Calculating Character Tables
Old and New

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Conjugacy Classes

Let G be a permutation group of degree at most 10^7. Algorithms by Sims, Seress, Kantor, and others allow to get easily:

- $_1, \ldots, C_k$; its conjug
- $_1 = 1, \ldots, h_k$; their card
- $_1 = 1_G, \ldots, g_k$. represent

Structure constants

$Z = Z(\mathbb{C}G)$ has dimension $k = k(G)$ and is generated by $\widehat{C_r} = \sum_{g \in C_r} g$, the conjugacy class sums. Moreover

$$\widehat{C_r}\widehat{C_s} = \sum_t c_{rst}\widehat{C_t},$$

where

$$c_{rst} = |\{(x, y) \,:\, x \in C_r, y \in C_s, xy = z\}|,$$

z is a fixed element in C_t, the structure constants.

Given $\chi \in \mathrm{Irr}(G)$, $C_i = g_i^G$,

$$\omega_\chi(\widehat{C_i}) = \frac{|C_i|\chi(g_i)}{\chi(1)}$$

defines an algebra homomorphism from Z to $\mathbb{C}$.

Common Eigenvectors

Theorem 1 (Burnside 1902)**.** *Let* $M_r = (c_{rst}) \in (\mathbb{Z})_k$ *and* $\chi \in \mathrm{Irr}(G)$*.*
Then $(\omega_\chi(\widehat{C_1}), \ldots, \omega_\chi(\widehat{C_k}))^{tr}$ *is a common (right) eigenvector for* $M_1, \ldots, M_k$*.*

Numerical approach (McKay 1967):

- ✎ r; calculate

- ✎ k-steps p

Set $e = \mathrm{Exp}(G)$ and ε a primitive e-th root of 1 then $\chi(g) \in \mathbb{Z}[\varepsilon]$.

Problem: Convert approximated eigenvalues into $\mathbb{Z}[\varepsilon]$.

Integer Relation Detection

Input: $v = (\alpha_1, \ldots, \alpha_n) \in K^n$, $K = \mathbb{R}$ or $\mathbb{Q}$ (approximation).

Output: No relation or $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that $\sum_j m_j \alpha_j = 0$.

- ász) discovered in 1982; LLL (Lens
- PSLQ (Pa
  Ferguson.

Algebraic dependence: $v = (1, \alpha, \ldots, \alpha^n)$;

Exact character values: $v = (\chi(g), 1, \varepsilon, \ldots, \varepsilon^{\phi(e)-1})$;

Dixon algorithm

Further Input: Power maps π_n defined by $g_i^n \in C_{i\pi_n}$,

Prime Field approach (Dixon 1967):

- $_p$, $p \equiv_e 1$ prime, $z \in F$, $|z| = e$; $F = \mathbb{F}$
- $\theta = f(z)$, for $f \in \mathbb{Z}[x]$; $\theta \in$ Hom
- $_r$ mod p; reduce M
- $_F(M_r \mod p)$ with Berlekamp's algorithm; factorize r

- $_1 = 1$) common eigenvectors v in $\binom{k}{2}$ steps; obtain all
- χ and $i' = i^{\pi_2}$, then $\sum_i \frac{v_i v_{i'}}{h_i} \equiv_p \frac{|G|}{\chi(1)^2}$; if $v = v$
- $\sqrt{|G|}$; modify p s
- $^\theta$. pin down

Assume $\chi(x) = \sum_{s=0}^{e-1} m_s \varepsilon^s \in \mathbb{Z}[\varepsilon]$. Apply Inverse Fourier Transform using power maps to get $em_s \equiv_p \sum_{n=0}^{e-1} \chi(x^n)^\theta z^{-sn}$ and m_s.

Schneider algorithm

Let $g_r^{-1} \in C_{r'}$.

Theorem 2 (Schneider 1990). $s : (x, y, z) \mapsto (x^{-1}, z, y)$ *is a bijection between*

$$\{(x, y, z) : x \in C_r, y \in C_s, z \in C_t, xy = z\}$$

and

$$\{(x', z, y) : x' \in C_{r'}, z \in C_t, y \in C_s, x'z = y\}.$$

Hence

$$c_{rst}|C_t| = c_{r'ts}|C_s|.$$

Thus $h_r\chi(g_{r'})\chi(g_s) = \sum_t \chi(g_t)c_{rts}$ *and* $(\chi(g_1), \ldots, \chi(g_k))$ *is a common (left) eigenvector for all* M_r.

Michler-P.-Weller algorithm

Let $G = \bigsqcup_{i=1}^{n} Hy_i$ and $G = \bigsqcup_{i=1}^{d} Hx_iH$.

