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On minimally irreducible groups of degree the product of two primes

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Abstract. We determine the minimally irreducible groups with proper non-soluble socle, whose degree is the product of two primes.

1 Introduction

Let V be a finite-dimensional complex vector space and G be a subgroup of $\mathrm{GL}(V)$ with character χ. We say that G is *χ-minimally irreducible* (or simply *minimally irreducible* when χ does not need to be specified) if G acts irreducibly on V but all proper subgroups of G act reducibly. It is known that any such group is finite (see [19]). A number of papers on minimally irreducible groups have appeared over the years, mostly by Russian authors and mostly focussed on soluble groups. Recently, Dalla Volta and Di Martino classified minimally irreducible non-soluble groups when $\dim V$ is prime ([3]). The soluble case had been dealt with by Suprunenko in the early 1970s (see [24]). As far as we know, the case when $\dim V$ is the product of two primes (and V is defined over an arbitrary algebraically closed field) has been considered only in a series of papers by Kopylova (see [13], [14], [15]), and only when G is soluble and $\dim V$ is the square of a prime. The case when G is simple and $\dim V$ is the square of a prime should not, in principle, be difficult, since all (quasi)-simple irreducible groups of prime-power degree are known (see [18]). However the case when G is simple and $\dim V$ is a product of two distinct primes might require considerable effort.

While some work is under way on the case when G has an abelian socle, in the present paper we consider the case when G is a non-simple group with a non-abelian socle. We classify the minimally irreducible subgroups G of $\mathrm{GL}(V)$, where $V = V(n, \mathbb{C})$, $n = pq$ (with p, q primes, not necessarily distinct), such that G has a proper non-abelian minimal normal subgroup N. Denote by χ the character of G associated to the representation space V. By the minimality assumption, the restriction $\chi_{|N}$ of χ to N is reducible. Let $\theta = \theta_1$ be the character afforded by an irreducible submodule of $V_{|N}$. By Clifford's Theorem, $\chi_{|N} = e\sum_1^t \theta_i$, where $e = (\theta, \chi_{|N})$ and $\theta_1, \dots, \theta_t$ are distinct irreducible characters of N (conjugate under the action of

G). As N is non-abelian, we may assume, say, that $\theta(1) = p$. Then two possibilities arise:

(i) $t = q$, $e = 1$ (i.e. $\chi_{|N}$ is the sum of q distinct irreducible characters of N of degree p);

(ii) $t = 1$, $e = q$ (i.e. $\chi_{|N}$ is the sum of q copies of θ).

We will refer to (i) and (ii) as to the *non-homogeneous* and *homogeneous* case, respectively. The non-homogeneous case will be dealt with in Section 3, and the homogeneous case in Section 4.

Notation. For any $d \in \mathbb{N}$, $[d]$ denotes the set of natural numbers from 1 to d and $\mathbb{Z}_d$ denotes the ring of residue classes modulo d. For $a \in \mathbb{Z}$, $\bar{a}_d$ (or simply $\bar{a}$ if the context is clear) denotes the residue class of a modulo d. The annihilator of an integer a in $\mathbb{Z}_d$ is the set $\mathrm{Ann}_{\mathbb{Z}_d}(a) = \{\bar{x} \in \mathbb{Z}_d \mid \overline{xa} = 0\}$. For $a \in \mathbb{Z}$, $\pi(a)$ is the set of prime divisors of a, whilst $\pi^*(a)$ is the multiset of such divisors, that is, an integer occurs in $\pi^*(a)$ as often as it divides a. In particular, for $a, b \in \mathbb{Z}$, $\pi^*(a) \setminus \pi^*(b) = \pi^*(a/\mathrm{gcd}(a, b))$.

We write $X : Y$ for a group that is the semidirect product of a normal subgroup X by a subgroup Y. For any finite group L, we denote by $\mathrm{Irr}_n(L)$ the set of irreducible complex characters of L of degree n, and set $a_n(L) = |\mathrm{Irr}_n(L)|$. By $\mathrm{cd}(L)$ we denote the set of irreducible character degrees of L.

Let $\mathbb{F}$ be a field. The symbol $(\mathbb{F})_n$ stands for the full matrix algebra of degree n over $\mathbb{F}$, 1_n denotes the identity matrix of size n, and e_{ij} (for $i, j \in [n]$) denotes the elementary matrix with 1 in the (i, j) position and zeros elsewhere. For $A \in (\mathbb{F})_n$, A' denotes the transpose, and A^* the inverse-transpose of A. A field of order u will usually be denoted by $\mathbb{F}_u$. A finite simple group S of Lie type defined over $\mathbb{F}_u$ will be denoted by $S(u)$, unless S is of unitary type, when S will be denoted by $S(\sqrt{u})$. It is well known (see [2]) that any automorphism of S is the product of inner, diagonal, field and graph automorphisms. In particular, the field automorphisms form a cyclic group generated by the automorphism induced by the Frobenius map $x \mapsto x^r$ $(x \in \mathbb{F}_u)$, which will be denoted by Fr. We write $e(S)$ for the lower bound for the minimal degree of an irreducible projective representation of S, as given in [12, Table 5.3.A].

2 Preliminary and auxiliary results

In this section we first refine some of the results in [3], and record some (possibly not well-known) properties of minimally irreducible groups. Then we set up the machinery needed in Sections 3 and 4.

2.1 Groups of prime degree. Our first lemma has rather general scope, and it will be used frequently.

Lemma 2.1. *Let G be a χ-minimally irreducible subgroup of* $\mathrm{GL}(V)$, *and suppose that N is a normal subgroup of G such that $\chi_{|N}$ splits into q distinct irreducible constituents, where q is a prime. Then there exists $y \in G$ of q-power order such that $G = N\langle y\rangle$.*

Proof. It is clear that there exists $y \in G \setminus N$ acting as a q-cycle on $\{\theta_1, \dots, \theta_q\}$, where $\chi_{|N} = \sum_1^q \theta_i$. It follows that $\langle N, y\rangle$ is irreducible, whence by the minimality assumption $\langle N, y\rangle = G$. We may choose y to have order a q-power by, if necessary, replacing the 'original' y by a suitable power.

The next lemma partially refines the 'reductions' in [3].

Lemma 2.2. *Let G be a χ-minimally irreducible subgroup of* $\mathrm{GL}(V)$, *where* $V = V(p, \mathbb{C})$ *and p is a prime. Then either G is soluble* (*of length* 2), *or G is simple and non-abelian. In the latter case, the Sylow p-subgroups of G have order p.*

Proof. Let N be a proper normal subgroup of G. Since $\chi_{|N}$ is reducible, N is abelian. Therefore, if G is not perfect, it must be soluble of length 2. So assume that G is perfect. Then, by Lemma 2.1, $\chi_{|N} = p\lambda$ and $N \leqslant Z(G)$. Therefore G is quasi-simple. Since $G = G'$ implies that $G \leqslant \mathrm{SL}(p, \mathbb{C})$, either $|Z(G)| = 1$ or $|Z(G)| = p$. Observe that, for any $P \in \mathrm{Syl}_p(G)$, $\chi_{|P}$ splits into p irreducible constituents of degree 1. Therefore P is abelian. By a result on transfer ([9, (IV.2.2)]), $P \cap Z(G) \cap G' \leqslant P'$. Hence $Z(G) = 1$. It follows that G is simple. Finally, observe that $|x^G|$ is a p'-number for every $x \in P$. Thus, by a theorem of Burnside ([10, p. 36]), either $x \in Z(G)$ or $\chi(x) = 0$. Suppose that $\chi_{|P} = \sum_i e_i\lambda_i$. Then

$$p = \sum_i e_i \leqslant \sum_i e_i^2 = (\chi_{|P}, \chi_{|P}) = \left(\sum_{x\in P} |\chi(x)|^2\right)/|P| = p^2/|P|.$$

We conclude that $|P| = p$.

The classification of soluble minimally irreducible groups by Suprunenko ([24]) implies the following lemma (see [3] for a direct proof):

Lemma 2.3. *Let G be a soluble minimally irreducible subgroup of* $\mathrm{GL}(V)$ *of prime degree. Then G is a minimally non-abelian group, i.e. a non-abelian group all of whose proper subgroups are abelian.*

Minimally non-abelian groups are known as Miller-Moreno groups, and were classified by Rédei in 1947. As already mentioned, the minimally irreducible groups of prime degree which are Miller-Moreno groups were determined in [24], via a direct matrix analysis. For the reader's sake and for later use, we give an independent proof. Furthermore, we pin down the normal subgroups of these groups, as needed later on to prove that minimal irreducibility is preserved under direct products. We recall that the so called *modular* p-group of order p^{n+1} is the group with presentation

$$G := \langle a, b \mid a^{p^n} = b^p = 1, [a, b] = a^{p^{n-1}}\rangle.$$

Theorem 2.4. *A soluble group G is minimally irreducible of prime degree p if and only if it has cyclic centre and is one of the following*:

(i) *an extra-special p-group of order p^3*;

(ii) *a modular p-group*;

(iii) *a semidirect product* $W : X$, *where* X *is a cyclic* p*-group and* W *is an irreducible* $\mathbb{F}_r[X/\mho_1(X)]$*-module with* r *a prime*, $r \neq p$.

Proof. Let G be minimally irreducible of prime degree p. By Lemma 2.3 every proper subgroup of G is abelian. Moreover, by faithfulness, $Z(G)$ is cyclic. We claim that G cannot have two normal subgroups of coprime indices. For, otherwise, the degree $\psi(1)$ of any irreducible character ψ of G must divide two coprime integers, hence $\psi(1) = 1$, and G is abelian, a contradiction.

Suppose first that G has two distinct maximal normal subgroups M, N. Then both M, N have index p, and $M \cap N = Z(G) \geqslant G'$. Therefore G is nilpotent of class 2, and hence is a p-group (otherwise G would be abelian). Furthermore $Z(G) = \Phi(G)$, as M, N are arbitrarily chosen. Since the irreducible characters of G are either linear or have degree p, comparing their number with the number of conjugacy classes we obtain that $|G'| = p$. If $|Z(G)| = p$, then G is extraspecial of order p^3. Assume that $|Z(G)| > p$. Since $Z(G)$ is cyclic, for any maximal normal subgroup A of G, either A is cyclic or $A = A_0 \times Z(G)$, with $|A_0| = p$. Suppose that no maximal subgroup of G is cyclic. Then $G = \langle a, b, Z(G) \rangle$ for suitable elements a, b of order p, whence it follows that $G/\mho_1(Z(G))$ is elementary abelian of rank 3, a contradiction, since $|G : \Phi(G)| = p^2$. On the other hand, if A is cyclic of index p, then G is one of five groups (see [9, (I.14.9)]). But among these only the modular p-groups have centre of index p^2. Conversely, all such groups are minimally non-abelian and have cyclic centre, and hence admit faithful minimally irreducible representations of degree p.

Next suppose that G has a unique maximal normal subgroup N. Clearly $|G : N| = p$ and G/G' is cyclic. It follows that G is not a p-group; for otherwise $N = \Phi(G)$ and G would be cyclic. Let P be a Sylow p-subgroup of G, choose $x \in P \backslash N$ and set $X = \langle x \rangle$. Clearly N is the direct product of its Sylow subgroups. If $SX \neq G$ for each Sylow subgroup S of N then $S \leqslant C_G(x)$ would force G to be abelian. Therefore $G = RX$ for some Sylow r-subgroup R of N, where $r \neq p$. Since $(|R|, |X|) = 1$, we have $R = [R, X] \times C_R(X)$ (see [9, (III.13.4)]). If $C_R(X) \neq 1$ then $[R, X]X$ is a proper subgroup of G and hence it is abelian; but this implies that R commutes with X, and therefore G would be abelian. Thus $C_R(X) = 1$. Set $W = \Omega_1(R)$. Then $W = R$, for otherwise W would centralize X, which in turn would imply $R = 1$. Thus $R = W$ is elementary abelian, and by Maschke's Theorem, W is a completely reducible $\mathbb{F}_r[X]$-module. If W is reducible then $W = W_1 \oplus W_2$ for proper submodules W_1, W_2, but this again forces G to be abelian. Therefore W must be irreducible. We also observe here that $\mho_1(X) = C_X(W)$, and recall that W is an irreducible $\mathbb{F}_r[X/\mho_1(X)]$-module if and only if $\dim(W) = o(\bar{r})$, where $\bar{r}$ is the residue class of r modulo p (see [9, (II.3.10)]). Conversely, let $G = W : X$, where X is a cyclic p-group, W is an irreducible $\mathbb{F}_r[X/\mho_1(X)]$-module, and r, p are distinct primes. By Itô's Theorem, any irreducible representation of G of degree > 1 has degree p. We claim that at least one of these is faithful, and if so is minimally irreducible. Note first that, if N is a normal subgroup of G, then either $W \leqslant N$, or $W \cap N = 1$ and $N \leqslant O_p(G) = \mho_1(X)$. Since $W = G'$, it follows that if $\chi \in \mathrm{Irr}_p(G)$, then $\ker \chi \leqslant O_p(G)$. If no such characters were faithful, then $\bigcap_{\mathrm{Irr}_p(G)} \ker \chi \geqslant \Omega_1(X)$. Set $H = G/\Omega_1(X)$. Then $pa_1(H) = |X| = a_1(G)$ and $a_p(H) = a_p(G)$. Therefore

$$a_1(G) + p^2 a_p(G) = |G| = p|H| = pa_1(H) + p^3 a_p(H) = a_1(G) + p^3 a_p(G),$$

whence $a_p(G) = 0$, against G being non-abelian. Next, let A be a proper subgroup of G. If $A \geqslant W$, then A is abelian. So we may assume that $AW = G$ and, up to conjugation, $X \leqslant A$. If $1 \neq w \in A \cap W$ then $W = [w, X] \leqslant A$. Hence $A \cap W = 1$ and $A \cong X$. We have now shown that any faithful representation of G of degree p is minimally irreducible.

Remarks. We call the groups G listed in Theorem 2.4 *Suprunenko groups*. We observe that if G is of type (i) or (ii), then a non-trivial subgroup N is normal in G if and only if $G' \leqslant N$. If G is of type (iii), then the proof above shows that $N \lhd G$ if and only if either $W \leqslant N$ or $N \leqslant \mho_1(X) = Z(G)$.

We will make use of the list of non-abelian simple groups S admitting an ordinary representation of prime degree p (cf. [4], [16]). These groups, together with the relevant values of p, are given in the first two columns of Table 1 below. We also need to describe the structure of $\mathrm{Out}(S)$. This information is taken from [12, §§2.2–2.4] and the Atlas (see [2, p. xvi]). It is well known that for groups S of Lie type $\mathrm{Out}(S)$ is obtained from the diagonal, field and graph automorphisms (see [23] and [1, (12.5)]). The next arithmetical lemma, which slightly extends [4, Lemma 3.1], allows us to pin down the field and diagonal automorphisms exactly, and hence the structure of $\mathrm{Out}(S)$, as recorded in the last column of Table 1. In particular, notice that the projective special linear groups and the unitary groups of Lie rank $\geqslant 2$ in Table 1 have no diagonal automorphisms.

