-n_1-n_2})$ belongs to $H_n^1(q) \leqslant H_n^d(q)$ and normalizes G. It follows that $x_1 \in G_1$ and $x_2 \in G_2$, and our claim follows in this case. Now let $H_{n_2}^{q-1}(q)$ be soluble, and suppose, by contradiction, that $d_2 < |\det(x_1)|$. As the determinant map takes a Carter subgroup of $H_{n_2}^{q-1}(q)$ onto a Carter subgroup of the image, G_2 cannot be a Carter subgroup of $H_{n_2}^{q-1}(q)$. Thus there exists y_2 in $H_{n_2}^{q-1}(q)\backslash G_2$ such that $G_2^{y_2} = G_2$. Clearly, there is a power y_1 of x_1 such that $\det(y_1) = \det(y_2)$. Using the matrix block-diag$(y_1, y_2^{-1}, I_{n-n_1-n_2})$ we conclude as above that $y_2 \in G_2$, a contradiction. We conclude that $d_2 = |\det(x_1)|$, and it follows easily that $d_1 = |\det(x_1)|$. □

LEMMA 2.2. *Let $n = 2m$, and let*

$$J = \begin{pmatrix} 0 & I_m \\ \pm I_m & 0 \end{pmatrix}.$$

Then the group

$$L = H_n^d(q, J, \tau) \cap \left\{ \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \,\middle|\, X, Y \in \mathrm{GL}_m(q) \right\}$$

does not contain any Carter subgroup of $H_n^d(q, J, \tau)$, except when $H_n^d(q) = \mathrm{SO}_2^+(q)$. In this case, $L = H_n^d(q)$.

Proof. It is readily checked that

$$L = \left\{ \begin{pmatrix} X & 0 \\ 0 & (X^{\tau t})^{-1} \end{pmatrix} \,\middle|\, X \in H \right\}$$

where $H = \mathrm{GL}_m(q)$ if $H_n^d(q)$ is symplectic or orthogonal, whereas $\mathrm{SL}_m(q) \leqslant H \leqslant \mathrm{GL}_m(q)$ if $H_n^d(q)$ is unitary. Assume that G is a Carter subgroup of $H_n^d(q)$ contained in $L \simeq H$. Then, by [**6**], G is the normalizer of a Sylow 2-subgroup of L: note that, if $r = 3$ and $H_n^d(q)$ is unitary, then $q \geqslant 9$.

If m is odd, assume first that $H_n^d(q) \neq \mathrm{SO}_n^+$. (In the orthogonal case the Witt index must be maximal because of the shape of J.) When m is odd and $H_n^d(q)$ is unitary, set

$$h = \begin{pmatrix} 0 & D \\ (D^\tau)^{-1} & 0 \end{pmatrix},$$

where $D = \mathrm{diag}(1, \ldots, 1, \alpha)$ with $\alpha \in \mathbb{F}_q$ such that $\alpha^{1-\sqrt{q}} = -1$. Otherwise, set $h = J$. In both cases, $h \in H_n^d(q)$ and h normalizes L. Therefore, $G^h = G^x$ for some $x \in L$. It follows that $hx^{-1} \in G$, and hence $h \in L$, a contradiction.

Finally, assume that m is odd, and that $H_n^d(q) = \mathrm{SO}_n^+$. In this case, if N denotes the normalizer of a Sylow 2-subgroup of $\mathrm{GL}_{m-1}(q)$, then by [**2**, Theorem 1],

$$G = N \times \mathbb{F}_q^* = \left\{ \text{block-diag}\left(Y, \eta, (Y^t)^{-1}, \eta^{-1}\right) \mid Y \in N,\ \eta \in \mathbb{F}_q^* \right\}.$$

Clearly also N^t is the normalizer of a Sylow 2-subgroup of $\mathrm{GL}_{m-1}(q)$; thus there exists $U \in \mathrm{GL}_{m-1}(q)$ such that $N^U = N^t$. In this case, we take

$$h = \begin{pmatrix} 0 & 0 & U & 0 \\ 0 & 1 & 0 & 0 \\ (U^t)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $h \in H_n^d(q)$ and normalizes G, a contradiction whenever $m > 1$. □

If T is any subgroup of G, then $\mathbb{F}_q T$ is a subalgebra of $\mathrm{Mat}_n(q)$ that satisfies the minimum condition on left ideals. Thus $\mathbb{F}_q T$ can be written in a unique way in the form $A_1 \oplus \ldots \oplus A_k$, where $k \geqslant 1$ and each A_i is an indecomposable two-sided ideal of $\mathbb{F}_q T$. Write $I_n = \sum_{i=1}^k e_i$; then each e_i is the unity of $A_i = e_i A = A e_i$, and $e_i e_j = 0$ if $i \neq j$. We shall call the e_is the *minimal central idempotents* of A. As they commute pairwise, they are simultaneously diagonalizable. Moreover, as they are pairwise orthogonal, up to conjugation in $\mathrm{Mat}_n(q)$ we may assume that $e_1 = \mathrm{diag}(I_{n_1}, 0_{n-n_1}), \ldots, e_k = \mathrm{diag}(0_{n-n_k}, I_{n_k})$, with $n_1 + \ldots + n_k = n$.