- G acts tra
- ${}_1, \ldots, \Lambda_d$ on $\Omega \times \Omega$; G has d c
- ${}_n \leq \mathrm{GL}_n(\mathbb{C})$; Set $\mathfrak{P} : G$
- ${}_i$ associated to Λ_i via $(A_i)_{(u,v)}$ equals Define the
  1 if $(u, v) \in \Lambda_i$, 0 otherwise.

- ${}_iA_j = \sum_{k=1}^{d} p_{ij}^{k} A_k$, where $p_{ij}^{l} = |\{w \in \Omega \,:\, (u,w) \in \Lambda_i, (w,v) \in$ A
  $\Lambda_j\}|$ and (u,v) is a fixed element in Λ_l.

- ${}_1, \ldots, A_d\rangle \leq (\mathbb{Z})_n$ be the adjacency algebra; Let $\mathcal{A} =$

- ${}_j = (p_{ij}^{k})$ and $\mathcal{D} = \langle D_1, \ldots, D_d\rangle \leq (\mathbb{Z})_d$. Let D

Theorem 3. *$\mathcal{A} = \mathrm{End}_G(\mathbb{C}\Omega)$. $\rho : A_j \mapsto D_j$ is the right regular representation and $\mathcal{A}$ and $\rho(\mathcal{A}) = \mathcal{D}$.*

Constituents of Permutation Characters

- $_H)^G = \mathrm{Tr}(\mathfrak{P})$ equals $\sum_{i=1}^{t} m_i\chi_i$, $m_i \in \mathbb{N}$, $\chi_i \in \mathrm{Irr}(G)$; $\pi = (1$
- $^T = \bigoplus_{i=1}^{t}((\mathbb{C})_{d_i} \otimes$ (Concrete
  $I_{m_i})$, $T \in \mathrm{GL}_n(\mathbb{C})$, $d_i = \chi_i(1)$;
- $^T = \mathrm{End}_G(\mathbb{C}\Omega)^T = \bigoplus_{i=1}^{t}(I_{d_i} \otimes (\mathbb{C})_{m_i})$; $\mathcal{A}$
- If $Z = Z($

Cyclic Algebras

Theorem 4. *Let R be a finite-dimensional semisimple commutative algebra over a separable field F. If $|F| >> 0$ then $R = F[r]$ is a cyclic algebra.*

- $\xrightarrow{\sim} \oplus_{i=1}^{s} F_i$, F_i field extensions of F. (Wedderb

- $_i = F[a_i]$. (Primitive

- $_F(a_i) \perp \min_F(a_j)$. If $|F| >>$

- $_i(x) \equiv \delta_{ij} \pmod{\min_F(a_j)}$. (Chinese

- $_i(x) = xp_i(x)$, then $a_i = q_i(a)$, where $a = a_1 \oplus \ldots \oplus a_s$. Set q
- $^{-1}(a)$, then $R = F[r]$. Set $r = \varphi$

Lagrange Polynomials

- Let ζ : Z … $_t$ be the right regular representation.
- $\zeta(Z \otimes \mathbb{Q})$ … $_0]$ and $Z \otimes \mathbb{Q} = \mathbb{Q}[z]$, where $\zeta(z) = z_0$.
- (Random … $_1, \ldots, z_t$ for $Z_0 = \zeta(Z)$, repeat $z_0 = \sum r_i z_i$, with r_i randomly chosen integers, until $\min(z_0)$ has distinct non-zero roots.
- From $\sigma(z$ … $_0) = \{\zeta_1, \ldots, \zeta_t\}$ build Lagrange polynomials L_i, $L_i(\zeta_j) = \delta_{ij}$.
- Then e … $_i = L_i(z)$ are the central primitive idempotents of $\mathcal{D} \otimes \mathbb{Q}$.

Character Values and Multiplicities

- There is a potents of $\mathcal{D}$.

- $_i$ equals m_i^2, where $m_i = (\chi_i, \pi)$. The rank

- $_i = \sum_j a_{ij} D_j$, $\chi_i(1) = |G : H| a_{i1}/m_i$. via $\mathcal{G}_{jl} = |\{y_p :$ If e $y_p g_l y_p^{-1} \in H x_j H\}|$.

- $_i(x) = \frac{1}{m_i} \sum_{j=1}^{d} \overline{a_{ij}} |\{g_p : g_p x g_p^{-1} \in H x_j H\}|$. χ

The Burnside-McKay-Dixon-Schneider algorithm is a particular case of the MPW algorithm when $H = 1$.

Tensor Powers

How could we use these constituents to get the full character table?