Lemma 2.5. *Let n be a positive integer, r be a prime and set $u = r^f$, where $f \geqslant 1$.*

(i) *If $(u^n - 1)/(u - 1)$ is a prime, then n is a prime, f is a power of n and* $\gcd(n, u - 1) = 1$.
(ii) *If $\frac{1}{2}(u^n - 1)$ is a prime, then either $nf = 1$, or n is an odd prime and $u = 3$.*
(iii) *If $(u^n + 1)/(u + 1)$ is a prime, then n is an odd prime,* $\gcd(n, u + 1) = 1$ *unless $(n, u) = (3, 2)$, and $f = 2^t n^s$ for some $t, s \geqslant 0$.*
(iv) *If $\frac{1}{2}(u^n + 1)$ is a prime, then nf is a 2-power.*

Proof. (i), (ii), (iv) are fairly standard and essentially contained in [4, Lemma 3.1]. The first part of (iii) is also in [4, Lemma 3.1]; so n is an odd prime. Denote by $\Phi_d(x)$ the dth cyclotomic polynomial (that is,

$$\Phi_d(x) = \prod_{\gcd(i,d)=1}^{i \in [d]} (x - \zeta^i),$$

where ζ is a primitive dth root of unity). Then $x^m - 1 = \prod_{d|m} \Phi_d(x)$ for each integer m, whence $p = \Phi_n(-u) = \Phi_{2n}(u)$. Suppose that q is a prime not dividing m. By the Möbius inversion formula, $\Phi_m(a^q) = \Phi_m(a)\Phi_{mq}(a)$ for each $a \in \mathbb{N}$ (cf. [17, Theorem 3.27]). Assume that f has a prime factor $q \notin \{2, n\}$, say $f = bq$. Then $p = \Phi_{2n}(r^{bq}) = \Phi_{2n}(r^b)\Phi_{2nq}(r^b)$, a contradiction, since p is a prime. Thus $f = 2^t n^s$

Table 1

S	p	range	T	$\mathrm{Out}(S)$
$\mathbb{A}_{r+1}$	r	$r \geqslant 7$	$\mathrm{PSL}(2,r)$	$\mathbb{Z}_2$
$\mathrm{PSL}(2,r)$	r	$r \geqslant 5$	—	$\mathbb{Z}_2$
$\mathrm{PSL}(2,u)$	$u-1$	$u = 2^n$	$N_S(U)$, $U \in \mathrm{Syl}_2(S)$	$\mathbb{Z}_n$
$\mathrm{PSL}(2,u)$	$u+1$	$u = 2^{2^s}$	—	$\mathbb{Z}_{2^s}$
$\mathrm{PSL}(2,u)$	$\frac{1}{2}(u+1)$	$11 < u = r^{2^s} \equiv 1 \bmod 4$	—	$\mathbb{Z}_2 \times \mathbb{Z}_{2^s}$
$\mathrm{PSL}(2,9)$	5		$\mathbb{A}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathrm{PSL}(2,u)$	$\frac{1}{2}(u-1)$	$u = r^n \equiv 3 \bmod 4$	$N_S(U)$, $U \in \mathrm{Syl}_r(S)$	$\mathbb{Z}_2 \times \mathbb{Z}_n$
$\mathrm{PSL}(n,u)$	$(u^n-1)/(u-1)$	$u = r^{n^s}$, $u > 2$, n odd	—	$\mathbb{Z}_2 \times \mathbb{Z}_{n^s}$
$\mathrm{PSp}(2^{t+1},u)$	$\frac{1}{2}(u^{2^t}+1)$	$u = r^{2^s}$	$\mathrm{PSp}(2^t,u^2).\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_{2^s}$
$\mathrm{PSp}(2n,3)$	$\frac{1}{2}(3^n-1)$	n odd	$\mathrm{PSL}(2,3^n)$	$\mathbb{Z}_2$
$\mathrm{PSU}(n,u)$	$(u^n+1)/(u+1)$	$u = r^f$, n odd, $f = 2^a n^b$	—	$\mathbb{Z}_{2^{a+1}n^b}$
$\mathrm{PSU}(3,3)$	7		$\mathrm{PSL}(2,7)$	$\mathbb{Z}_2$
$\mathrm{PSp}(6,2)$	7		$\mathbb{S}_8$, $E_{2^6} : \mathrm{PSL}(3,2)$	1
M_{11}	11		—	1
M_{12}	11		$\mathrm{PSL}(2,11)$	$\mathbb{Z}_2$
M_{24}	23		$\mathrm{PSL}(2,23)$	1
Co_i, $i = 2,3$	23		—	1

for some $t, s \geqslant 0$. Finally assume that $\gcd(n, u+1) > 1$. Since n is a prime, n divides $(u+1)$; write $u = -1 + n^a k$, where $\gcd(n,k) = 1$. Using the binomial expansion we see that $n^{a+1} \mid (u^n+1)$. Since $(u^n+1)/(u+1)$ is a prime, it follows that $n = (u^n+1)/(u+1)$. Also $n(u+1) \mid (u+1)^2$. Now $u^2 - u - 2 > 0$ if $u > 2$. Thus $(u+1)^2 < u^3 + 1 \leqslant u^n + 1$, unless $(n,u) = (3,2)$.

Remarks. Several comments are in order concerning the data in Table 1.

(1) We always denote by $u = r^f$, where r is a prime, the order of the field over which the relevant group of Lie type is defined, except in the unitary case, when u denotes the square root of the field order. Moreover, n always denotes a prime, unless $S = \mathrm{PSL}(2, r^n)$, in which case n may equal 1. The third column in Table 1 indicates further restrictions imposed on u and n. (The occurrences and ranges in Table 1 are fewer than in [4] since the latter paper takes into account all projective, not just ordinary, representations.)

(2) Concerning the cases $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u \pm 1)$, we recall that it is conjectured that there exist infinitely many pairs $(2p+1, p)$ of so-called Sophie Germaine's primes and infinitely many pairs $(2p-1, p)$ of so-called Legendre primes. We also note that u need not be a 3-power when $\frac{1}{2}(u+1)$ is prime.

(3) The group $\mathbb{A}_5 \cong \mathrm{PSL}(2,5) \cong \mathrm{PSL}(2,4)$ has a (monomial) representation of degree $5 = 4+1$ and $\mathbb{A}_6 \cong \mathrm{PSL}(2,9)$ has (primitive) representations of degree $5 = \frac{1}{2}(9+1)$. The group $\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$ has an irreducible (monomial) representation of degree $7 = 2^3 - 1$. However, if $n \geqslant 5$, $\mathrm{PSL}(n,2)$ has no irreducible representation of degree $2^n - 1$ (cf. [4]).

(4) The entries $S = \mathrm{PSp}(2^{t+1},u)$, $p = \frac{1}{2}(u^{2^t}+1)$ and $S = \mathrm{PSp}(2n,3)$, $p = \frac{1}{2}(3^n-1)$

in Table 1 correspond to the so-called 'Weil representations'. The general situation (see [28]) can be described as follows. The symplectic group $\mathrm{Sp}(2n,u)$, for any n and any odd u, has exactly two complex representations ρ, ρ^* of degree u^n. Each of these decomposes into two irreducible constituents; we have $\rho = \xi + \psi$, $\rho^* = \xi^* + \psi^*$, where ξ, ξ^* have degree $\frac{1}{2}(u^n+1)$ and ψ, ψ^* have degree $\frac{1}{2}(u^n-1)$. Let $Z = \langle -1_{2n} \rangle$ be the centre of $\mathrm{Sp}(2n,u)$. Then, by [28, Lemma 2.6], $Z \leqslant \ker\xi$ (ξ^*) if and only if $Z \not\leqslant \ker\psi$ (ψ^*). Thus only one of the values $\frac{1}{2}(u^n \pm 1)$ occurs as a character degree of $\mathrm{PSp}(2n,u)$ and, whichever it is, it is known to be the minimal non-trivial character degree of $\mathrm{PSp}(2n,u)$ (cf. [27, Corollary 5.4]). It is also well known that one obtains standard embeddings $\mathrm{Sp}(2m,u^d) \hookrightarrow \mathrm{Sp}(2n,u)$ via the trace map whenever $n = md$, $d > 1$ (see [9, (II.9.24)]). In particular, $\mathrm{SL}(2,u^n))$ embeds in $\mathrm{Sp}(2n,u)$. Since $\mathrm{PSL}(2,u^n)$ and $\mathrm{PSp}(2n,u)$ share the same minimal character degree, it follows that any Weil representation of S restricts irreducibly to $\mathrm{PSL}(2,u^n)$ (hence *a fortiori* to any overgroup $\mathrm{PSp}(2m,u^d)$). Finally, note that the extra constraints on n and u in Table 1 are only due to the requirement that p be a prime.

(5) The entry $\mathrm{PSU}(n,u)$ for $p = (u^n+1)/(u+1)$ and n odd also corresponds to 'Weil representations'. Here $(u^n+1)/(u+1)$ prime forces $(n,u+1) = 1$, so that $\mathrm{PSU}(n,u) \cong \mathrm{SU}(n,u)$. The latter group, for any odd value of n and any value of u has exactly u irreducible representations of degree $(u^n+1)/(u+1)$. This is known to be the next to minimal non-trivial character degree of $\mathrm{SU}(2n,u)$ (cf. [27, §4]). It is also known that these representations are minimally irreducible, except for $\mathrm{PSU}(3,3)$ (cf. [3], and (6) below).

(6) Whenever S is not minimally irreducible, we list in the fourth column of Table 1 examples of proper irreducible subgroups of S. The necessary information is essentially gathered from [3]. However the case $S = \mathrm{PSU}(3,3)$ is not thoroughly treated there. This group has exactly three irreducible representations of degree 7 and a unique conjugacy class of maximal subgroups $T \cong \mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$. While one of these representations restricts to the Steinberg character of $\mathrm{PSL}(2,7)$, so that $\mathrm{PSU}(3,3)$ is not minimally irreducible with respect to it, the other two restrict to the permutation action of $\mathrm{PSL}(3,2)$ on the points of the Fano plane $\mathrm{PG}(2,2)$. In the latter two cases, $\mathrm{PSU}(3,3)$ is indeed minimally irreducible.

For later use, the group $\mathrm{PSL}(2,9) \cong \mathbb{A}_6$ deserves a few more words. The group $\mathbb{A}_6$ has two conjugacy classes of subgroups isomorphic to $\mathbb{A}_5$, say $K^{\mathbb{A}_6}$ and $L^{\mathbb{A}_6}$, where K is a point stabilizer in the natural permutation representation π, while L is a 2-transitive subgroup obtained from the action of $\mathbb{A}_5$ on its six Sylow 5-subgroups. The group $\mathrm{PSL}(2,9)$ has two Weil representations θ, θ^* of degree 5. One of them, say θ, corresponds to the deleted natural permutation representation of $\mathbb{A}_6$, that is, $\theta = \pi - 1_{\mathbb{A}_6}$. The representation θ^* is obtained from θ via the diagonal automorphism ∂ of $\mathrm{PSL}(2,9)$. Indeed ∂, viewed as an automorphism of $\mathbb{A}_6$, maps 3-cycles to products of pairs of disjoint 3-cycles. It follows that ∂ interchanges the classes $K^{\mathbb{A}_6}$ and $L^{\mathbb{A}_6}$, $\pi^* = \pi^\partial$ differs from π and therefore $\theta^* = \pi^* - 1_{\mathbb{A}_6}$ is the other Weil representation. Clearly $\theta_{|K}$ is reducible, while $\theta_{|L}$ is irreducible (similarly for θ^*, with the roles of K, L interchanged).

Lemma 2.6. *Let S and p be as in Table* 1. *Then* $p \nmid |\mathrm{Out}(S)|$.

Proof. We see from the last column in Table 1 that $\pi(\mathrm{Out}(S)) \subseteq \{2, n\}$. However, since S is simple, it cannot have ordinary complex representations of degree 2. Thus we only have to rule out the case $p = n$. Clearly, if $S = \mathrm{PSL}(2, 2^n)$ and $p = 2^n - 1$, then $p > n$. If $S = \mathrm{PSL}(2, r^n)$ and $p = \frac{1}{2}(r^n - 1)$, then $r^n \geqslant 3^n > 2n + 1$. If $S = \mathrm{PSL}(n, u)$ and $p = (u^n - 1)/(u - 1)$, then again $p > n$. Finally, suppose that $S = \mathrm{PSU}(n, u)$ and $p = (u^n + 1)/(u + 1)$. Then $p = n$ implies $u^n \equiv -1 \bmod n$. Since $u^{n-1} \equiv 1 \bmod n$ by Fermat's Little Theorem, it follows that $n \mid (u + 1)$. Arguing as at the end of the proof of Lemma 2.5, we obtain $(n, u) = (3, 2)$; but in this case $S = \mathrm{PSU}(3, 2)$, which is not simple.

2.2 Technical tools. Suppose that G is a finite group admitting an irreducible character χ of degree pq, with p, q primes. Moreover, assume that N is a normal subgroup of G such that $\chi_{|N} = \sum_1^q \theta_i$, where $\theta_1, \ldots, \theta_q$ are distinct irreducible characters of N of degree p. (Note that, if G is also χ-minimally irreducible, then G/N must be a cyclic q-group by Lemma 2.1.) Our first proposition limits considerably the structure of a proper irreducible subgroup M of G. In particular, the structure of $H = M \cap N$ is quite restricted.

Proposition 2.7. *Let G and N be as above, and furthermore assume that G/N is a cyclic q-group. Let M be a maximal irreducible subgroup of G, and set $H = M \cap N$. Then $MN = G$ and $\chi_{|H} = \sum_1^t \eta_i$, where $\eta_1, \ldots, \eta_t$ are distinct irreducible characters of H, and either*

(1) $t = q$ *and* $\eta_i = \theta_{i|H}$, *or*

(2) $p = q$, $t = q^2$ *and* $\eta_i(1) = 1$ (*in which case H is abelian*).

Proof. Clearly M is a maximal subgroup of G. First we show that $M \not\geqslant N$. Set $\theta = \theta_1$ and assume the contrary. Then, since G/N is a cyclic q-group, $|G : M| = q$ and $M \equiv I_G(\theta)$. This contradicts the irreducibility of M, since by Clifford's Theorem M acts transitively on $\{\theta_1, \ldots, \theta_q\}$. Thus $MN = G$. Let $\mu = \chi_{|M}$. Since $H \trianglelefteq M$ we have $\mu_{|H} = e \sum_1^t \eta_i$, where $\eta_1, \ldots, \eta_t$ are distinct irreducible characters of H. Set $\eta = \eta_1$ and $R = I_M(\eta)$. Since $M/H \cong G/N$ is cyclic, η extends to a character ϕ of R. Then, by Gallagher's Theorem, $\eta^R = \sum_{\beta \in \mathrm{Irr}(R/H)} \phi\beta$, and by the Clifford correspondence $\eta^M = \phi^M = \sum_\beta (\phi\beta)^M$, where the characters $(\phi\beta)^M$ are the (distinct) irreducible constituents of ϕ^M. Since $(\eta^M, \mu) = (\eta, \mu_{|H}) \neq 0$, μ is a constituent of ϕ^M. It follows that

$$et\eta(1) = \mu(1) = (\phi\beta)^M(1) = |M : R|(\phi\beta)(1) = t\eta(1).$$

Hence $e = 1$. Moreover $t > 1$, since $\mu(1) = pq$ and $\eta(1) \leqslant \theta(1) = p$. Since $t = |M : R|$ is a q-power, we conclude that only two possibilities arise, namely either $t = q$ and $\eta(1) = p$, or $t = q^2$, $\eta(1) = 1$ and $p = q$.

Note that in the setting of Proposition 2.7, not only are the constituents η_i distinct, but they also form a single orbit under the action of M. The next proposition is a partial converse to 2.7, enabling one to build irreducible subgroups of G of degree pq from given irreducible subgroups of N of degree p.