THEOREM 2.3. *Let G be a Carter subgroup of $H_n^d(q, J, \tau)$, and let T be any subgroup of G such that $G \leqslant TC_G(T)$. Assume that $\mathbb{F}_q T$ is decomposable, and let $E = \{e_1, \ldots, e_k\}$ be the set of minimal central idempotents of $\mathbb{F}_q T$, of respective ranks $n_1, \ldots, n_k$. Then the following statements hold.*

(1) *$\mathbb{F}_q G$ is decomposable and $\mathbb{F}_q G \leqslant \mathrm{Mat}_{n_1}(q) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(q)$.*

(2) *The map σ, defined by $X \mapsto J X^{\tau t} J^{-1}$, permutes E by acting as $(e_1, e_2) \ldots (e_{2h-1}, e_{2h})$, for some $h \geqslant 0$, and fixing $e_{2h+1}, \ldots, e_k$.*

(3) *If we set $m_i = n_{2i-1} + n_{2i}$ for $i \leqslant h$ and $m_i = n_i$ for $i > 2h$, the identity*

$$J = \text{block-diag}(J_1, \ldots, J_h, J_{2h+1}, \ldots, J_k), \text{ with } J_i \in \mathrm{Mat}_{m_i}(q),$$

holds.

(4) *The case $h \geqslant 1$ can occur only if $H_n^d(q, J, \tau)$ is orthogonal. Moreover, in this case, $\mathrm{Mat}_{m_i}(q) = \mathrm{Mat}_2(q)$, and the projection G_i of G coincides with $\mathrm{SO}_2^+(q)$, for each $i \leqslant h$.*

TABLE 1.

PH'	A_ℓ	B_ℓ or C_ℓ	D_ℓ	${}^2A_\ell$	${}^2D_\ell$
d	$(\ell+1, q-1)$	$(2, q-1)$	$(4, q^\ell - 1)$	$(\ell+1, \sqrt{q}+1)$	$(4, q^\ell+1)$
$\lvert D\rvert$	$\frac{1}{d}(q-1)^\ell$	$\frac{1}{d}(q-1)^\ell$	$\frac{1}{d}(q-1)^\ell$	$\frac{1}{d}\prod_{j=1}^{\ell}(\sqrt{q}-\eta_j)$	$\frac{1}{d}\prod_{j=1}^{\ell}(\sqrt{q}-\eta_j)$

Proof. (1) From $\mathbb{F}_qT \leqslant \mathbb{F}_qG$ it follows that the e_i belong to $\mathbb{F}_qG$, and from the assumption that $G \leqslant TC_G(T)$ it follows that the e_i are central in $\mathbb{F}_qG$. Thus

$$\mathbb{F}_qG = \bigoplus_{i=1}^{k} e_i(\mathbb{F}_qG)e_i \leqslant \mathrm{Mat}_{n_1}(q) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(q). \tag{2.1}$$

In particular, (2.1) gives a decomposition of $\mathbb{F}_qG$ as a direct sum of 2-sided ideals $e_i(\mathbb{F}_qG)e_i$, $i \leqslant k$. Hence $\mathbb{F}_qG$ is decomposable.

(2) By the definition of $H_n^d(q)$, every $X \in T$ satisfies the relation $\sigma(X) = X^{-1}$; hence σ fixes $\mathbb{F}_qT$. From the uniqueness of the central idempotents it follows that σ also fixes the set E, inducing a permutation of order less than or equal to 2.

(3) For $i \leqslant h$, σ interchanges e_{2i-1} with e_{2i}. This is equivalent to the fact that J interchanges, by conjugation, e_{2i-1} with e_{2i}. For $i > 2h$, σ fixes e_i. This is equivalent to the fact that J centralizes e_i.