Theorem 5 (Brauer-Burnside). *Let $\chi \in \mathrm{Irr}(G)$ be faithful and $m = |\{\chi(g) : g \in G\}|$. If $\psi \in \mathrm{Irr}(G)$, then $\psi \le \chi^i$ for some $i = 0, \ldots, m-1$.*

(Proof by Dr. Spiga) Let $\lambda_2, \ldots, \lambda_m$ be the values of χ different from $\chi(1)$. Then $\rho = \prod_{j=2}^{m}(\chi - \lambda_j)$ is a non-zero multiple of the regular character for G, so $(\psi, \rho) \neq 0$ and $(\psi, \chi^j) \neq 0$ for some $0 \le j < m$.

Schur and Murnaghan functions

- $_m$) parametrized by partition of m. $\lambda \in \text{Irr}(\text{S}$

- $\lambda(g) = \frac{1}{m!}\sum_{\sigma\in \text{Sym}_m}\lambda(\sigma)\prod_{k=1}^{m}\chi(g^k)^{a_k(\sigma)}$, $a_k(\sigma)$ =number of χ k-cycles of σ, is a constituent of χ^m.

- $_d(\mathbb{C})$ on $V^{\otimes m}$, $d = \chi(1)$. irreducible

- $_2(\chi) = \pm 1$ get further decomposition If Schur-F considering $\text{O}_d(\mathbb{C})$ or $\text{Sp}_d(\mathbb{C})$.

Example 1. $m = 2$, $\chi^{id}(g) = \frac{\chi(g)^2+\chi(g^2)}{2}$, $\chi^{sgn}(g) = \frac{\chi(g)^2-\chi(g^2)}{2}$.

Inducing from Local Subgroups

- $^{G} : \lambda \in \mathrm{Irr}(C), C = \langle c \rangle \leq G]$ has finite index (Artin) Th
  in $\mathbb{Z}[\mathrm{Irr}(G)]$.

- Induce fro

- Irrationalit
  groups.

LLL and Integral Lattices

- ${}_1, \ldots, b_k \in \mathbb{R}^n$ linearly independent. b

- ${}^*_1, \ldots, b^*_k$ pairwise orthogonal, $b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{ij} b_j^*$, (Gram-Sc

  where $\mu_{ij} = \frac{(b_i, b_j^*)}{(b_j^*, b_j^*)}$.

- ${}_1, \ldots, b_k$ is LLL-reduced if We say th

  1. $|\mu_{ij}| \leq \frac{1}{2}$;
  2. $||b_i^* + \mu_{ii-1} b_{i-1}^*||^2 \geq \frac{3}{4} ||b_{i-1}^*||^2$.

Theorem 6 (Lenstra, Lenstra and Lovász 1982). *Given a basis b for a lattice L in $\mathbb{R}^n$ there exists a polynomial-time algorithm transforming b into an LLL-reduced basis.*

✡ $\sqrt{2^m}\,\min L$. If L has ra

✡ (Ajtai 199

✡ (Pohst 19

✡ (Euclid 30

Character Lattices

- Input a fin
- Construct ^{2k}, $k = k(G)$, with inner product matrix $\mathrm{diag}(|C_G(g_1)|^{-1}, \ldots, |C_G(g_k)|^{-1}) \otimes I_2$.
- Extract S $^{\perp}$.
- Apply LLL
- Look for v

Plesken Algorithm

- Assume { $_1, \ldots, \psi_r\}$ are linearly independent virtual characters and $\{\chi_1, \ldots, \chi_s\}$ are the unknown irreducibles.
- Get m $_{ij} \in \mathbb{Z}$ such that $\psi_i = \sum_j m_{ij}\chi_j$.
- $A = ((\psi$ $_i, \psi_j)_G) \in (\mathbb{Z})_r$, $M = (m_{ij}) \in (\mathbb{Z})_{r \times s}$. Then $MM^{tr} = A$.
- (Plesken
  and $M \in (\mathbb{Z})_{r \times s}$ without zero columns satisfying $MM^{tr} = A$.
- $r \leq$ Tr(A

$_{r \times k}$ have rows χ_i and ψ_i If $r = s$, t

^{-1}Y leads to irreducibles, $M \in$ Plesken(A). Check wh

D$_r$-type Lattices

- ✓ $_i$ of length 2. Extract vi

- ✓ $_i$: $(\psi_i, \psi_i) = 2$] is isometric to D$_r$. Check if L

- ✓ $_{r-1} + \psi_r = \pm 2\chi_r$, $\chi_r \in$ Irr(G). If $r \geq 5$ th

- ✓ $_i + \psi_j$, $\{i, j\} = \{1, 3\}, \{1, 4\}, \{3, 4\}$. If $r = 4$ th

- ✓ Find emb

Character Table of Sporadic Groups

- 122760. The O'Na
- $_1$, then $d = d(H\backslash G/H) = 7 = \dim(\mathcal{D}) = \dim(Z(\mathcal{D}))$. If $H = G$
- $_H)^G) = [10944, 26752, 32395, 32395, 37696, 52668, 85064]$ $cd(G|(1$
- 10944 is