Proposition 2.8. *Let G and N be as above, and H be a subgroup of N such that $\chi_{|H} = \sum_1^q \eta_i$ (with $\eta_i = \theta_{i|H}$) is a decomposition of $\chi_{|H}$ into distinct irreducible constituents. Set $\theta = \theta_1$. If $N_G(H) \not\leq I_G(\theta)$, then $N_G(H)$ has a cyclic q-subgroup $X \not\leq I_G(\theta)$ and, for any such X, $M = HX$ is irreducible with respect to $\chi_{|M}$.*

Proof. Since $N_G(H) \not\leq I_G(\theta)$ we have $G = N_G(H)I_G(\theta)$. Let $Q \in \mathrm{Syl}_q(N_G(H))$. Then $Q \not\leq I_G(\theta)$, since otherwise

$$|G|_q = \frac{|N_G(H)|_q\,|I_G(\theta)|_q}{|N_G(H) \cap I_G(\theta)|_q} = |I_G(\theta)|_q.$$

Therefore we may pick $x \in Q \backslash I_G(\theta)$. Set $X = \langle x \rangle$ and $M = HX$. As in Proposition 2.7, set $\mu = \chi_{|M}$ and $\eta = \eta_1$. Then $\mu_{|H} = \sum_1^q \eta_i$. Moreover, since $x^q \in I_G(\theta)$, we have $x^q \in R = I_M(\eta)$ and $q = |M : R|$. Arguing as in Proposition 2.7, we see that η extends to a character ϕ of R and $\eta^M = \sum_{\beta \in \mathrm{Irr}(R/H)} (\phi\beta)^M$, where the characters $(\phi\beta)^M$ are the (distinct) irreducible constituents of ϕ^M. Since

$$\mu(1) = q\eta(1) = |M : R|\eta(1) = (\phi\beta)^M(1),$$

we obtain that $\chi_{|M} = (\phi\beta)^M$ for some $\beta \in \mathrm{Irr}(R/H)$, and therefore $\chi_{|M}$ belongs to $\mathrm{Irr}(M)$.

We close this subsection with a technical result needed later. It gives a sufficient condition for certain extensions of a centreless group to admit a faithful irreducible character of degree pq.

Proposition 2.9. *Let $G = NCY$, where $N \trianglelefteq G$, $C = C_G(N)$ and $Y = \langle y \rangle$ is a cyclic q-group, q a prime. Suppose that $N \cap C = 1$ and there exist $\theta \in \mathrm{Irr}_p(N)$, p a prime, and $\gamma \in \mathrm{Irr}_1(C)$ such that the orbit of $\theta\gamma$ under Y has size q and $\bigcap_{i=0}^{q-1} \ker(\theta\gamma)^{y^i} = 1$. Then G has a faithful irreducible character χ of degree pq such that $\chi_{|NC} = \sum_{i=0}^{q-1} (\theta\gamma)^{y^i}$.*

Proof. By assumption $\theta\gamma \in \mathrm{Irr}_p(NC)$. Denote by I the inertia subgroup of $\theta\gamma$: then I has index q in G and, since I/NC is cyclic, $\theta\gamma$ extends to $\psi \in \mathrm{Irr}_p(I)$. It follows that $\chi = \psi^G \in \mathrm{Irr}_{pq}(G)$. We now show that χ is faithful. First note that $1 = (\chi, \psi^G) = (\chi_{|I}, \psi)$; thus $\theta\gamma$ is an irreducible constituent of $\chi_{|NC}$. Since Y acts transitively on the constituents of $\chi_{|NC}$, it follows by degree considerations that $\chi_{|NC} = \sum_{i=0}^{q-1} (\theta\gamma)^{y^i}$. Let $K = \ker \chi$. By our assumption, $K \cap NC = 1$. Thus for $g \in K$ and $x \in N$ we have $[g, x] \in K \cap N = 1$, whence $g \in C \cap K = 1$. We conclude that χ is faithful.

3 The non-homogeneous case

The general assumption in this section is that G has a proper non-abelian minimal normal subgroup N and an irreducible character χ whose restriction to N splits into pairwise distinct irreducible constituents. We emphasize again that, if G is χ-minimally irreducible, then $G = NY$, by Lemma 2.1, where Y is a cyclic q-group. Whether or not G is χ-minimally irreducible, N is the direct product of isomorphic simple groups, and the following holds:

Lemma 3.1. *Let G, N and χ be as above. Furthermore assume that $\chi \in \mathrm{Irr}_{pq}(G)$, where p, q are primes, $\chi_{|N} = \sum_1^q \theta_i$, where $\theta_i \in \mathrm{Irr}_p(N)$. Set $N = \prod_1^m S_i$, where each S_i is isomorphic to a non-abelian simple group S. Then either $m = 1$ (i.e. N is simple), or $m = q$.*

Proof. By the minimality of N, G acts transitively on the set $\{S_1, \dots, S_m\}$. Set $\theta = \theta_1$, $\theta = \sigma_1 \dots \sigma_m$, where $\sigma_i \in \mathrm{Irr}(S_i)$ for each i. As $\theta(1) = p$, we may assume that σ_i is trivial for each $i > 1$. Let $g \in I_G(\theta)$, and $1 \neq s_1 \in S_1$: then $\theta^g(s_1) = \theta(s_1) = \sigma_1(s_1)$. On the other hand,

$$\theta^g(s_1) = \theta(s_1^g) = \theta(s_i') = p\sigma_i(s_i'),$$

where $1 \neq s_i' \in S_i$ for some i. If $i > 1$ then $\sigma_i(s_i') = 1$, and hence $\sigma_1(s_1) = p$. This is clearly impossible since N is a minimal normal subgroup of G, and therefore $\ker \chi \cap N = 1$. Thus $S_1^g = S_1$; in other words, $I_G(\theta)$ is contained in the stabilizer of S_1 under the action of G on the simple factors of N. Since $|G : I_G(\theta)| = q$ and q is a prime, either $\mathrm{Stab}_G(S_1) = I_G(\theta)$, in which case $m = q$, or $\mathrm{Stab}_G(S_1) = G$, in which case $m = 1$.

3.1 Case I. In this subsection, we deal with the case when $m = 1$. Thus $N = S$ is a non-abelian simple group. Our first result establishes a significant restriction on the parameter q.

Lemma 3.2. *Suppose that G satisfies the following assumptions:*

(i) *$G = SY$, where S is a normal non-abelian simple group and Y is a cyclic q-subgroup;*

(ii) *G has an irreducible complex character χ of degree pq, with p, q primes, such that $\chi_{|S} = \sum_1^q \theta_i$, where $\theta_1, \dots, \theta_q$ are distinct irreducible characters of S of degree p.*

Then $\{\theta_1, \dots, \theta_q\}$ is an orbit of $\mathrm{Irr}_p(S)$ under the action of some cyclic q-subgroup of $\mathrm{Out}(S)$. In particular, q divides $|\mathrm{Out}(S)|$.

Proof. By Clifford theory, the irreducible constituents $\theta_1, \dots, \theta_q$ of $\chi_{|S}$ are G-conjugate. Since each of them is a class function on S, and since $G = SY$ it follows that they are also Y-conjugate. Thus Y cannot act on S by inner automorphisms. We conclude that Y maps to a non-trivial q-subgroup of $\mathrm{Out}(S)$. The claim follows.

Lemma 3.2 prompts us to determine for which S the set $\mathrm{Irr}_p(S)$ has non-trivial orbits under the action of $\mathrm{Out}(S)$. To this end, we inspect the irreducible simple groups of prime degree in Table 1, and pin down the action of the outer automorphisms of S on the corresponding characters. While most cases are fairly straightforward, the case when $S = \mathrm{PSL}(n,u)$, $p = (u^n - 1)/(u - 1)$ under the action of field automorphisms requires some arithmetical results. We collect them in a series of preliminary lemmas.

Suppose that $u = r^{n^s}$, where r and n are primes. Set $a_t = r^{n^t} - 1$ for $t \geqslant 0$, and $a_{-1} = 1$. Note that $b_t = a_{t+1}/a_t$ is an integer, since $a_t | a_v$ whenever $t \leqslant v$. In fact, $b_{-1} = \Phi_1(r)$ and $b_t = \Phi_n(r^{n^t})$ if $t \geqslant 0$. In view of this, we will make use of a nice property of cyclotomic polynomials:

Lemma 3.3. *For any two polynomials $f, g \in \mathbb{Z}[x]$ denote by* $\mathrm{rem}(f, g)$ *the remainder in the Euclidean division algorithm. Suppose that n is prime. Then, for any $d \in \mathbb{N}$,*

$$\mathrm{rem}(\Phi_n(x^{n^d}), \Phi_n(x)) = \mathrm{rem}(\Phi_n(x^{n^d}), \Phi_1(x)) = n.$$

In particular, for any integer a we have

$$\gcd(\Phi_n(a^{n^d}), \Phi_n(a)) = \gcd(n, \Phi_n(a)), \quad \gcd(\Phi_n(a^{n^d}), \Phi_1(a)) = \gcd(n, a - 1).$$

Proof. Since $x^n - 1 = \Phi_1(x)\Phi_n(x)$, it is enough to show that

$$\mathrm{rem}(\Phi_n(x^{n^d}), x^n - 1) = n$$

for any $d \in \mathbb{N}$. As $x^n \equiv 1 \bmod (x^n - 1)$ we have

$$\Phi_n(x^{n^d}) = \sum_{i=0}^{n-1} x^{in^d} \equiv n \bmod (x^n - 1),$$

and we are done.

We are now in a position to show the following:

Lemma 3.4. *If $t \neq v$, then* $\gcd(b_t, b_v) = 1$.

Proof. We know that $b_{-1} = \Phi_1(r)$ and $b_v = \Phi_n(r^{n^v})$ if $v \geqslant 0$. Thus, if $-1 = t < v$, then $\gcd(b_t, b_v) = \gcd(\Phi_1(r), \Phi_n(r^{n^v}))$. By Lemma 3.3 this equals $\gcd(n, r - 1)$. By Lemma 2.5 (i), we have $\gcd(n, u - 1) = 1$, whence $\gcd(n, r - 1) = 1$. Now assume that $-1 < t < v$ and set $d = v - t$, $a = r^{n^t}$. Then

$$\gcd(b_t, b_v) = \gcd(\Phi_n(a), \Phi_n(a^{n^d})) = \gcd(n, \Phi_n(a)).$$

Suppose that $\gcd(n, \Phi_n(a)) = n$. Then $a^n \equiv 1 \bmod n$, that is, $r^{n^{t+1}} \equiv 1 \bmod n$. Since $r^{n-1} \equiv 1 \bmod n$, it follows that $r \equiv 1 \bmod n$, a contradiction.

For application to the action of Fr on $\mathrm{Irr}_p(S)$, we separate the cases when n is odd and when $n = 2$.

Case 1. Suppose that n is odd. Consider the additive group $R = \mathbb{Z}_{u-1}$. Then multiplication by r defines an endomorphism μ of R of order n^s. We claim that μ, in its permutation action on R, has orbits of all possible lengths n^l with $0 \leqslant l \leqslant s$. First we record the following elementary facts (the map $\overline{cx}_a \mapsto \bar{x}_b$ provides the isomorphism in (i)).

Lemma 3.5. *Let $a, b, c \in \mathbb{N}$.*

(i) *If $a = bc$, then $\mathrm{Ann}_{\mathbb{Z}_a}(b) = c\mathbb{Z}_a \cong \mathbb{Z}_b$.*

(ii) *If b and c are factors of a, then $b\mathbb{Z}_a \cap c\mathbb{Z}_a = \mathrm{lcm}(b, c)\mathbb{Z}_a$.*

By Lemma 3.4, $\pi(a_s)$ is the disjoint union of the sets $\pi(b_t)$. Therefore the following definition makes sense.

Definition 3.6. Let $c \in \pi(a_s)$. The unique integer t such that $c \in \pi(b_{t-1})$ is called the weight of c and denoted by $w(c)$.

Now observe that if $\bar{i}, \bar{j} \in R$ and $\bar{i} = \overline{jk}$ for a unit $\bar{k}$, then clearly $\bar{i}$ and $\bar{j}$ lie in orbits of the same size under the action of μ. Thus we only need to consider those $\bar{j} \in R$ such that $\pi(j) \subseteq \pi(a_s)$.

Lemma 3.7. *The number of orbits of length n^l under the action of μ on R equals $(r^{n^{l-1}}(r^{(n-1)n^{l-1}} - 1))/n^l$ if $0 < l \leqslant s$, and $r - 1$ if $l = 0$. In particular, μ has orbits of all possible lengths n^l with $0 \leqslant l \leqslant s$. More precisely, given $\bar{j} \in R$ such that $\pi(j) \subseteq \pi(a_s)$, then $|\bar{j}^{\langle\mu\rangle}| = n^{w(c)}$, where c is a prime of maximal weight in $\pi^*(a_s) \backslash \pi^*(j)$.*

Proof. By definition, $u - 1 = a_s = \prod_{-1 \leqslant t < s} b_t$. Set $c_l = \prod_{l \leqslant t < s} b_t$, so that $u - 1 = c_l a_l$. Now $\bar{j} \in R$ lies in an orbit of length at most n^l if and only if a_l is annihilated by $\bar{j}$. By Lemma 3.5, $\mathrm{Ann}_R(a_l) = c_l R \cong \mathbb{Z}_{a_l}$. Set

$$m_{s,l} = |\{\bar{j} \in R - \{0\} \mid |\bar{j}^{\langle\mu\rangle}| = n^l\}|;$$

thus $a_l = m_{s,l} + a_{l-1}$. Since the number of cycles of length n^l is exactly $m_{s,l}/n^l$, we get the first part of the statement. The last part follows because the integers b_t are pairwise coprime, by Lemma 3.4.

Case 2. Suppose that $n = 2$. In this case also we must have $r = 2$. Setting $R = \mathbb{Z}_{u-1}$ as above, we let μ act by multiplication on the quotient set $\tilde{R} = R/\sim$, where $\bar{j} \sim \bar{i}$ if and only if $j \equiv \pm i \bmod (u - 1)$.

Lemma 3.8. *The number of orbits of length n^l under the action of μ on $\tilde{R}$ is $2^{2^{l-1}-l}(2^{2^{l-1}}-1)$ if $0 < l < s$, is $2^{2^{s-1}-s}(2^{2^{s-1}-1}-1)$ if $l = s > 1$, and is 2 if $l = 0$. In particular, μ has orbits of all possible lengths 2^l with $0 \leqslant l \leqslant s$, unless $(s, l) = (1, 1)$. More precisely, given $\bar{j} \in R$ such that $\pi(j) \subseteq \pi(a_s)$, and given an element c of maximal weight in $\pi^*(a_s)\backslash\pi^*(j)$, then $|[\bar{j}]_{\sim}^{\langle\mu\rangle}|$ equals $2^{w(c)-1}$ if $\pi^*(a_s)\backslash\pi^*(j) \subseteq \pi^*(b_{w(c)-1})$, and $2^{w(c)}$ otherwise.*

Proof. Since $a_t = 2^{2^t} - 1$ we have $b_t = 2^{2^t} + 1$. Assume first that $0 \leqslant l < s$. The μ-orbit of $[\bar{j}]_{\sim} \in \tilde{R}$ has size at most 2^l if and only if $2^{2^l} j \equiv \pm j \bmod (u-1)$, that is, if and only if $\bar{j} \in \mathrm{Ann}_R(a_l) \cup \mathrm{Ann}_R(b_l)$. By Lemma 3.5,

$$\mathrm{Ann}_R(a_l) \cap \mathrm{Ann}_R(b_l) = (a_s/a_l)R \cap (a_s/b_l)R = \mathrm{lcm}(a_s/a_l), (a_s/b_l))R.$$

But $b_l \mid a_s/a_l$ and $\gcd(b_l, (a_s/b_l)) = 1$ by Lemma 3.4, and thus

$$\mathrm{lcm}((a_s/a_l), (a_s/b_l)) = a_s.$$

It follows that $|\mathrm{Ann}_R(a_l) \cup \mathrm{Ann}_R(b_l)| = a_l + b_l - 1$ for $0 \leqslant l < s$. Set

$$n_{s,l} = |\{\bar{j} \in R \mid |[\bar{j}]_{\sim}^{\langle\mu\rangle}| = 2^l\}|.$$

Then $n_{s,l} = 2^{2^{l-1}+1}(2^{2^{l-1}}-1)$ if $1 \leqslant l < s$, whereas $n_{s,0} = 3$. Next assume that $l = s$. Then $\mathrm{Ann}_R(a_l) = R$, whence $n_{s,s} = 2^{2^{s-1}+1}(2^{2^{s-1}-1}-1)$ if $s > 1$, whereas $n_{1,1} = 0$. We now have to identify elements of R via $\sim$. Since $u-1$ is odd, each pair of non-zero opposite elements in R are identified via $\sim$. Thus there is exactly one fixed point under μ in $\tilde{R} - \{0\}$, whereas the number of $\langle\mu\rangle$-orbits of size $2^l > 1$ equals $n_{s,l}/2^{l+1}$. Finally, notice that $\overline{a_{w(c)}j} = 0$. If $\pi^*(a_s)\backslash\pi^*(x) \subseteq \pi^*(b_{w(c)-1})$, then

$$\overline{b_{w(c)-1}j} = 0 \neq \overline{a_{w(c)-1}j}$$

and the orbit of $[\bar{j}]_{\sim}$ has size $2^{w(c)-1}$. Otherwise $\overline{b_{w(c)-1}j} \neq 0$ and the orbit size equals $2^{w(c)}$.