(4) With respect to the partition $n = m_1 + \ldots + m_h + m_{2h+1} + \ldots + m_k$, we are in the situation of Lemma 2.1. In particular, for each $i \leqslant h$ the projection G_i of G into $\mathrm{Mat}_{m_i}(q)$ is a Carter subgroup of $H_{m_i}^{d_i}(q, J_{m_i}, \tau)$, for some d_i. On the other hand, by point (1), G_i consists of matrices of shape

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

By Lemma 2.2, this can happen only if $m_i = 2$ and $G_i = \mathrm{SO}_2^+(q)$. □

LEMMA 2.4. (1) *Let $R \neq 1$ be a Sylow r-subgroup of $H_n^d(q)$, and assume that $N_{H_n^d(q)}(R)$ is nilpotent. Then:*

$q = 2$, or

$q = 4$ and $H_n^d(q) \in \{\mathrm{SU}_2(4),\ \mathrm{U}_2(4),\ \mathrm{SU}_3(4)\}$, or

$q = 3, 9$ and $H_n^d(q) \in \{\mathrm{Sp}_2(3),\ \mathrm{SU}_2(9),\ 2.\mathrm{SU}_2(9)\}$.

(2) *Let P be a non-abelian Sylow p-subgroup of $H_n^d(q)$, with p odd and $p \neq r$. Then $N_{H_n^d(q)}(P)$ is nilpotent only when $n = 1$.*

Proof. (1) Note that $R \leqslant H'$, where H' denotes the derived subgroup of $H = H_n^{q-1}(q)$. Thus it is sufficient to show that $N_{H'}(R)$ is not nilpotent in the cases not mentioned in the statement. Clearly, that is equivalent to showing this fact for the projective image PH' of H'. When PH' is a group of Lie type, of rank $\ell = \ell(n)$, we know that R is normalized by the diagonal subgroup D. Since $C_D(R)$ is contained in the centre of PH', which is trivial, it is enough to show that D is non-trivial in the cases not mentioned in the statement. For this purpose we will use the information assembled in Table 1, where the η_j are the eigenvalues of the isometry τ, which induces a symmetry of the Dynkin diagram.

If PH' is of type D_ℓ, we assume that $\ell \geqslant 4$. In the non-twisted case, elementary arithmetic shows that D is trivial only if either $q = 2$, or $\ell = 1$ and $q = 2, 3$. In the twisted case, we may assume that $\ell \geqslant 2$, since $\mathrm{SL}_2(q) \simeq \mathrm{SU}_2(q^2)$. In particular, we can suppose that $\eta_1 = -1$. Again, we use elementary arithmetic to deduce that $|D| > 1$.

Recalling that $\mathrm{SL}_2(q) \simeq \mathrm{Sp}_2(q) \simeq \mathrm{SU}_2(q^2)$, we note that the groups H that are left out of the above analysis are essentially the orthogonal groups in dimension less than or equal to 6 (see, for example, [**10**, Proposition 2.9.1]). Using the isomorphisms

$$\mathrm{SO}_3(q) \simeq \mathrm{PGL}_2(q), \quad P\Omega_4^-(q) \simeq \mathrm{PSL}_2(q^2), \quad P\Omega_4^+(q) \simeq \mathrm{PSL}_2(q) \times \mathrm{PSL}_2(q),$$
$$P\Omega_5(q) \simeq \mathrm{PSp}_4(q), \quad P\Omega_6^+(q) \simeq \mathrm{PSL}_4(q), \text{ and } P\Omega_6^-(q) \simeq \mathrm{PSU}_4(q^2),$$

it is easy to see that we do not find any exceptions to our claim (a more careful check being necessary when $H \leqslant \mathrm{O}_4^+(3)$).

(2) Our claim follows from [**12**, Theorem 3.3], if H is symplectic or orthogonal in odd dimension. In the remaining cases it is shown in [**12**, Theorem 3.4] that, if P is a Sylow p-subgroup of H, then there exists a p' element of the derived subgroup H' that normalizes P but does not centralize $P \cap H'$. □

Theorem 2.5. *Let $H_n^d(q)$ and G be counterexamples to Theorem* 1.1, *with n minimal. Then $\mathbb{F}_q G$ is indecomposable.*

Proof. We assume, by contradiction, that $\mathbb{F}_q G$ *is* decomposable, and we let $E = \{e_1, \dots, e_k\}$ be the set of minimal central idempotents of $\mathbb{F}_q G$, of respective ranks $n_1, \dots, n_k$. Let $h \geqslant 0$ be the number of non-trivial orbits of the permutation σ on E. By Theorem 2.3, and in the same notation, we have

$$\mathbb{F}_q G \leqslant \underbrace{\mathrm{Mat}_2(q) \oplus \ldots \oplus \mathrm{Mat}_2(q)}_{h \geqslant 0 \text{ times}} \oplus \mathrm{Mat}_{m_{2h+1}}(q) \oplus \ldots \oplus \mathrm{Mat}_{m_k}(q).$$

As usual, for each i, we denote by G_i the projection of G into $\mathrm{Mat}_{m_i}(q)$. By Lemma 2.1, we see that each G_i is a Carter subgroup of $H_{m_i}^{d_i}(q, J_i, \mathrm{id})$, where $d_i = |\{\det(x) \mid x \in G_i\}|$.