In order to state and prove Theorem 3.9 below, we set up the following *ad hoc* notation: in Case 1, for $\bar{j} \in R$ set $s(j) = w(c)$, as defined in Lemma 3.7; in Case 2, for $[\bar{j}]_{\sim} \in \tilde{R}$ set $s(j) = w(c) - 1$ or $w(c)$, according to Lemma 3.8.

We are now ready to prove our 'Orbit Theorem'.

Theorem 3.9. *Let S be a non-abelian simple group having an irreducible complex character θ of prime degree p. Suppose that the orbit θ^X of θ under the action of some cyclic q-subgroup X of* Out(S), *where q is a prime, is non-trivial. Then $(S, p, |\theta^X|, |X|)$ is one of the following.*

(1) $(\mathrm{PSL}(2,u), \frac{1}{2}(u \pm 1), 2, 2)$. *Here θ is a Weil character and $X = \langle\partial\rangle$, where ∂ is the diagonal automorphism of S.*

(2) $(\mathrm{PSL}(2,u), u-1, q, q)$. *Here* $u = 2^q$, *and* S *has* $u/2$ *irreducible characters of degree* $u-1$. *Set* $\mathrm{Irr}_p(S) = \{\theta_j \mid 1 \leqslant j \leqslant u/2\}$. *Then* $\theta = \theta_j$ *for each* $j \neq \frac{1}{3}(u+1)$ *and* $X = \langle \mathrm{Fr} \rangle$.

(3) $(\mathrm{PSL}(n,u), (u^n - 1)/(u-1), n^t, n^{t+s-s(j)})$. *Here* $n = q \geqslant 2$ *is a prime and*

$$u = r^{n^s} > 2.$$

If $n > 2$, S *has* $u-2$ *irreducible characters of degree* p, *parametrized by* $\mathbb{Z}^*_{u-1}$, *whereas, if* $n = 2$, *it has* $\frac{1}{2}(u-2)$ *irreducible characters of degree* p, *parametrized by* $\mathbb{Z}^*_{u-1}/\sim$. *Set* $\mathrm{Irr}_p(S) = \{\theta_j\}$, *where* j *ranges over* $[u-2]$ *if* $n > 2$ *and over* $[(u-2)/2]$ *if* $n = 2$. *Then* $\theta = \theta_j$ *and* $X = \langle \mathrm{Fr}^{n^{s(j)-t}} \rangle$, *where* Fr *is the Frobenius automorphism of* S (*of order* q^s), $0 < t \leqslant s(j) \leqslant s$, *and* $(n,u) \neq (2, 2^2)$.

(4) $(\mathrm{PSL}(n,u), (u^n-1)/(u-1), 2, 2)$. *Here* n *is an odd prime*, $u = r^{n^s} > 2$. *Then* $\mathrm{Irr}_p(S) = \{\theta_j \mid 1 \leqslant j \leqslant u-2\}$, $\theta = \theta_j$ *for each* $j \neq \frac{1}{2}(u-1)$, *and* $X = \langle \gamma \rangle$, *where* γ *is the graph automorphism of* S.

(5) $(\mathrm{PSp}(2n,u), \frac{1}{2}(u^n \pm 1), 2, 2)$. *Here* θ *is a Weil character and* $X = \langle \partial \rangle$, *where* ∂ *is the diagonal automorphism of* S.

(6) $(M_{12}, 11, 2, 2)$. *Here* θ *is one of two deleted permutation characters and*

$$X = \mathrm{Out}(S) \cong \mathbb{Z}_2.$$

Proof. We inspect the simple irreducible groups of prime degree p (see Table 1).

(a) $S = \mathbb{A}_{p+1}$. This case is ruled out, since the only representation of degree $p > 5$ is the deleted permutation representation (see [4, Lemma 3.2]).

(b) $S = \mathrm{PSL}(2,u)$, $p = u$. It is a well known result (dating back to I. Schur) that there is only one irreducible representation of degree u (the so-called Steinberg representation). Thus this case is also ruled out.

(c) $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u \pm 1)$. There are exactly two representations of S for each of the stated values of p (see [6, Chapter 38], where the explicit character values are given). These so-called Weil representations are fixed by the field automorphism, but interchanged by the diagonal automorphism of S. Thus we get (1).

(d) $S = \mathrm{PSL}(2,u)$, u even, $p = u-1$. Here p prime forces $u = 2^q$, with q a prime. Let $b \in S$ be of order $u+1$. Then (cf. [6, Chapter 38]) $\mathrm{Irr}_p(S) = \{\theta_j \mid 1 \leqslant j \leqslant u/2\}$, and for all $j, m \in \{1, \ldots, u/2\}$ we have $\theta_j(b^m) = -(\alpha^{jm} + \alpha^{-jm})$, where $\alpha \in \mathbb{C}^*$ is a primitive $(u+1)$th root of 1. As u is even, $\mathrm{Inn}(S)$ has a complement in $\mathrm{Aut}(S)$, namely the cyclic group of order q generated by Fr. Moreover the action of Fr on θ_j is completely determined by the values of θ_j on the powers b^m ($m \in \{1, \ldots, u/2\}$). Observe that the eigenvalues of both b^2 and b^{Fr} are the squares of the eigenvalues of b, and hence their invariant factors, and so also the minimum and characteristic polynomial, coincide. It follows that b^2 and b^{Fr} are conjugate in $\mathrm{SL}(2,u)$, and therefore θ_j is Fr-invariant if and only if

$$\theta_j^{\mathrm{Fr}}(b^m) = \theta_j((b^m)^{\mathrm{Fr}}) = \theta_j(b^{2m}) = \theta_j(b^m),$$

that is, if and only if $\alpha^{jm} + \alpha^{-jm} = \alpha^{2jm} + \alpha^{-2jm}$ for all $m \in \{1, \dots, u/2\}$. Setting $m = 1$, this in turn implies $(\alpha^j - 1)(\alpha^{3j} - 1) = 0$, which occurs, as $j \leqslant u/2$, if and only if $j = \frac{1}{3}(u+1)$. Thus $\theta_{(u+1)/3}$ is the only character in $\mathrm{Irr}_p(S)$ fixed by Fr, while the remaining ones lie in $(u/2 - 1)/q$ Fr-orbits of size q.

(e) $S = \mathrm{PSL}(n,u)$, $p = (u^n - 1)/(u-1)$, $u > 2$. By Lemma 2.5 (i), n is a prime, $u = r^{n^s}$ for some prime r and $\gcd(n, u-1) = 1$. In particular, $\mathrm{PSL}(n,u) \cong \mathrm{SL}(n,u)$. Let us consider the action of S on the projective space $\mathrm{PG}(n-1,u)$ and let P be the stabilizer in S of a point of $\mathrm{PG}(n-1,u)$. Up to conjugation, we may assume that P consists of the $n \times n$ matrices

$$g(\beta, a, B) = \begin{pmatrix} \beta & a \\ 0 & B \end{pmatrix},$$

where $\beta \in \mathbb{F}_u^*$, $a \in \mathbb{F}_u^{n-1}$, $B \in \mathrm{GL}(n-1,u)$ and $\beta \det B = 1$. The map

$$g(\beta, a, B) \mapsto g(\beta, 0, B)$$

is an epimorphism from P to $K \cong \mathrm{GL}(n-1,u)$ with elementary abelian kernel $A \cong \mathbb{F}_u^{n-1}$, and $P = A : K$. It is well known that $\mathrm{GL}(n,u)' = \mathrm{SL}(n,u)' = \mathrm{SL}(n,u)$, unless $(n,u) = (2,2)$ or $(2,3)$ (cf. [9, (II.6.10)]). Now A is an irreducible K-module, and hence either $[A,K] = 1$ or $[A,K] = A$. But the former holds if and only if $(n,u) = (2,2)$ or $(2,3)$, and therefore is ruled out since S is simple. It follows that $P' = K'[A,K] = K'A$, whence $|P : P'| = |K : K'| = u - 1$, unless $(n,u) = (3,2)$, in which case $|P : P'| = 2$. Since by assumption $u > 2$, we conclude that

$$|\mathrm{Irr}_1(P)| = u - 1.$$

It is known (cf. [4], [5]) that inducing to S the non-principal linear characters of P one obtains all irreducible characters of S of degree p. Indeed, all such characters are monomial (by [4, Theorem 1.2]), and moreover the only subgroups of S having index p are the point stabilizers and their duals (cf. [27, Theorem 2.1]; see also [3, Remark 3.1.1]). However the dual of P is obtained by applying to P the invert-transpose automorphism of S, hence inducing non-principal linear characters from the dual of P to S gives rise to the same set of characters obtained from P.

In order to determine the orbits under outer automorphisms of S, we need a more detailed description of $\mathrm{Irr}_p(S)$. First observe that $g(\beta, a, B)P' \mapsto \beta$ is an isomorphism from P/P' to $\mathbb{F}_u^*$. Let $\mathbb{F}_u^* = \langle v \rangle$, choose a primitive $(u-1)$th root of unity ω in $\mathbb{C}$, and define $\rho : v^i \mapsto \omega^i$, $\lambda : g(v^i, a, B) \mapsto \rho(v^i) = \omega^i$. Then clearly $\mathrm{Irr}_1(P) = \langle \lambda \rangle$. We now set $\lambda_j := \lambda^j$ and $\theta_j := \lambda_j^S$ (where j is taken $\mathrm{mod}(u-1)$, $j \neq 0$), and proceed to evaluate θ_j on convenient elements of P. We need a suitable left transversal to P in S. For each $i \in [n]$, set $\varepsilon = (-1)^{\lceil i/2 \rceil}$, $J_i = \mathrm{antidiag}(\varepsilon, 1, \dots, 1)$ of size i and

$$t(i,v) = \begin{pmatrix} J_i & 0 \\ V(v) & 1_{n-i} \end{pmatrix},$$

where $v \in \mathbb{F}_u^{n-i}$ (column space) and $V(v)$ is the $(n-i) \times i$ matrix with first column v and zeros elsewhere. Then $T = \{t(i, v) \mid i \in [n], v \in \mathbb{F}_u^{n-i}\}$ is the required transversal. Indeed, $|T| = p$, and $t(i, v)$ maps the standard basis vector e_1 of $\mathbb{F}_u^n$ to $e_i + (0_i, v)^t$. Now we choose $c = \mathrm{diag}(v^{1-n}, v1_{n-1}) \in P$ and compute $\theta_j(c) = \sum_{t \in T} \lambda_j^0(t^{-1}ct)$. Setting $D_i = \mathrm{diag}(v^{1-n}, v1_{i-1})$, we get

$$t^{-1}ct = \begin{pmatrix} J_i^t D_i J_i & 0 \\ V(v)(v1_{n-i} - J_i^t D_i J_i) & v1_{n-i} \end{pmatrix}.$$

Since $J_i^t D_i J_i$ is obtained by reflecting D_i along the second main diagonal, it follows that $t^{-1}ct \in P$ if and only if either $i > 1$ or $(i, v) = (1, 0)$. Thus, for suitable diagonal matrices B_i,

$$\begin{aligned}\theta_j(c) &= \sum_{2}^{n-1} u^{n-i} \lambda_j(g(v, 0, B_i)) + \lambda_j(g(v^{1-n}, 0, B_1)) \\ &= ((u^{n-1} - 1)/(u-1))\omega^j + \omega^{j(1-n)}.\end{aligned}$$

We are now ready to prove (3). Since under our assumptions $\gcd(n, u-1) = 1$, S has no diagonal automorphisms: thus $\mathrm{Out}(S) \cong \Sigma : \Gamma$, where Σ and Γ denote the groups of field and graph automorphisms respectively (in particular, Γ is generated by the invert-transpose automorphism if $n > 2$, and is trivial if $n = 2$). Suppose first that $n > 2$. Now Σ is generated by the Frobenius automorphism Fr of order n^s and acts on $\mathrm{Irr}_p(S)$ via $\theta_j \to \theta_j^{\mathrm{Fr}}$, where $\theta_j^{\mathrm{Fr}}(s^{\mathrm{Fr}}) = \theta_j(s)$ for $s \in S$. We claim that $\theta_j^{\mathrm{Fr}^{-1}} = \theta_{rj}$. Observe that $\theta_j(c^{\mathrm{Fr}}) = \theta_j(c^r) = \theta_{rj}(c)$. Hence, to prove our claim, it will be enough to show that the value $\theta_j(c)$ uniquely determines θ_j. Set $\xi = \omega^{-j}$, $\psi = \omega^{-i}$ and observe that $\theta_i(c) = \theta_j(c)$ if and only if

$$\frac{u^{n-1} - 1}{(u-1)(\psi^{-1} - \xi^{-1})} = \xi^{n-1} - \psi^{n-1}.$$

If $\xi \neq \psi$, then

$$\frac{\xi^{n-1} - \psi^{n-1}}{\psi^{-1} - \xi^{-1}} = \xi\psi \sum_{k=0}^{n-2} \xi^{n-2-k} \psi^k,$$

whose absolute value is at most $n-1$. However $(u^{n-1} - 1)/(u-1) > n-1$ as $n > 2$. Therefore $\theta_i(c) = \theta_j(c)$ if and only if $i = j$. We conclude that the action of Σ on $\mathrm{Irr}_p(S)$ is permutationally equivalent to multiplication by r on $\mathbb{Z}_{u-1}^*$, as described in Lemma 3.7. Namely, there are Σ-orbits on $\mathrm{Irr}_p(S)$ of all sizes between 1 and n^s depending on j (an explicit algorithm was provided there to determine $s(j)$, where $|\theta_j^\Sigma| = n^{s(j)}$). Now suppose that $n = 2$, which in turn forces $r = 2$. In this case, θ_j is uniquely determined by its values on the powers of c (see [6, Theorem 38.2]). The same computations as above show that $\theta_i = \theta_j$ if and only if $i \equiv \pm j \bmod (u-1)$; in other words, $\mathrm{Irr}_p(S)$ is parametrized by $\mathbb{Z}^*_{u-1}/\sim$. In particular, the action of Σ on $\mathrm{Irr}_p(S)$ is permutationally equivalent to multiplication by 2 on $\mathbb{Z}^*_{u-1}/\sim$, as described in Lemma 3.8. Namely, all possible orbit sizes 2^l occur, where $0 \leqslant l \leqslant s$, unless $(s,l) = (1,1)$, i.e. $S = \mathrm{PSL}(2,4)$. Moreover, for any given $[\bar{j}]_\sim$, the integer $s(j)$ such that $|\theta_j^\Sigma| = n^{s(j)}$ can be explicitly computed. Note that $\theta_{(u-1)/3}$ is the unique Σ-invariant element of $\mathrm{Irr}_p(S)$; in particular, $\mathrm{PSL}(2,4)$ has a unique character of degree 5. Finally, observe that, for a given $X \leqslant \Sigma$, we have $|\theta_j^X| = n^t$ if and only if $|X : C_X(\theta_j)| = n^t$. Since Σ is cyclic and $|C_\Sigma(\theta_j)| = n^{s-s(j)}$, the latter occurs if and only if $|X| = n^{t+s-s(j)}$. (In particular, notice that, if θ_j is not Σ-invariant, then there exists a subgroup X of Σ such that θ_j^X has exactly size n.) The proof of (3) is complete.

It remains to consider the action of Γ on $\mathrm{Irr}_p(S)$. Here $n > 2$ and $\Gamma = \langle \gamma \rangle$, where γ is the invert-transpose automorphism. The matrix c defined above is inverted by γ and hence $\theta_j^\gamma = \theta_{-j}$. It follows that under the action of γ the set $\mathrm{Irr}_p(S)$ splits into $\frac{1}{2}(u-2)$ orbits of size 2 if u is even, and into $\frac{1}{2}(u-3)$ orbits of size 2 and a single fixed character if u is odd. Thus we have proved (4).

(f) $S = \mathrm{PSp}(2n,u)$, $n \geqslant 2$. Here the relevant representations of S are the Weil representations listed in Table 1 and discussed in the subsequent remarks. Since u is odd, $\mathrm{Out}(S) \cong \Sigma \times \langle \partial \rangle$, where Σ denotes the group of field automorphisms and ∂ is a diagonal automorphism of order 2 (cf. [12, Chapter 2]). Let $V = V(2n,u)$ be the underlying space of $\mathrm{Sp}(2n,u)$, and

$$M = \langle e_i \,|\, 1 \leqslant i \leqslant n \rangle, \quad N = \langle e_i \,|\, n+1 \leqslant i \leqslant 2n \rangle$$

be two maximal totally isotropic subspaces of V such that $V = M \oplus N$. We may order the basis $(e_i \,|\, 1 \leqslant i \leqslant 2n)$ of V so that the alternating form defining $\mathrm{Sp}(2n,u)$ is represented by the matrix

$$\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}.$$

Then ∂ is realizable by transforming via the matrix $\Delta = \mathrm{diag}(v1_n, 1_n)$, where v is a non-square in $\mathbb{F}_u^*$ (note that conjugating by Δ has order 2 modulo $\mathrm{Sp}{\cdot}Z(\mathrm{GL}(2n,u))$). We claim that, if θ is one of the Weil irreducible characters of $\mathrm{Sp}(2n,u)$, then $\theta^\partial \neq \theta$. Recall that $\mathrm{Sp}(2n,u)$ has two distinct classes of conjugate transvections. (This is well known. It suffices to look at transvections with the same centre and axis. This allows us to reduce to the case $n = 1$, and examine when

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

are conjugate under $\mathrm{SL}(2,u)$. This happens if and only if $\tau \in (\mathbb{F}_u^*)^2$.) In particular, $t = 1_{2n} + e_{1,n+1}$ and $t_v = 1_{2n} + ve_{1,n+1}$ are two symplectic transvections, not conjugate under $\mathrm{Sp}(2n,u)$, and such that $t^\partial = t_v$. We now want to evaluate θ on t and t_v. Recall that $\mathrm{Sp}(2n,u)$ has two characters ρ, ρ^* of degree q^n such that $\rho = \xi + \psi$, $\rho^* = \xi^* + \psi^*$, for which explicit formulas are given in [26, Proposition 2]. Computation shows that

$$\rho(t) = u^{n-1} \sum_{\alpha \in \mathbb{F}_u} \lambda(c^2) \quad \text{and} \quad \rho(t_v) = u^{n-1} \sum_{\alpha \in \mathbb{F}_u} \lambda(vc^2),$$

where $\lambda \in \mathrm{Irr}_1(\mathbb{F}_u, +)$. Let ψ be the quadratic character of $(\mathbb{F}_u^*, \cdot)$ and consider the 'Gaussian sum' $G_\lambda = G(\psi, \lambda) = \sum_{c \in \mathbb{F}_u^*} \psi(c)\lambda(c)$ (see [17, p. 192]). Denoting by $\square$ the subset of non-zero squares in $\mathbb{F}_u$, we have $G_\lambda = \sum_{c \in \square} \lambda(c) - \sum_{c \in \square} \lambda(vc)$. By the orthogonality relations $\sum_{c \in \mathbb{F}_u} \lambda(c) = 0$, and hence $G_\lambda = \sum_{c \in \mathbb{F}_u} \lambda(c^2)$. Setting $\lambda_v(c) = \lambda(vc)$, by [17, Theorem 5.12 (i)], we have $G_{\lambda_v} = -G_\lambda$. It follows that $\rho(t) = -\rho(t_v)$. In particular, since $|G|^2 = u$ by [17, Theorem 5.11], we see that $\rho(t) \neq \rho(t_v)$. Now, by [28, Lemma 2.6] we have $\rho(t) = \xi(t) + \psi(t)$, $\xi(t) = 1 + \psi(t)$, $\rho(t_v) = \xi(t_v) + \psi(t_v)$ and $\xi(t_v) = 1 + \psi(t_v)$. This readily implies that

$$\xi(t) \neq \xi(t_v) = \xi^\partial(t) \quad \text{and} \quad \psi(t) \neq \psi(t_v) = \psi^\partial(t).$$

The same holds for ξ^* and ψ^*. We conclude that $\xi^\partial = \xi^*$, $\psi^\partial = \psi^*$.

We now turn to the Frobenius action, and observe that $t^{\mathrm{Fr}} = t$. Since by the above

$$\theta^*(t) \neq \theta^*(t_v) = \theta(t) = \theta(t^{\mathrm{Fr}}) = \theta^{\mathrm{Fr}^{-1}}(t),$$

we conclude that Fr stabilizes θ.

(g) $S = \mathrm{PSU}(n,u)$, $p = (u^n+1)/(u+1)$, $n \geqslant 3$. Here $(u^n+1)/(u+1)$ prime forces $\gcd(n, u+1) = 1$. Thus $\mathrm{PSU}(n,u) \cong \mathrm{SU}(n,u)$, and (cf. [12, Chapter 2]) $\mathrm{Out}(S) \cong \Sigma$, the group of field automorphisms. Let $\mathbb{F}_{u^2}^* = \langle v \rangle$, $\mu = v^{u-1}$ and $\xi = \omega^{u-1}$, where ω is a primitive (u^2-1)th root of 1 in $\mathbb{C}$. It was proven in [28, §4] that $\mathrm{Irr}_p(S) = \{\zeta_n^i \mid 1 \leqslant i \leqslant u\}$ and, for $g \in S$,

$$\zeta_n^i(g) = \frac{(-1)^n}{u+1} \sum_{l=0}^{u} \xi^{-il}(-u)^{\dim \ker(g - \mu^{-l}1)}. \qquad (*)$$

We claim that $\mathrm{Irr}_p(S)$ is fixed elementwise by $\Sigma = \langle \mathrm{Fr} \rangle$. Indeed, the value $\zeta_n^i(g)$ only depends on $\dim \ker(g - \mu^{-l}1)$. Since the rank of a matrix A is the same as the rank of A^{Fr}, we have

$$\dim \ker(g - \mu^{-l}1) = \dim \ker(g^{\mathrm{Fr}} - \mu^{-lr}1).$$

We conclude from $(*)$, with μ replaced by μ^{Fr}, that each ζ_n^i is Fr-invariant. Thus the case $S = \mathrm{PSU}(n,u)$ is ruled out.

(h) $S = \mathrm{PSp}(6,2)$. This case is ruled out, since $\mathrm{Out}(\mathrm{PSp}(6,2))$ is trivial (see [12, Chapter 2]).

(i) The sporadic groups in the list all have trivial outer automorphism groups, except for $S = M_{12}$. In the latter case $\mathrm{Out}(S)$ has order two, and S has precisely two irreducible complex representations of degree 11, which are interchanged by an outer automorphism.

Another ingredient needed to control outer automorphisms concerns the embeddings of automorphism groups of symplectic groups in symplectic groups of larger degree over smaller fields. Let $\mathbb{F}$ be a field of order u contained in a field $\mathbb{E}$ of order u^d. As already mentioned, a standard embedding of the group $\mathrm{Sp}(2m, u^d)$ in $\mathrm{Sp}(2md, u)$ is obtained via the trace map of $\mathbb{E}$ over $\mathbb{F}$. It is convenient to have explicit control of the embedding. This can be done exploiting the so-called regular representation of $\mathbb{E}$ over $\mathbb{F}$. Fix an $\mathbb{F}$-basis b for $\mathbb{E}$. Then, given $x \in \mathbb{E}$, right multiplication by x induces an $\mathbb{F}$-linear transformation on $\mathbb{E}$. Denoting by $R(x)$ the matrix of x with respect to b, the map $x \mapsto R(x)$ is a representation of $\mathbb{E}$ as a subalgebra of $(\mathbb{F})_d$, the regular representation of $\mathbb{E}$ over $\mathbb{F}$. Moreover it is known (cf. [22]) that b may be chosen so that $R(x)$ is a symmetric matrix for every $x \in \mathbb{E}$. Then the following holds:

Lemma 3.10. *Let b be an $\mathbb{F}$-basis of $\mathbb{E}$ such that the regular representation R is symmetric. Define the map $R_0 : (\mathbb{E})_{2m} \to (\mathbb{F})_{2md}$ as follows: $R_0(A) = (R(a_{ij}))$ for $A = (a_{ij}) \in (\mathbb{E})_{2m}$. Then $R_0(\mathrm{Sp}(2m, u^d)) \leqslant \mathrm{Sp}(2md, u)$.*

Proof. It is readily checked that R_0 is an $\mathbb{F}$-algebra monomorphism. Without loss of generality, we may assume that $g = (g_{ij}) \in \mathrm{Sp}(2m, u^d)$ if and only if $g e_{2m} g' = e_{2m}$, where $e_{2m} = (e_{12} - e_{21}) \otimes 1_m$. Since

$$R_0((g_{ij}))' = (R'(g_{ji})) = (R(g_{ji})) = R_0((g_{ji})),$$

we see that $R_0(g') = R_0(g)'$. Thus $R_0(g)R_0(e_{2m})R_0(g)' = R_0(e_{2m})$. Since

$$R_0(e_{2m}) = e_{2md},$$

we deduce that $R_0(\mathrm{Sp}(2m, u^d)) \leqslant \mathrm{Sp}(2md, u)$.

We can now determine when a group G satisfying the assumptions of Lemma 3.2 is χ-minimally irreducible. It turns out that minimal irreducibility depends heavily on the image of Y in $\mathrm{Out}(S)$, which can be determined from Theorem 3.9. In particular, the two families of symplectic groups in Table 1 behave quite differently.

Theorem 3.11. *Let p and q be primes. Suppose that G has a non-abelian simple normal subgroup S, and that there exist $\theta \in \mathrm{Irr}_p(S)$ and $X \leqslant \mathrm{Out}(S)$ such that the orbit of θ under the action of X has length q. Set $\theta^X = \{\theta_1, \dots, \theta_q\}$ and $C = C_G(S)$. If G is χ-minimally irreducible for some character χ such that $\chi_{|S} = \sum_1^q \theta_i$, then the following holds.*

(1) *$G = SY$, where Y is a cyclic q-group and $\{\theta_1, \dots, \theta_q\}$ is a Y-orbit.*

(2) *The triple (S, p, q) is one of the following:*

(a) $(\mathrm{PSL}(2,u), \frac{1}{2}(u+1), 2)$, $u \equiv 1 \bmod 4$;

(b) $(\mathrm{PSL}(2,2^q), 2^q - 1, q)$;

(c) $(\mathrm{PSL}(n,u), (u^n - 1)/(u-1), n)$, $n = q \geqslant 2$, $u > 2$;

(d) $(\mathrm{PSL}(n,u), (u^n - 1)/(u-1), 2)$, *n an odd prime*, $u > 2$;

(e) $(\mathrm{PSp}(2n,u), \frac{1}{2}(u^n+1), 2)$, $(n,u) \neq (2,3)$;

(f) $(M_{12}, 11, 2)$.

(3) $[C, \mho_1(Y)] = 1$.

(4) *Any linear constituent of* $\chi_{|C}$ *is faithful.*

Conversely, suppose that G is a group with a simple normal subgroup S satisfying (1)–(3). *Then G has an irreducible character* χ *such that* $\chi_{|S} = \sum_1^q \theta_i$. *Furthermore, if* χ *satisfies* (4), *then* χ *is faithful and G is* χ*-minimally irreducible.*

Proof. (1) follows immediately from Lemma 2.1. To determine whether or not $G = SY$ is χ-minimally irreducible, we test for the existence of maximal irreducible subgroups M of G behaving as in Proposition 2.7. In particular, we have information on $H = M \cap S$. Namely, we know that H is a proper subgroup of S, and either H is abelian and $p = q$, or H is irreducible of degree p. However, by Lemma 2.6 and Lemma 3.2, the former case cannot occur. It remains to examine the simple groups S appearing in Theorem 3.9, seeking proper irreducible subgroups H satisfying the constraints of Proposition 2.7. In particular, for any such H we have $(\theta_i)_{|H} = \eta_i$ for $1 \leqslant i \leqslant q$ and hence $|\mathrm{Irr}_p(H)| \geqslant q$. If no such subgroup exists, then G is minimally irreducible. Otherwise, we apply Proposition 2.8 to prove that G does contain proper irreducible subgroups.

Running through the groups S listed in Theorem 3.9, we first observe that, according to [3], the group $\mathrm{PSL}(2,u)$ is minimally irreducible of degree $p = \frac{1}{2}(u+1)$ if $u > 11$, and the group $\mathrm{PSL}(n,u)$ is minimally irreducible of degree

$$p = (u^n - 1)/(u-1)$$

if $n \geqslant 2$, $u > 2$. We are left to consider the remaining groups $\mathrm{PSL}(2,u)$, the symplectic groups and the Mathieu group M_{12}.

(a) $S = \mathrm{PSL}(2,5)$, $p = 3$. Here $S \cong \mathbb{A}_5$, $|\mathrm{Irr}_3(S)| = 2$ and the only irreducible subgroups $H < S$ are isomorphic to $\mathbb{A}_4$. However $|\mathrm{Irr}_3(\mathbb{A}_4)| = 1$. Therefore, by Proposition 2.7, G is minimally irreducible of degree 6 (an example of minimal order being $\mathrm{P\Gamma L}(2,5) \cong \mathbb{S}_5$).

(b) $S = \mathrm{PSL}(2,9)$, $p = 5$. Here $|\mathrm{Irr}_5(S)| = 2$ and the only irreducible subgroups $H < S$ are isomorphic to $\mathbb{A}_5$. However $|\mathrm{Irr}_5(\mathbb{A}_5)| = 1$. By Proposition 2.7 again, G is minimally irreducible of degree 10.

(c) $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u-1)$. Here $q = 2$ and $|\mathrm{Irr}_p(S)| = 2$. Denote by H a standard Borel subgroup of S (say, the image in S of the set of upper triangular matrices). Now H is a Frobenius group of order up and hence its character degree pattern is $[1^p, p^2]$ (see [9, (V.16.13)]). Set $\mathrm{Irr}_p(H) = \{\eta, \eta^*\}$, $\mathrm{Irr}_p(S) = \{\theta, \theta^*\}$. Since H is non-abelian, θ and θ^* both restrict irreducibly to H. In fact, $\theta_{|H} = \eta \neq \eta^* = \theta^*_{|H}$ (cf. [6, Chapter 38]). Without loss of generality we may assume that $CY/C = \langle \partial \rangle$, where ∂ is the diagonal automorphism of S realized by conjugation via a diagonal matrix. It follows that H is Y-invariant, and moreover $Y \not\leq I_G(\theta)$. Thus we may apply Proposition 2.8 to H, and $M = HY$ is χ-irreducible. Notice that M is soluble and hence is a proper subgroup of G. We conclude that G is not minimally irreducible.