By virtue of [**7**], we may suppose that $H_n^d(q)$ is not a symplectic group.

1. Assume that $H_n^d(q)$ is orthogonal. If the above decomposition has just one summand, then $h = 1$ and $n = 2$, by the assumption that $k > 1$. It follows that $H_n^d(q) = \mathrm{SO}_2^+(q)$, and this case is not a counterexample to Theorem 1.1. Thus there is more than one summand. By the inductive hypothesis, each G_i is the normalizer of a Sylow 2-subgroup P_i of $H_{m_i}^{d_i}(q)$. Also, by point (3) of Lemma 2.1, the group $H_{m_i}^{d_i}(q)$ must coincide with the full orthogonal group $\mathrm{O}_{m_i}^{\varepsilon_i}(q)$. It follows that $h = 0$ and each G_i is a 2-group. We conclude that G itself is a 2-group, and again $H_n^d(q)$ and G are not counterexamples to Theorem 1.1.

2. Assume that $H_n^d(q)$ is unitary. In this case, $h = 0$. We first show that the minimal central idempotents of $\mathbb{F}_q G$ have pairwise different ranks. To this purpose, let e_1 and e_2 have the same rank $n_1 = n_2$. As observed in Lemma 2.1, we may suppose that $J_1 = J_2$. By the inductive hypothesis we may assume that, for $i = 1, 2$, G_i is the normalizer of a Sylow p_i-subgroup of $H_{n_i}^{d_i}(q)$, except possibly when $n_1 = 3$ and $q = 4$. In the first case, by point (3) of Lemma 2.1, we have $d_1 = d_2 = \sqrt{q} + 1$, and hence $G_1 = G_2$. In the second case, we have $G_1 = G_2$ except when $1 = d_1 < d_2 = 3$. If so, G_1 must be the normalizer of a Sylow 2-subgroup of $\mathrm{SU}_3(4)$. Noting that $\mathrm{U}_3(4)$ is soluble, we can apply point (3) of Lemma 2.1 and obtain $d_1 = d_2 = 3$, a contradiction. So, in all cases, we may suppose that $G_1 = G_2$.

Now consider the matrix

$$x = \begin{pmatrix} 0 & (-I_{n_1})^{n_1} & 0 \\ I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n-2n_1} \end{pmatrix}.$$

Clearly, $x \in H_n^1(q) \leqslant H_n^d(q)$, and normalizes G. Hence $x \in G$, which is impossible.

Next we show that q is odd, and that G is the normalizer of a Sylow 2-subgroup of $H_n^d(q)$, which is not in contrast with Theorem 1.1.

Suppose that q is even. Note that there is no index i such that $H_{n_i}^{d_i}(q) = \mathrm{U}_3(4)$; in fact, direct computation shows that, in this case, $\mathbb{F}_q G_i$ would be decomposable. Thus, by the inductive hypothesis, G_i is the normalizer of a Sylow 2-subgroup R_i of $H_{n_i}^{d_i}(q)$. In particular, by point (3) of Lemma 2.1 , $d_i = \sqrt{q}+1$, for each i. Now we fix the notation so that $n_1 < n_2 < \ldots < n_k$. If $R_k = 1$, then $H_{n_k}^{d_k}(q) = G_k$ must be nilpotent. However, this gives $n_k = 1$, a contradiction. If $R_k \neq 1$, by Lemma 2.4 we must have $\mathrm{U}_{n_k}(q) = \mathrm{U}_2(4)$, and hence $H_n^d(q) \leqslant \mathrm{U}_3(4)$, which does not contradict Theorem 1.1.

We are left with q odd. Thus, by induction, G_i is the normalizer of a Sylow 2-subgroup P_i of $\mathrm{U}_{n_i}(q)$, for each i. By the structure of the Sylow 2-normalizers of $\mathrm{U}_{n_i}(q)$ (see [**2**]), the group algebra $\mathbb{F}_q G_i$ is indecomposable only if n_i is a power of 2. But then, by the same paper, $\widehat{P} = P_1 \times \ldots \times P_k$ is a Sylow 2-subgroup of $\mathrm{U}_n(q)$. Thus G is a subgroup of the normalizer N of $\widehat{P}$ in $H_n^d(q)$. Setting $P = \widehat{P} \cap H_n^d(q)$, we see that P is a Sylow 2-subgroup of $H_n^d(q)$, and that $G \leqslant N_{H_n^d(q)}(P)$. In particular, as P is a subdirect product of $\widehat{P}$, the centre of P is contained in the centre of $\widehat{P}$, which has shape $Z_1 \times \ldots \times Z_k$, where each Z_i is cyclic. We may assume that the ranks $n_2, \ldots, n_k$ of $e_2, \ldots, e_k$ are greater than 1. It follows that the involutions $U_2, \ldots, U_k$, which take $e_2, \ldots, e_k$ to their respective opposites and fix the remaining ones, have determinant 1. Moreover, they are the unique involutions in the centre of $\widehat{P}$, whose eigenspaces relative to the eigenvalue -1 have respective dimensions $n_2 < \ldots < n_k$. Now let $x \in N_{H_n^d(q)}(P)$. By the previous considerations, x centralizes each involution $U_2, \ldots, U_k$. It follows that $x \in \mathrm{Mat}_{n_1}(q) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(q)$, and hence $x = x_1 \ldots x_k$, where $x_i \in \mathrm{U}_{n_i}(q)$. Again, as P is a subdirect product of $\widehat{P}$, each x_i must normalize the projection P_i of $\widehat{P}$. Thus $x_i \in G_i$ for each i; that is, $x \in G$.

We conclude that $G = N_{H_n^d(q)}(P)$, where P is a Sylow 2-subgroup of $H_n^d(q)$. □

We state the following theorem in the notation in which it will be used in the proof of Theorem 1.1. We recall that a map $\sigma : \mathrm{Mat}_m(q^h) \longrightarrow \mathrm{Mat}_m(q^h)$ is called an *involution* if it is an anti-isomorphism of order 2.

Theorem 2.6. *Let σ be an involution of $\mathrm{Mat}_m(q^h)$, and set*

$$\mathrm{U}_m^\sigma(q^h) = \{X \in \mathrm{Mat}_m(q^h) \mid X\sigma(X) = I_m\}.$$

Then there exist an automorphism τ_1 of $\mathbb{F}_{q^h}$, of order less than or equal to 2, and a reflexive matrix $J_1 \in \mathrm{Mat}_m(q^h)$ such that σ is the map $X \mapsto J_1\, X^{\tau_1\, t}\, J_1^{-1}$. Moreover,

(1) *$\mathrm{U}_m^\sigma(q^h) = H_m^{q^h-1}(q^h, J_1, \tau_1)$ coincides with one of the classical groups under consideration, or*

(2) *q is even, $\tau_1 = \mathrm{id}$, J_1 is symmetric but not alternating and one of the following holds:*

  (a) *$m = 2\ell+1$ and $\mathrm{U}_m^\sigma(q) \simeq \mathrm{Sp}_{2\ell}(q)$;*

  (b) *$m = 2\ell+2$ and $\mathrm{U}_m^\sigma(q)$ is a split extension $N \cdot \mathrm{Sp}_{2\ell}(q)$, where N is a 2-group (we set $\mathrm{Sp}_0(q) = 1$).*

Proof. The centre Z of $\mathrm{Mat}_m(q^h)$ is preserved by σ. Therefore, σ acts on Z as an automorphism τ_1 of order less than or equal to 2. We identify Z with $\mathbb{F}_{q^h}$, and we consider the map $\mathrm{Mat}_m(q^h) \longrightarrow \mathrm{Mat}_m(q^h)$ defined by:

$$X \mapsto \sigma(X^{\tau_1 t}).$$

This map is a Z-algebra automorphism. So, by the Noether–Skolem theorem (see [**14**, 12.6]), there exists $J_1 \in \mathrm{GL}_m(q^h)$ such that $\sigma(X^{\tau_1 t}) = J_1 X J_1^{-1}$, for all $X \in \mathrm{Mat}_m(q^h)$. It follows that $\sigma(X) = J_1 X^{\tau_1 t} J_1^{-1}$. It remains to show that J_1 is reflexive.

From

$$X = \sigma^2(X) = J_1 J_1^{\tau_1 t^{-1}} X J_1^{\tau_1 t} J_1^{-1},$$

we obtain $J_1^{\tau_1 t} = \gamma J_1$, with $\gamma \in \mathbb{F}_{q^h}$. Applying $\tau_1 t$ to the last equation, we deduce that $\gamma^{\tau_1 + 1} = 1$. If $\tau_1 = \mathrm{id}$, then $\gamma = \pm 1$ and either J_1 is symmetric, or q is odd and J_1 is alternating. So we assume that $\tau_1 \neq \mathrm{id}$. Then we may apply Hilbert's Satz 90 (see [**13**, p. 93]) and deduce the existence of $\mu \in \mathbb{F}_{q^h}$, such that $\gamma = \mu/\mu^{\tau_1}$. In this case, we replace J_1 with μJ_1, which is hermitian, and we are done.