(d) $S = \mathrm{PSL}(2, 2^q)$, $p = 2^q - 1$. We show that in this case no proper subgroup H of S meets the constraints of Proposition 2.7 by examining the list of subgroups of $\mathrm{PSL}(2, u)$ (cf. [9, (II.8.27)]). Dihedral subgroups are immediately ruled out since S has no irreducible ordinary representation of degree 2. The groups $\mathbb{A}_4$ and $\mathbb{S}_4$ are also ruled out, the former because it has only one non-linear irreducible character, the latter because obviously $2^{2q} \not\equiv 1 \bmod 16$. The character degree pattern $[1, 3^2, 4, 5]$ of $\mathbb{A}_5$ forces $p = 3$, $q = 2$, whence $u = 4$, and therefore $S = H$. Now suppose that H is a semidirect product of an elementary abelian 2-group of order 2^m by a cyclic group of order t, where $t \mid \gcd(2^m - 1, 2^q - 1)$. Since $p = 2^q - 1$, either $t = 1$, in which case H is abelian, or $t = p$ and $m = q$. In the latter case, H is a Frobenius group of order $2^q p$, and therefore $|\mathrm{Irr}_p(H)| = 1$. Since $\mathrm{PGL}(2, u) \cong \mathrm{PSL}(2, u)$ for u even, we are only left to consider the case $H \cong \mathrm{PSL}(2, 2^m)$, where m divides $2q$. Since $H < S$, either $m = 1$ or $m = 2$. Now $H \cong \mathrm{PSL}(2, 2)$ is ruled out, since it has a single non-linear irreducible character of degree 2, which cannot be a character of S. Otherwise $m = 2$, and we are back to the case $H \cong \mathbb{A}_5$. This exhausts the list of subgroups of $\mathrm{PSL}(2, 2^q)$. We conclude that G is a minimally irreducible group of degree $(2^q - 1)q$.

(e) $S = \mathrm{PSp}(2n, 3)$, $p = \frac{1}{2}(3^n - 1)$. Considering the embedding of $\mathrm{PSL}(2, 3^n)$ in $\mathrm{PSp}(2n, 3)$ realized via the regular representation of $\mathbb{F}_{3^n}$ over $\mathbb{F}_3$, we set $H = R_0(\mathrm{PSL}(2, 3^n))$. Observe that the irreducible Weil characters θ, θ^* of S restrict irreducibly to H (since $\frac{1}{2}(3^n - 1)$ is the minimal non-trivial degree for both groups). We have shown in the proof of Theorem 3.9 that the diagonal automorphism ∂ induced by $\mathrm{diag}(-1_n, 1_n)$ interchanges θ and θ^*. Notice that

$$\mathrm{diag}(-1_n, 1_n) = R_0(\mathrm{diag}(-1, 1)),$$

and -1 is not a square in $\mathbb{F}^*_{3^n}$, since $\frac{1}{2}(3^n - 1)$ is odd. It then follows (cf. [6, Chapter 38]) that $\theta_{|H} \neq (\theta^{\partial})_{|H}$. Since $\mathrm{Out}(S) \cong \mathbb{Z}_2$, the subgroup Y maps onto $\mathrm{Out}(S)$. As in (c), we may assume that $CY/C = \langle \partial \rangle$. Thus H is Y-invariant, and applying Proposition 2.8 we obtain that $M = HY$ is an irreducible subgroup of G. Since S is a composition factor of G but not of M, we conclude that G is not minimally irreducible.

(f) $S = \mathrm{PSp}(2n, u)$, $p = \frac{1}{2}(u^n + 1)$. Here $n = 2^a \geqslant 2$, $u = r^{2^b}$, r an odd prime. We will prove that no maximal subgroup H of S meets the constraints of Proposition 2.7, concluding that G is minimally irreducible of degree $2p$, unless $S = \mathrm{PSp}(4, 3)$. Now, a maximal subgroup H of S belongs either to one of the 'Aschbacher classes' $\mathscr{C}_1, \ldots, \mathscr{C}_8$ or to a certain class $\mathscr{S}$ of almost simple groups. These classes are described in detail in [12], which will also serve as our reference for the orders of finite classical groups. A case-by-case analysis shows the following:

$\mathscr{C}_1$: H is the stabilizer either of a totally singular subspace of dimension $m \leqslant n$, or of a non-degenerate subspace of dimension $2m < 2n$. In the former case,

$$H \cong E.\mathbb{Z}_{u-1}.(\mathrm{PGL}(m,u) \times \mathrm{PSp}(2(n-m),u)),$$

where E is an elementary abelian r-group of order $u^{2nm-((3m-1)m/2)}$. Since

$$u^n + 1 > u^i \pm 1$$

for $i < n$, $p \nmid |H|$. In the latter case,

$$H \cong \mathbb{Z}_2.(\mathrm{PSp}(2m,u) \times \mathrm{PSp}(2(n-m),u)),$$

and hence H has order $\frac{1}{2}u^{m^2+(n-m)^2}\prod_{i=1}^{n-m}(u^{2i}-1)\prod_{i=1}^{m}(u^{2i}-1)$. As before, we conclude that $p \nmid |H|$.

$\mathscr{C}_2$: H is either of 'type' $\mathrm{Sp}(2m,u) \wr \mathbb{S}_t$, $n = mt$ with $t \geqslant 2$, or of 'type' $\mathrm{GL}(n,u).\mathbb{Z}_2$. In the former case, $H \cong (\mathbb{Z}_2)^{t-1}.(\mathrm{PSp}(2m,u) \wr \mathbb{S}_t)$ and hence

$$|H| = \frac{t!}{2}\left(u^{m^2}\prod_1^m (u^{2i}-1)\right)^t.$$

Since $p \nmid |\mathrm{PSp}(2m,u)|$, we have $p \mid |H|$ if and only if $t \geqslant p$. But the latter implies that $n = mt > \frac{1}{2}(u^n+1)$, which is impossible, since $2n-1 < 3^n$ for each integer n. If H has type $\mathrm{GL}(n,u).\mathbb{Z}_2$, then $H \cong \mathbb{Z}_{(u-1)/2}.\mathrm{PGL}(n,u).\mathbb{Z}_2$, and hence $|H| = u^{\binom{n}{2}}\prod_1^n(u^i-1)$ is not divisible by p.

$\mathscr{C}_3$: H is isomorphic to either $\mathbb{Z}_{(u+1)/2}.\mathrm{PGU}(n,u).\mathbb{Z}_2$ or $\mathrm{PSp}(n,u^2).\mathbb{Z}_2$. In the former case $|H| = u^{\binom{n}{2}}\prod_1^n(u^i-(-1)^i)$. Since $p > \frac{1}{2}(u^i-(-1)^i)$ for $i \leqslant n$ and n even, $p \nmid |H|$. Now suppose that $H \cong \mathrm{PSp}(n,u^2).\mathbb{Z}_2$. Then S has a single conjugacy class of subgroups isomorphic to H (cf. [12, §4.3]). Thus, by Lemma 3.10, without loss of generality we may assume that $\mathrm{PSp}(n,u^2) = R_0(\mathrm{Sp}(n,u^2))/\langle -1_{2n}\rangle$. Set $\mathbb{E} = \mathbb{F}_{u^2}$, $\mathbb{F} = \mathbb{F}_u$. It is known that $\mathbb{Z}_2$ realizes as an inner automorphism of S the field automorphism σ of $\mathrm{PSp}(n,u^2)$ induced by the involutory automorphism $f \mapsto f^u$ of $\mathbb{E}$ (cf. [12, §4.3, p. 116]). We need an explicit matrix s_0 realizing σ. Observe that any such s_0 belongs to $C_{(\mathbb{F})_{2n}}(D)$, where $D = R_0((\mathbb{F})_n)$. Since $\dim_{\mathbb{F}}(C_{(\mathbb{F})_{2n}}(D)) = 2^2$ (see [9, (V.14.5)]), and obviously any matrix of shape $\mathrm{diag}(s,\dots,s)$ with $s \in (\mathbb{F})_2$ centralizes D, it follows that $s_0 = \mathrm{diag}(s,\dots,s)$ for some $s \in \mathrm{GL}(2,\mathbb{F})$. Since s_0 realizes σ, clearly $R(x)^s = R(x^u)$ for all $x \in \mathbb{E}$. We claim that s is conjugate in $\mathrm{GL}(2,u)$ to $\mathrm{diag}(1,-1)$. First note that 1 is an eigenvalue of s, since s centralizes $R(1)$. On the other hand, since $\mathbb{E} = \mathbb{F}(\alpha)$ where $\alpha^2 = b \in \mathbb{F}\backslash\mathbb{F}^2$, we have $R(\alpha^u) = -R(\alpha)$. Thus -1 is also an eigenvalue of s. We now use the above information on s_0 to prove that the irreducible Weil characters θ, θ^* of $\mathrm{PSp}(2n,u)$ restrict to the same character of H. In the notation of the proof of Theorem 3.9 (f) and of [26], clearly s_0 stabilizes both M and N. Denote by X the $\mathrm{Sp}(2n,u)$-module affording the Weil representation $\mathscr{R}$ with character ρ, by $(e_v \mid v \in N)$ a basis for X, and by X_1, X_{-1} the eigenspaces of $\mathscr{R}(-1_{2n})$

relative to the eigenvalues 1, -1 respectively. Then, given any subset W of N such that $W \cap (-W) = \varnothing$ and $W \cup (-W) = N \backslash \{0\}$, the sets

$$(e_0, e_v + e_{-v} \mid v \in W) \quad \text{and} \quad (e_v - e_{-v} \mid v \in W)$$

are bases for X_1 and X_{-1} respectively. It is shown in [26, Proposition 3] that $\mathscr{R}_{|X_1} = \Xi$ and $\mathscr{R}_{|X_{-1}} = \Psi$ are the irreducible constituents of $\mathscr{R}$, with characters ξ and ψ respectively. Since $\mathscr{R}(s_0)e_v = \left(\frac{\det(s_{0_{|N}})}{\mathbb{F}}\right) e_{s_0(v)} = \pm e_{s_0(v)}$ by [26, Proposition 2], it follows that $\mathscr{R}(s_0)$, $\Xi(s_0)$ and $\Psi(s_0)$ are all monomial matrices. Parallel results hold for $\mathscr{R}^*$. Then it is easy to check that $\rho(s_0) = \rho^*(s_0) = \pm u^{n/2}$,

$$\xi(s_0) = \xi^*(s_0) = \pm\left(1 + \frac{1}{2}(u^{n/2} - 1)\right),$$

and $\psi(s_0) = \psi^*(s_0) = \pm\frac{1}{2}(u^{n/2} - 1)$. In particular, $\theta(s_0) = \theta^*(s_0) \neq 0$ for $\theta \in \{\xi, \psi\}$. Now set $T = \mathrm{PSp}(n, u^2)$, $\eta = \theta_{|H}$, $\eta^* = \theta^*_{|H}$, $\tau = \theta_{|T}$, $\tau^* = \theta^*_{|T}$. We know (cf. Theorem 3.9) that $\theta^* = \theta^{\partial}$, where the diagonal automorphism ∂ is realized by the matrix $\mathrm{diag}(v1_n, 1_n) = R_0(\mathrm{diag}(v1_{n/2}, 1_{n/2}))$ with v a non-square in $\mathbb{F}^*$. However v is a square in $\mathbb{E}$, and hence $\mathrm{diag}(v1_{n/2}, 1_{n/2})$ belongs to $\mathrm{Sp}(n, u^2)\cdot Z(\mathrm{GL}(n, u^2))$. Thus $\tau = \tau^{R_0(\mathrm{diag}(v1_{n/2}, 1_{n/2}))} = \tau^*$. Since η and η^* restrict to the same character of T and H/T is cyclic of order 2, we have $\eta = \eta^*\beta$ for some $\beta \in \mathrm{Irr}(H/T)$. However, by the above, $\eta(s_0) = \eta^*(s_0) \neq 0$. Thus $\beta = 1$ and $\eta = \eta^*$. Therefore H does not meet the conditions of Proposition 2.7.

$\mathscr{C}_4$: $H \cong (\mathrm{PSp}(2m, u) \times \mathrm{PO}_t^{\varepsilon}(u)).\mathbb{Z}_2$, $n = mt$, $t \geqslant 4$. Thus any odd prime factor of $|H|_{r'}$ must divide $u^i \pm 1$, for some $i \leqslant \max(m, t/2)$. Since $m, t/2 < n$, none of these primes can equal p.

$\mathscr{C}_5$: $H \cong \mathrm{PSp}(2n, u_0).\mathbb{Z}_2$, where $u = u_0^2$. Hence $|H|$ is coprime to p.

$\mathscr{C}_6$: In this case u is an odd prime, and $H \cong \mathbb{Z}_{2^{2a+2}}.\mathrm{O}_{2a+2}^-(2).c$, where $c \leqslant 2$. Thus $|H|_{2'} = (2^{a+1} + 1)\prod_1^a (4^i - 1)$. Therefore if $p = \frac{1}{2}(u^{2^a} + 1) \mid |H|$, then p must divide either $2^{a+1} + 1$ or $4^i - 1$. Assume first that $a = 1$. Since $u \geqslant 3$,

$$p \geqslant \frac{1}{2}(3^2 + 1) = 5 > 4 - 1,$$

which forces $p = 5$, $u = 3$. Now $S = \mathrm{PSp}(4, 3)$ has a single conjugacy class of maximal subgroups $H \in \mathscr{C}_6$, isomorphic to $E_{2^4} : \mathbb{A}_5$. Computation using the GAP package [7] shows that the Weil representations of S remain irreducible and distinct under restriction to H. Since Y is a 2-group, at least one of the $27 = |S : H|$ conjugates of H is Y-invariant. Thus, applying Proposition 2.8, we obtain an irreducible proper subgroup HY of G and conclude that G is not minimally irreducible of degree 10. Next, assume that $a > 1$. It is readily seen that $p > 4^a - 1 > 2^{a+1} + 1$. Thus, apart from the exceptional case $S = \mathrm{PSp}(4, 3)$, $|H|$ is coprime to p.

$\mathscr{C}_7$: $H \cong \mathrm{PSp}(2m, u)^t.\mathbb{Z}_2^t.\mathbb{S}_t$, where $2n = (2m)^t$, ut odd, $t \geqslant 3$, $(m, u) \neq (1, 3)$. Arguing as in the first instance of the case $\mathscr{C}_2$, we see that $|H|$ is coprime to p.

$\mathscr{C}_8$: No subgroup of S belongs to $\mathscr{C}_8$, since u is odd.

We conclude that H can only belong to the class $\mathscr{S}$, as defined in [12, §1.2]. In particular, H is an almost simple group with simple non-abelian socle T. It follows that $\eta = \theta_{|H}$ restricts irreducibly to T. Hence T admits an irreducible representation of degree p, and again we are led to examine Table 1. Since T is embedded in $\mathrm{PGL}(2n, u)$, in most cases it will be enough to show that the minimal degree of a faithful projective representation of T exceeds $2n$. In particular, if T is a group of Lie type of characteristic coprime to u (the so-called 'cross characteristic' situation), this minimal degree was estimated in [16] and is listed (with some corrections, and denoted by $R_{r'}(T)$) in [12, Table 5.3.A]. We perform a case-by-case analysis.