(1) The following facts are well known (see, for example, [**4**]). If $\tau_1 = \mathrm{id}$, J_1 is symmetric and q is odd, then $\mathrm{U}_m^\sigma(q^h) = \mathrm{O}_m(q^h)$. If $\tau_1 = \mathrm{id}$ and J_1 is alternating, then $\mathrm{U}_m^\sigma(q^h) = \mathrm{Sp}_m(q^h)$. If $\tau_1 \neq \mathrm{id}$ and J_1^t is hermitian, then $\mathrm{U}_m^\sigma(q^h) = \mathrm{U}_m(q^h)$.

(2) This claim is proved in [**7**]. □

LEMMA 2.7. *Let T be a subgroup of $H_n^d(q)$ such that $C = \mathbb{F}_q T$ is a subfield of* $\mathrm{Mat}_n(q)$, *of order q^n. Then every $\varphi \in \mathrm{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$ is realizable by conjugation via an element of $N_{H_n^1(q)}(C^*)$ except when $H_n^{q-1}(q) = \mathrm{O}_n^-(q)$, with q odd. In this case n is even, and the same fact holds for every automorphism φ that lies in the subgroup of index* 2 *of* $\mathrm{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$. *In particular, $N_{H_n^d(q)}(T) > T$ whenever $n > 1$ and $H_n^d(q) \neq \mathrm{SO}_2^-(q)$.*

Proof. It is well known that $N_{\mathrm{GL}_n(q)}(C^*)$ induces on C, by conjugation, the full group $\mathrm{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$. If $H_n^d(q) = H_n^d(q, J, \tau)$, denote by σ the map $X \mapsto J X^{\tau t} J^{-1}$. From $\sigma(T) = T$ it follows that σ fixes C, inducing an automorphism of order less than or equal to 2. If $\sigma_{|C}$ has order 1, then the relation $X^2 = \sigma(X)X = 1$ for each $X \in T$ implies that $|T| \leqslant 2$. Hence $\mathbb{F}_q T = \mathbb{F}_q$, which gives $n = 1$, and our claim is trivially true. So we may assume that $\sigma_{|C}$ has order 2, so that $|C : C_\sigma| = 2$. Moreover, by the definition of σ, we see that $\mathbb{F}_q I_n \leqslant C_\sigma$ when $\tau = \mathrm{id}$, and $\mathbb{F}_q I_n \cap C_\sigma = \mathbb{F}_{\sqrt{q}} I_n$ when $\tau \neq \mathrm{id}$. Since $\mathrm{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$ is cyclic, these facts imply that n is even when $\tau = \mathrm{id}$, whereas n is odd when $\tau \neq \mathrm{id}$. It is shown in [**7**] that

$$N_{\mathrm{GL}_n(q)}(C^*) = C^* N_{H_n^{q-1}(q)}(C^*). \tag{2.2}$$

Clearly, this disposes of the symplectic case, because all the elements have determinant 1. Assume that $H_n^{q-1}(q) = \mathrm{U}_n(q)$. Then n is odd, and hence

$$s = (1 + \sqrt{q},\ 1 + \sqrt{q} + \ldots + \sqrt{q}^{n-1}) = 1. \tag{2.3}$$

Now let $C^* = \langle c \rangle$. From $\sigma(c) = c^{\sqrt{q^n}}$ we deduce that

$$c^{\sqrt{q^n}-1} \sigma(c^{\sqrt{q^n}-1}) = c^{\sqrt{q^n}-1} (c^{\sqrt{q^n}-1})^{\sqrt{q^n}} = I_n.$$

This implies that $c^{\sqrt{q^n}-1} \in \mathrm{U}_n(q)$. As $\det(c)$ has order $q-1$, $\det(c^{\sqrt{q^n}-1})$ has order $\sqrt{q}+1$, by virtue of (2.3). This shows that $C^* \cap \mathrm{U}_n(q)$ contains matrices of any possible determinant in $\mathrm{U}_n(q)$, and we are done, by (2.2). Finally, we assume that $H_n^{q-1}(q) = \mathrm{O}_n^{\varepsilon}(q)$, and we consider the determinant map:

$$N \cap H_n^{q-1}(q) = N_{\mathrm{O}_n^{\varepsilon}(q)}(C^*) \longrightarrow \{\pm 1\}.$$

As the kernel of this map has index at most 2, the proof is complete, by (2.2). □

LEMMA 2.8. *Let $G = P \times O_{p'}(G)$ be a Carter subgroup of a finite group H, where p is a prime and P is a Sylow p-subgroup of G. If $C_H(P)$ is nilpotent, then $G = N_H(P)$. In particular, P is a Sylow p-subgroup of H.*

Proof. $Z(P) \times O_{p'}(G)$ is a Carter subgroup of $C_H(P)$. So, when this last group is nilpotent, we have $Z(P) \times O_{p'}(G) = C_H(P)$. Noting that $N_H(P)$ normalizes $C_H(P)$, and hence normalizes $O_{p'}(C_H(P)) = O_{p'}(G)$, we conclude that $N_H(P)$ normalizes G. □

3. *Proof of the main theorem*

Proof of Theorem 1.1. Let $H = H_n^d(q)$ and G be counterexamples to our theorem, with n minimal. Clearly, $n > 1$ and G is not a 2-group. Moreover, we may assume that $\mathbb{F}_q G$ is indecomposable, by Theorem 2.5, and that $H_n^d(q) \not\cong \mathrm{Sp}_n(q)$, by [**7**].