(i) $T = \mathbb{A}_t$, $t = p+1$. Here $t = \frac{1}{2}(u^n + 3)$. As $u^n \equiv 1 \bmod 4$, $a = \frac{1}{4}(u^n - 1)$ is an integer greater than 1. Hence, by Bertrand's Postulate, there exists a prime s such that $a < s < 2a$. Clearly $s \mid |\mathbb{A}_t|$. We claim that

$$s \nmid |\mathrm{PSp}(2n, u)| = \frac{1}{2} u^{n^2} \prod_1^n (u^{2i} - 1),$$

and therefore $\mathbb{A}_t$ is not a subgroup of $\mathrm{PSp}(2n, u)$, unless $(n, u) = (2, 3)$. First observe that $a > \frac{1}{2}(u^{n-1} + 1)$ and therefore $s > \frac{1}{2}(u^i \pm 1)$ for all $i < n$, unless $(n, u) = (2, 3)$. As for $i = n$, note that $p = \frac{1}{2}(u^n + 1) > s$, and obviously $r \nmid 2a$. Suppose that s divides u: then $u > a$, which again forces $(n, u) = (2, 3)$. The latter instance is a true exception and has already been dealt with above, while examining the class $\mathscr{C}_6$ (in fact, $\mathbb{A}_6 \cong \mathrm{PSL}(2, 9)$ does embed in $\mathrm{PSp}(4, 3)$).

(ii) $T = \mathrm{PSL}(2, t)$, $t = p$. Here we are in cross characteristic, and according to [12, Table 5.3.A],

$$R_{r'}(T) \geqslant \frac{t-1}{\gcd(2, t-1)} = \frac{1}{4}(u^n - 1),$$

unless $t = 4$ or 9. However $2n \geqslant \frac{1}{4}(u^n - 1)$ only when $(n, u) = (2, 3)$. Since t is prime, we are back to the single and already treated exception $S = \mathrm{PSp}(4, 3)$.

(iii) $T = \mathrm{PSL}(2, t)$, $t = p+1$. Since p is odd, t is even, hence we are again in cross characteristic. Then $R_{r'}(T) \geqslant t - 1 = \frac{1}{2}(u^n + 1)$, unless $t = 4$ or 9. However $4n - 1 < u^n$ for all $n \geqslant 2$. Since t is even, the only exception to be considered is $t = 4$. But $p = 3$ forces $u^n = 5$, a contradiction.

(iv) $T = \mathrm{PSL}(2, t)$, $t = p - 1$. As in the previous case $R_{r'}(T) \geqslant t - 1$. However $4n + 3 < u^n$ unless $(n, u) = (2, 3)$. Moreover the exceptional case $t = 4$ forces $p = 5$, hence $u^n = 9$ and once again $(n, u) = (2, 3)$.

(v) $T = \mathrm{PSL}(m, t)$, $m > 2$, $t > 2$, $p = (t^m - 1)/(t - 1)$. It is readily seen that $\gcd(t, u) = 1$, and hence we are again in cross characteristic. From [12, Table 5.3.A] we have $R_{r'}(T) \geqslant t^{m-1} - 1$, unless $(m, t) = (3, 4)$. It is easy to check that if $m > 2$, $t > 2$ then $2(t^{m-1} - 1) > (t^m - 1)/(t - 1)$. Since

$$\frac{t^m - 1}{t - 1} = \frac{u^n + 1}{2} \geqslant \frac{3^n + 1}{2}$$

and $\frac{1}{2}(3^n + 1) > 4n$ for all $n \geqslant 2$, we see that $2n < t^{m-1} - 1$, and we are done. Finally, the exceptional case is ruled out, since $(m, t) = (3, 4)$ forces $p = 21$.

(vi) $T = \mathrm{PSp}(2m, t)$, $m = 2^b \geqslant 1$, $p = \frac{1}{2}(t^m + 1)$. Here $p = \frac{1}{2}(t^m + 1)$ implies $t^m = u^n$ and hence we are in the natural characteristic case. Comparing the order of the maximal unipotent subgroups of T and S, we get $t^{m^2} | u^{n^2}$, whence $m \leqslant n$. Since m and n are both 2-powers, $n = md$ for some d (in fact, $d = 2$, by the maximality of H in S). In this case, however, H fails to be an absolutely irreducible subgroup of $\mathrm{PGL}(2n, u)$, as required for membership in $\mathscr{S}$ (cf. [12, §1.2]). Indeed, H is in $\mathscr{C}_3$, which has been dealt with above. Thus this case is ruled out.

(vii) $T = \mathrm{PSp}(2m, 3)$, $p = \frac{1}{2}(3^m - 1)$. Here $p = \frac{1}{2}(u^n + 1)$ implies $3^m = u^n + 2$, whence $\gcd(3, u) = \gcd(2, u) = 1$; therefore we are in cross characteristic. Since t is odd, $R_{r'}(T) = \frac{1}{2}(t^m - 1)$ from [12, Table 5.3.A]. However $2n < \frac{1}{2}(u^n + 1)$ for every odd u and every $n \geqslant 2$.

(viii) $T = \mathrm{PSU}(m, t)$, $p = (t^m + 1)/(t + 1)$. This is again a cross characteristic case. Note that, as $(t^m + 1)/(t + 1)$ is prime, m is an odd prime by Lemma 2.5. Thus, from [12, Table 5.3.A],

$$R_{r'}(T) \geqslant \frac{t(t^{m-1} - 1)}{t + 1} = \frac{1}{2}(u^n - 1).$$

However $4n + 1 \leqslant u^n$, with equality if and only if $(n, u) = (2, 3)$. (It is easy to check that the equation $p = 5 = (t^m + 1)/(t + 1)$ has no solutions.)

(ix) In the remaining cases listed in Table 1, p is $7, 11$ or 23, and hence $u^n = 13, 21$ or 45. Clearly none of these values is admissible, since u is a prime power and $n \geqslant 2$.

This exhausts our analysis, and allows to conclude that the relevant group G is minimally irreducible of degree $2p$, unless $S = \mathrm{PSp}(4, 3)$.

(g) $S = M_{12}$, $p = 11$. It is well known (see [2], p. 33) that the only maximal subgroups of S of order divisible by 11 are isomorphic either to $\mathrm{PSL}(2, 11)$ or to M_{11}. Since both these groups have just one irreducible character of degree 11, it follows immediately that G is minimally irreducible of degree 22.

The above arguments take care of all triples (S, p, q) listed in part (2) of the statement of Theorem 3.11. As for (3), we first note that $C = C_G(S)$ embeds in an epimorphic image of $\mho_1(Y)$. Indeed, since $C \leqslant I = \mho_1(Y)S$ and $C \cap S = 1$,

$$C \lesssim \mho_1(Y)S/S \cong \mho_1(Y)/(\mho_1(Y) \cap S).$$

Now if γ is a linear constituent of $\chi_{|C}$ then $\ker \gamma$ is a characteristic subgroup of C, and hence a normal subgroup of G. It follows that $\ker \gamma = \ker \chi \cap C = 1$. Finally, $\chi_{|S} = \sum_{i=1}^{q} \theta_i$ implies $\chi_{|S \times C} = \sum_{i=1}^{q} \theta_i \gamma_i$, where the characters γ_i are (not necessarily distinct) linear constituents of $\chi_{|C}$. In particular, y^q stabilizes a faithful character γ of C. Hence $[C, \mho_1(Y)] = 1$, and (4) is proven.

Conversely, suppose that a group G satisfies the conditions (1)–(3). Choose a linear character γ of C. Since $[C, \mho_1(Y)] = 1$, the irreducible character $\theta\gamma$ of $S \times C$ has a Y-orbit of size q. It follows from Proposition 2.9 that $\theta\gamma$ extends to $\psi \in \mathrm{Irr}_p(I_G(\theta\gamma))$,

$\chi = \psi^G \in \mathrm{Irr}_{pq}(G)$ and $\chi_{|S} = \sum_1^q \theta_i$. If, furthermore, χ satisfies (4), then $\ker(\theta\gamma) = 1$. Therefore χ is faithful, again by Proposition 2.9.

We would like to refine our knowledge of the group SY in Theorem 3.11 by characterizing CY. We know that CY is metacyclic, since both C, Y are cyclic q-groups. As $[C, \mho_1(Y)] = 1$, Y acts either trivially or as a group of order q on C. Moreover, C being embeddable in an image of $\mho_1(Y)$, we have $|C|q \leqslant |Y|$. Finally, $\mho_1(Y)C/\mho_1(Y)$ has index q in $CY/\mho_1(Y)$. Thus (e.g., see [9, (I.14.9)]) either $CY/\mho_1(Y)$ is abelian, or it is a modular, dihedral, semidihedral or generalized quaternion q-group. The structure of CY can then be worked out using the techniques and results of [11] and [8]. Our additional constraints on CY permit an explicit self-contained proof:

Proposition 3.12. *Let $Q = CY$, where C, Y are cyclic q-subgroups, C is normalized by Y, $[C, \mho_1(Y)] = 1$ and $|C| < |Y|$. Then Q is one of the following metacyclic groups:*

(1) *cyclic*: $\langle a \mid a^{q^l} = 1\rangle$, $l \geqslant 1$;

(2) *abelian of type* (l, m): $\langle a, b \mid a^{q^l} = b^{q^m} = [a, b] = 1\rangle$, $m > l \geqslant 1$;

(3) *generalized modular*: $\langle a, b \mid a^{q^l} = b^{q^m} = 1,\ a^{q^{l-1}} = [a, b]\rangle$, $1 < l \neq m$;

(4) *generalized dihedral*: $\langle a, b \mid a^{2^l} = b^{2^m} = 1,\ a^b = a^{-1}\rangle$, $1 < l < m$;

(5) *generalized semidihedral*: $\langle a, b \mid a^{2^l} = b^{2^m} = 1,\ a^b = a^{-1+2^{l-1}}\rangle$, $2 < l < m$;

(6) *non-split semidihedral*: $\langle a, b \mid a^{2^l} = b^{2^m},\ a^{2^{l+1}} = 1,\ a^b = a^{-1+2^l}\rangle$, $1 < l < m$.

Groups of different type or of the same type but with different parameters are not isomorphic.

Proof. Set $Y = \langle y\rangle$, $C = \langle c\rangle$, $|C \cap Y| = q^t$, $|C : C \cap Y| = q^r$, and $|Y : C \cap Y| = q^s$. By assumption, $s > r$. By elementary arguments (see [20, 5.3.4]), we may assume that $c^y = c^\varepsilon$, where either $\varepsilon = \pm 1$ or $\varepsilon = \pm 1 + q^{r+t-1}$ (the minus sign occurring only when $q = 2$), and $c^{q^r} = y^{q^s}$.

If $\varepsilon = 1$, then either $r = 0$ and Q is cyclic, or Q is abelian of type $(r, s + t)$.

Assume that $\varepsilon = 1 + q^{r+t-1}$. Suppose first that $t = 0$. If $r = 1$, then $|C| = q$. However in this case y cannot act non-trivially on C. Thus $r > 1$ and Q is generalized modular. So suppose that $t > 0$, and set $b = c^{-1}y^{q^{s-r}}$, $a = y$. Then $b^{q^r} = 1$ and $a^b = a^{1+q^{s+t-1}}$, since $y^{q^{s-r}} \in Z(Q)$. Therefore Q is again a generalized modular group.

Now assume that $q = 2$ and either $\varepsilon = -1$ or $\varepsilon = -1 + 2^{r+t-1}$. Define recursively $[C; 0Y] = C$, $[C; (i+1)Y] = [[C; iY], Y]$. An easy induction on i shows that

$$[C; iY] = \mho_i(C).$$

Thus $\mho_{r+1}(C) = [\mho_r(C), Y] = [\mho_s(Y), Y] = 1$, whence $t \leqslant 1$. If $t = 0$, then Q is generalized dihedral if $\varepsilon = -1$ and generalized semidihedral if $\varepsilon = -1 + 2^{r-1}$. Now assume that $t = 1$. If $\varepsilon = -1$, set $a = cy^{2^{s-1}}$ and $b = y$. If $r > 1$, then an easy computation shows that $a^{2^r} = y^{2^s}$ and $a^b = a^{-1+2^r}$; hence Q is non-split semidihedral. If $r = 1$, then $o(a) = 2$ and $b^a = b^{1+2^s}$; therefore Q is generalized modular. Finally, suppose that $\varepsilon = -1 + 2^r$. If $r > 1$, then again Q is non-split semidihedral. If $r = 1$ then $\varepsilon = 1$, and we are back in Case (2).

As for the parameters l, m in Cases (2)–(6), we observe that $(l, m) = (r, s)$, except in Case (2), where $(l, m) = (r, s + t)$, and in Case (3) whenever $t > 0$ and the roles of C and Y are interchanged. This explains the given ranges of l and m.

As for the last statement, consider first types (1) to (3). Clearly groups of different type are non-isomorphic. Within a given type, comparison of the order of Q and the structure of Q/Q' and $Z(Q)$ shows that isomorphism forces equality of the parameters. Thus, to prove our claim, we only need to examine the case $q = 2$ and types (3)–(6). The result then follows from [8, Lemma 4.1] and Theorem 4.6.

Remark. In Theorem 3.11, the subgroup $C = C_G(S)$ need not lie in a cyclic supplement to S. Consider $S = \mathrm{PSL}(2,9)$ and $\mathbb{M}_{10} = S\langle \partial\,\mathrm{Fr}\rangle$. It is well known that $\mathbb{M}_{10}$ is a non-split extension of S. (Indeed, writing γ_g for conjugation via an element g of $\mathrm{GL}(2,9)$ and by v a primitive element of $\mathbb{F}_9^*$, we have $\partial = \gamma_\delta$, where $\delta = \mathrm{diag}(v, 1)$. Set $t = s\delta$ for some $s \in \mathrm{SL}(2,9)$, and suppose that $\gamma_t\,\mathrm{Fr}$ acts as an involution. Then $tt^{\mathrm{Fr}} = \lambda 1_2$, whence $t^{\mathrm{Fr}}t = \lambda^{\mathrm{Fr}} 1_2$ and $\lambda = \lambda^{\mathrm{Fr}}$, since t and t^{Fr} commute. Since $\det(t) = v$, we get $v^4 = \lambda^2 = 1$, a contradiction.) Let z generate a cyclic group of order 4 and ψ denote the isomorphism sending z^2 to $\partial\,\mathrm{Fr}\,S \in \mathbb{M}_{10} \backslash S$, and define G to be the subgroup of $\mathbb{M}_{10} \times \langle z\rangle$ consisting of all pairs (m, x) such that $mS = \psi(x^2)$. We claim that $C_G(S)$ is not in a cyclic supplement to S in G. First observe that G is indeed a subgroup of $\mathbb{M}_{10} \times \langle z\rangle$ containing S and that $\{(1,1), (1,z^2), (\partial\,\mathrm{Fr}, z), (\partial\,\mathrm{Fr}, z^3)\}$ is a transversal for S in G. If $(c, w) \in C_G(S)$, then $c \in C_{\mathbb{M}_{10}}(S) = 1$, since S is centreless and $\partial\,\mathrm{Fr}$ is outer. Thus $C_G(S) = \langle (1, z^2)\rangle$. Let X be a cyclic supplement to S. Then X is generated by (x, z), where $x \in \mathbb{M}_{10} \backslash S$. Since $x^2 \neq 1$, we have $C_G(S) \nleq X$.

3.2 Case II. In this subsection, we deal with the case $m = q$. Our general assumptions are as follows: $G = NY$, where N is a minimal normal subgroup of G and $Y = \langle y\rangle$ is a cyclic q-group; G has an irreducible character χ of degree pq, whose restriction to N splits into q distinct irreducible constituents θ_i of degree p; $N = \prod_1^q S_i$, where each S_i is isomorphic to a given non-abelian simple group S. For simplicity, we write $N = S^{\times q}$.