Suppose that T is any subgroup of G such that $G = TC_G(T)$. It follows from Theorem 2.3 that the algebra $\mathbb{F}_q T$ is indecomposable. If we make the further assumption that the order of T is not divisible by r, then $\mathbb{F}_q T$ is a simple algebra. Additionally, by a well-known theorem of Wedderburn, there exists a factorization $n = mht$ such that (up to conjugation):

$$\mathbb{F}_q T = \mathrm{Mat}_m(q^h) \otimes I_t, \qquad C_{\mathrm{Mat}_n(q)}(\mathbb{F}_q T) = I_m \otimes \mathrm{Mat}_t(q^h). \tag{3.1}$$

Let σ denote the involution of $\mathrm{Mat}_n(q)$ defined by $X \mapsto JX^{\tau t}J^{-1}$. The condition $\sigma(X) = X^{-1}$ for each $X \in T$ implies that $\sigma(\mathbb{F}_q T) = \mathbb{F}_q T$. It follows that σ also fixes $C_{\mathrm{Mat}_n(q)}(\mathbb{F}_q T)$, inducing an (anti)automorphism of order less than or equal to 2. We set

$$K_m = H_n^d(q) \cap \mathbb{F}_q T, \qquad K_t = H_n^d(q) \cap C_{\mathrm{Mat}_n(q)}(\mathbb{F}_q T). \tag{3.2}$$

If $\sigma_{|\mathbb{F}_q T} = \mathrm{id}$, then K_m is an elementary abelian 2-group. Otherwise, by Theorem 2.6, either: (i) K_m is one of the classical groups under consideration, or (ii) q is even and $K_m = 2^a \cdot \mathrm{Sp}_{m-s}(q^h)$ with $a \geqslant 0$, $s \in \{1, 2\}$. In this case, the centre of K_m is a 2-group. Moreover, a Carter subgroup of K_m can only be a 2-group, by [**7**]. Clearly, the same facts apply to K_t.

Note that, by Lemma 2.4, a Sylow r-subgroup of H is never self-normalizing. Thus $O_{r'}(G) \neq 1$.

Case 1. Suppose that $O_{r'}(G)$ is abelian. In this case, we set $T = O_{r'}(G)$. So $m = 1$, $\mathbb{F}_q T = \mathbb{F}_{q^h} \otimes I_t$ is contained in $C = C_{\mathrm{Mat}_n(q)}(\mathbb{F}_q T)$, and $G \leqslant C_{H_n^d(q)}(T)$. Hence G is a Carter subgroup of K_t. In particular, $\sigma_{|C} \neq \mathrm{id}$ and K_t is not of type (ii), since G cannot be a 2-group. Consider first the case $t = 1$, $h = n$. It follows that $\mathbb{F}_q T = C_{\mathrm{Mat}_n(q)}(\mathbb{F}_q T) = \mathbb{F}_{q^n}$ is a subfield of $\mathrm{Mat}_n(q)$ and $G = T \leqslant \mathbb{F}_{q^n}^*$. By Lemma 2.7, we have the contradiction that $T < N_{H_n^d(q)}(T)$, except when $H_n^d(q) = \mathrm{SO}_2^-(q)$. But this group is not a counterexample to Theorem 1.1.

Let us now consider the case where $t>1$. Because T is scalar over $\mathbb{F}_{q^h}$, it follows that T is contained in the centre of K_t, a group of type (i). It follows that either $R=\{1\}$, $G=T=K_t$, or G is the normalizer of a non-trivial Sylow r-subgroup R of K_t. The first alternative is ruled out by the fact that K_t should be scalar, with $t>1$, and this can happen only if K_t is an orthogonal 2-group. So suppose that $G=N_{K_t}(R)$. By Lemma 2.4, we know that $q^h\in\{2,3,4\}$. In particular, $h\leqslant 2$. However, $h=2$ implies that $q=2$, and $H_n^d(2)$ could only be orthogonal. Since we are not considering such groups, we have $h=1$ in all cases. Thus T is scalar over $\mathbb{F}_q$, and $G=R\times T$ is the normalizer of a Sylow r- subgroup of $H_n^d(q)$. From Lemma 2.4 we have $r=2$ or $H_n^d(q)\in\{\mathrm{Sp}_2(3),\ \mathrm{SU}_2(9),\ 2.\mathrm{SU}_2(9)\}$, which is consistent with Theorem 1.1.

We conclude that $O_{r'}(G)$ cannot be abelian; hence there exists a non-abelian Sylow p-subgroup P of G, with $p\neq r$.

Case 2. We set $T=P$, and we write $G=T\times S$. We note that $\sigma_{|\mathbb{F}_qT}\neq \mathrm{id}$ as T is non-abelian. As proved in [**7**, Main theorem], $T\times(\mathbb{F}_qT\cap S)$ is a Carter subgroup of K_m, and $Z(T)\times S$ a Carter subgroup of K_t. Thus, from $\mathbb{F}_qT\cap S\leqslant Z(\mathbb{F}_qT)\cap H_n^d(q)\leqslant Z(K_m)$ it follows that $T\times(\mathbb{F}_qT\cap S)=N_{K_m}(T)$. In particular, T must be an absolutely irreducible Sylow p-subgroup of K_m, with nilpotent normalizer. Since Carter subgroups of groups of type (ii) are 2-groups, if K_m or K_t were of type (ii), then T would be a 2-group and we would have $p=r=2$, against the assumption. Thus both K_m and K_t are of type (i), and Lemma 2.4 implies that $p=2$. By the absolute irreducibility of T over $\mathbb{F}_{q^h}$, m is a 2-power. Since T is non-abelian, $1<m$ and $t<n$. From $r\neq p=2$ and the inductive hypothesis, we have $Z(T)\times S=N_{K_t}(P_t)$ with either: (a) $P_t\in\mathrm{Syl}_2(K_t)$, or (b) $P_t\in\mathrm{Syl}_3(K_t)$.

In case (a), $P_t=Z(T)$ is scalar; it follows that $Z(T)\times S=K_t$ is nilpotent. As $K_t=C_H(T)$, by Lemma 2.8 we obtain $G=N_H(T)$, as desired. Note that when $H_n^d(q)$ is orthogonal and $n>2$, then the centralizer of T in $H_n^d(q)$ is a 2-group. This can be deduced using the description of T given in [**2**], and was proved in [**11**]. Thus, in the orthogonal case, G is a 2-group.

Finally let us consider case (b), in which $Z(T)\times S$ is the normalizer of a Sylow 3-subgroup of K_t, and $K_t\in\{\mathrm{Sp}_2(3),\ \mathrm{SU}_2(9),\ 2.\mathrm{SU}_2(9)\}$. Then S has order 3, and T must be a Sylow 2-subgroup of $N_{H_n^d(q)}(S)$. From $|N_{H_n^d(q)}(S):C_{H_n^d(q)}(S)|\leqslant 2$ it follows that $N_{H_n^d(q)}(S)=C_{H_n^d(q)}(S)$. Thus the generator s of S is not conjugate to s^{-1} in $H_n^d(q)$. This can only happen, however, if $m=1$ and $H_n^d(q)=K_t$. In fact, we assume that $m=2^a$ with $a>1$. Up to conjugation, we may assume that

$$J=\begin{pmatrix}0 & J_0\\ \pm J_0 & 0\end{pmatrix}\quad\text{and}\quad s=\begin{pmatrix}I & 0\\ I & I\end{pmatrix},$$

where $J_0\otimes I_t$ is the form fixed by K_m. We may also assume that

$$J_0=\begin{pmatrix}0 & I\\ \pm I & 0\end{pmatrix}.$$

Let $A=\text{block-diag}(I,-I)$. We have $AJ_0A^t=-J_0$. It follows that the matrix block-diag$(A,-A)$ belongs to $H_n^1(q)$ and inverts s, a contradiction. □

Corollary 3.1. *Let X be a minimal counterexample to the conjecture of conjugacy of Carter subgroups. Then there exists a prime p that divides the order of every Carter subgroup of X. Moreover, p^2 divides $|X|$.*

Proof. By proofs given in [**3**], the socle $\mathrm{Soc}(X)$ is simple non-abelian. From the minimality of X it follows that $X = \mathrm{Soc}(X)\, K$, for every Carter subgroup K of X. Combining Theorem 1.1 with the results in [**15**], we see that X cannot be simple. So there exists a p that divides $|X : \mathrm{Soc}(X)|$. But any such p must divide $|K|$, since $X/\mathrm{Soc}(X) \simeq K/(K \cap \mathrm{Soc}(X))$. Clearly, p^2 divides $|X|$; otherwise, up to conjugation, all the Carter subgroups of X would have to be contained in the normalizer of the same cyclic subgroup of order p. □

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