As observed in Lemma 3.1, by the minimality of N the group G acts transitively on the set $\{S_1, \ldots, S_q\}$. In particular, y acts as a q-cycle on $\{S_1, \ldots, S_q\}$. Clearly the group $C_G(N)Y/C_G(N) \cong Y/(C_G(N) \cap Y)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$. Therefore, since $\mathrm{Aut}(S^{\times q}) \cong \mathrm{Aut}(S) \wr \mathbb{S}_q$, without loss of generality we may write $y \equiv (\omega_1, \ldots, \omega_q)\beta \bmod C_G(N)$, where $\omega_i \in \mathrm{Aut}(S)$ for each i, and $\beta = (1, \ldots, q) \in \mathbb{S}_q$. Setting $\Omega_0 = 1$, $\Omega_i = \Omega_{i-1}\omega_i$ (and taking the indices of the ω's modulo q), we see that $y^i \equiv (\Omega_i, \Omega_1^{-1}\Omega_{i+1}, \ldots, \Omega_{q-1}^{-1}\Omega_{i+q-1})\beta^i \bmod C_G(N)$. In particular, each automorphism of S associated to y^i is a string of length i in the ω's, the first being Ω_i and each other being obtained from the previous one by shifting all indices upwards by 1. (To stress the dependence of Ω_i on y, sometimes we write $\Omega_i(y)$.) Also, since y has q-power order, the same holds for Ω_q, as one may readily check by considering y^q. There is some flexibility in the choice of y. In fact, we may replace y by xy, for some $x \in N$. Then, denoting by $\bar{x}$ the coset of $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ containing x, we clearly get $\overline{\Omega_i(y)} = \overline{\Omega_i(xy)}$. Suppose that $xy = zt$, with $zt = tz$, z q-regular, t q-singular. Set $y_1 = t^{|z|}$. Our first concern is to control the role of $Y_1 = \langle y_1 \rangle$.

Lemma 3.13. *Let y_1 be defined as above. Then the following assertions hold.*

(i) *$\langle y_1 \rangle$ supplements N in G (and y_1 acts as a q-cycle on $\{S_1, \dots, S_q\}$).*

(ii) $\overline{\Omega_q(y)} = \overline{\Omega_q(y_1)}$.

Proof. Let $xy = tz$ be the above decomposition of xy into its q-regular and q-singular parts, and set $m = o(z)$. Then $y_1 N = (xyN)^m = (yN)^m$ and hence

$$o(y_1 N) = o(yN) = |G/N|,$$

and (i) is proven. As observed above, $\Omega_i(y) \equiv \Omega_i(ny) \bmod \mathrm{Inn}(S)$ for each $n \in N$. Thus, since y_1 differs from y^m by an element of N, to prove (ii) we only need to compare $\Omega_q(y)$ with $\Omega_q(y^m)$ modulo $\mathrm{Inn}(S)$. We count the occurrences of ω_i as a factor of the q automorphisms associated to y^m. By induction on m, this turns out to be exactly m. Since $\mathrm{Out}(S)$ is abelian for all groups listed in Table 1, it follows that $\Omega_q(y^m) \equiv \Omega_q(y)^m \bmod \mathrm{Inn}(S)$, and we are done.

Our next target is to establish criteria, in terms of conditions on subgroups of N, which show that a given group G is *not* χ-minimally irreducible. Keeping the notation introduced in Lemma 3.1, let $\theta = \theta_1 \in \mathrm{Irr}(N)$ and suppose that $\theta = \sigma_1 \dots \sigma_q$, where $\sigma_i \in \mathrm{Irr}(S)$ for each i. As $\theta(1) = p$, we may assume that σ_i is trivial for each $i > 1$ and set $\sigma = \sigma_1$. Thus $\theta(s_1, \dots, s_q) = \sigma(s_1)$. Furthermore, considering the action of Y on $\{\theta_1, \dots, \theta_q\}$, we may assume that $\theta_{i+1} = \theta^{y^i}$, and therefore $\theta_{i+1}(s_1, \dots, s_q) = \sigma^{\Omega_i}(s_{i+1})$. The subgroups of $N = S^{\times q}$ which will play a role in our analysis are defined as follows:

Definition 3.14. Let $T_1, \dots, T_q$ be subgroups of S. Then the subgroup

$$H = \prod_i T_i = \{(t_1, \dots, t_q) \mid t_i \in T_i\}$$

is called a *product* subgroup of $S^{\times q}$.

Definition 3.15. Let $\alpha_1, \dots, \alpha_q$ be automorphisms of S. Then the subgroup

$$H = \{(s^{\alpha_1}, \dots, s^{\alpha_q}) \mid s \in S\}$$

is called a *full diagonal* subgroup of $S^{\times q}$. (Observe that H is obviously isomorphic to S.)

The following criterion holds for a product subgroup:

Proposition 3.16. *Suppose that T is a subgroup of S, and for all $i \geqslant 0$ set $T_{i+1} = T^{\Omega_i}$ and $H = \prod_i T_i$. Then H is Y-invariant if and only if T is Ω_q-invariant. If this happens and if $\sigma_{|T}$ is irreducible, then $\chi_{|HY}$ is irreducible.*

Proof. By definition $T_{i+1} = T_i^{\omega_i}$. Since

$$(t_1, \dots, t_q)^y = (t_q^{\omega_q}, t_1^{\omega_1}, \dots) \quad \text{for each} \quad (t_1, \dots, t_q) \in H,$$

we see that H is Y-invariant if and only if T is Ω_q-invariant. Observe that $\theta_{|H} \in \mathrm{Irr}_p(H)$ if and only if $\sigma_{|T} \in \mathrm{Irr}_p(T)$. Set $\eta_i = \theta_{i|H}$. Since $\ker \eta_i = \prod_{j \neq i} T_j$, the constituents η_i are pairwise distinct. Now apply Proposition 2.8 to H and Y.

(Notice that, in the above setting, the Y-invariance of H does not require Ω_q to be q-singular.)

Next, we single out a criterion for a full diagonal subgroup

$$H = \{(s^{\alpha_1}, \dots, s^{\alpha_q}) \mid s \in S\}$$

of N to be invariant under the action of a cyclic supplement X to N in G. Note that, since $S^{\alpha_1} = S$, we may assume $\alpha_1 = 1$. Set $X = \langle x \rangle$. Then, as seen above, x may be read as $(\omega_1, \dots, \omega_q)\beta \bmod C_G(N)$, where $\beta = (1, \dots, q)$, and the following holds:

Lemma 3.17. *There exists in G a full diagonal X-invariant subgroup H if and only if $\Omega_q(x)$ is a q-power in* $\mathrm{Aut}(S)$.

Proof. By the above, we may assume that $S_i^x = S_{i+1}$, where the indices are taken modulo q, and $H = \{(s, s^{\alpha_2}, \dots, s^{\alpha_q}) \mid s \in S\}$. Set $\rho = (\alpha_q \omega_q)^{-1}$. A direct computation shows that H is X-invariant if and only if $\alpha_i = \rho \alpha_{i-1} \omega_{i-1}$ for $2 \leqslant i \leqslant q$. An easy induction shows that this is equivalent to requiring that $\alpha_i = \rho^{(i-1)} \Omega_{i-1}$. In particular, each α_i with $i < q$ is uniquely determined by the ω's and α_q, and moreover $\Omega_q = (\alpha_q \omega_q)^q$. Conversely, assume that $\Omega_q(x)$ is a q-power in $\mathrm{Aut}(S)$. Then there exists $\alpha_q \in \mathrm{Aut}(S)$ such that $\Omega_q = (\alpha_q \omega_q)^q$. Set $\alpha_1 = 1$ and $\alpha_i = \rho^{i-1} \Omega_{i-1}$ for $i = 2, \dots, q-1$. Then $H = \{(s, s^{\alpha_2}, \dots, s^{\alpha_q}) \mid s \in S\}$ is X-invariant.

Finally, we single out the pairs (H, Y), where $Y = \langle y \rangle$ is a q-cyclic supplement to N in G and H is a full diagonal Y-invariant subgroup of N, such that HY is a χ-irreducible subgroup of G. By Lemma 3.17, Y-invariance is equivalent to requiring that $\Omega_q(y)$ is a q-power in $\mathrm{Aut}(S)$. Set $\bar{Q} = \langle \overline{\Omega_q(y)} \rangle$. As observed at the beginning of this subsection, $\bar{Q}$ does not depend on the choice of the q-cyclic supplement Y to N.

Proposition 3.18. *In the notation of Lemma* 3.17, *denote by* $\mathrm{Out}(S)^q$ *the set of q-powers in* $\mathrm{Out}(S)$ *and set* $\bar{I} = I_{\mathrm{Aut}(S)}(\sigma)/\mathrm{Inn}(S)$. *Then there exists a pair (H, Y), where Y is a*

q-cyclic supplement to N in G and H is a full diagonal Y-invariant subgroup of N, if and only if $\bar{Q} \subseteq \mathrm{Out}(S)^q$. Moreover, HY is irreducible with respect to χ (in particular, G is not χ-minimally irreducible) if and only if $\bar{Q} = \langle \bar{\rho}^q \rangle$, and (1) *$\bar{\rho}^q \in \bar{I}$ but* (2) *$\bar{\rho} \notin \bar{I}$.*

Proof. Suppose that N has a Y-invariant full diagonal subgroup H. Then $\Omega_q(y)$ is a q-power by Lemma 3.17, and hence $\bar{Q} \subseteq \mathrm{Out}(S)^q$. Conversely, if the latter condition holds, then $\overline{\Omega_q(y)}$ is a q-power for any generator y of a q-cyclic supplement to N. Thus $s\Omega_q(y) = \rho^q$, where $\rho \in \mathrm{Aut}(S)$ and $s \in \mathrm{Inn}(S) \cong S$. Set $x = (s, 1, \dots, 1)$. Then $\Omega_q(xy) = \rho^q$, and by Lemma 3.17 there exists an (xy)-invariant full diagonal subgroup H. Let y_1 be defined as in Lemma 3.13 (so that y_1 is essentially the q-part of xy). Then H is y_1-invariant, and y_1 generates a q-cyclic supplement to N in G. We now turn to the irreducibility of HY. Without loss of generality, we may assume that $y \equiv (\omega_1, \dots, \omega_q)\beta$ modulo $C_G(N)$, where $\omega_i \in \mathrm{Aut}(S)$ for each i, and $\beta = (1, \dots, q) \in \mathbb{S}_q$. Since

$$\eta_{i+1}(h) = \theta_{i+1}(h) = \sigma^{\Omega_i}(s^{\alpha_{i+1}}) = \sigma^{(\alpha_q\omega_q)^i}(s),$$

to ensure that $\eta_1, \dots, \eta_q$ are distinct and apply Proposition 2.8, we only need to control the action of $\alpha_q\omega_q$ on σ. As $\theta = \theta^{y^q}$ we have $(\alpha_q\omega_q)^q \in I_{\mathrm{Aut}(S)}(\sigma)$. Thus the orbit of σ under $\alpha_q\omega_q$ has order q if and only if $\alpha_q\omega_q \notin I_{\mathrm{Aut}(S)}(\sigma)$. Therefore by Proposition 2.8, if H satisfies conditions (1), (2), then HY is irreducible with respect to χ. Conversely, if HY is χ-irreducible, then arguments in the proof of Proposition 2.7 (with $M = HY$) show that $\eta_1, \dots, \eta_q$ are distinct, and therefore (1), (2) hold. Since $H \cong S$, clearly HY is a proper subgroup of G.

Our next goal is to show that if S is not σ-minimally irreducible, then with one single exception G is not χ-minimally irreducible.

Theorem 3.19. *Suppose that $G = NY$ and $N = S^{\times q}$ satisfy the assumptions of this subsection. Suppose furthermore that S is not minimally irreducible with respect to the character σ of degree p defined above. Then G is not χ-minimally irreducible, unless $S = \mathrm{PSL}(2, 9)$, $p = 5$, $q = 2$ and $\Omega_2 \equiv \partial$ or $\partial \circ \mathrm{Fr} \bmod \mathrm{Inn}(S)$, where ∂ and* Fr *denote the diagonal and Frobenius automorphism of S.*

Proof. We inspect Table 1, looking at the groups that are not minimally irreducible and testing the subgroups T listed in the fourth column of the Table for Ω_q-invariance; then apply Proposition 3.16. We only need T to be preserved by Ω_q up to conjugacy in S. For, suppose that, given $\Omega_q(y)$, we find an irreducible subgroup T of S such that $T^{\Omega_q} = T^s$, for some $s \in S$. Let

$$x = (1, \dots, 1, s^{-\omega_q^{-1}}) \in N.$$

Then $\Omega_q(xy) = \Omega_q(y)s^{-1} \in N_{\mathrm{Aut}(S)}(T)$ (here s^{-1} is to be understood as an inner automorphism of S). Moreover, $H = \prod T^{\Omega_i}$ is xy-invariant. If $xy = zt$ is the decomposition of xy into its q-regular part z and q-singular part t, and $m = o(z)$, then H

is $(xy)^m$-invariant. By Lemma 3.13, $Y_1 = \langle y_1 \rangle$ is a q-cyclic supplement to N in G: hence Y_1 is not contained in $I_G(\theta)$, and Proposition 2.8 gives that $M = HY_1$ is a proper irreducible subgroup of G (M is proper since H is normal in M but not in G, as each normal subgroup of N is a product of copies of S). In particular, the required condition on T and $\Omega_q(y)$ will be plainly satisfied if S has only one conjugacy class of subgroups isomorphic to T.

We now proceed with a case-by-case analysis of the subgroups T listed in Table 1, either exploiting the fact that S has a unique conjugacy class of subgroups isomorphic to T, or proving that each coset of $\mathrm{Aut}(S)/\mathrm{Inn}(S)$ has an element stabilizing T, apart from the exception mentioned above.

(i) $S = \mathbb{A}_u$, $u = p + 1 > 6$, $T \cong \mathrm{PSL}(2, p)$. Here $\mathrm{PSL}(2, p)$ is embedded in $\mathbb{A}_{p+1}$ via its natural action on the $p + 1$ points of $\mathbb{P}^1(p)$. It is easy to check that the diagonal automorphism induced by the matrix

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix},$$

where $\langle \lambda \rangle = \mathbb{F}_u^*$, acts on $\mathbb{P}^1(p)$ as a cycle of length $p - 1$, hence as an odd permutation. Thus T is preserved by an outer automorphism of S, and we are done.

(ii) $S = \mathrm{PSL}(2, u)$, $p = u - 1$, $u = 2^n$ for some prime n. Here we choose $T = N_S(U)$, where $U \in \mathrm{Syl}_2(S)$. Clearly there is in S only one conjugacy class of subgroups isomorphic to T.

(iii) $S = \mathrm{PSL}(2, 9) \cong \mathbb{A}_6$, $T \cong \mathbb{A}_5$, $p = 5$. Using the notation of the Remarks following Table 1, we recall that S has two conjugacy classes K^S and L^S of subgroups isomorphic to $\mathbb{A}_5$, which are interchanged by the diagonal automorphism ∂ and fixed by the Frobenius automorphism Fr. If $\Omega_q \equiv \mathrm{Fr} \bmod \mathrm{Inn}(S)$, we may apply Proposition 3.16 with $T = K$ or L. Suppose that $\Omega_q \equiv \partial$ or $\partial \circ \mathrm{Fr} \bmod \mathrm{Inn}(S)$: we claim that in this case G is minimally irreducible. First we observe that Ω_q being q-singular entails $q = 2$. Assume that M is a maximal irreducible subgroup of G. By Proposition 2.7, $MN = G$ and $H = M \cap N$ is either abelian or irreducible of degree 5. Moreover M can be assumed Y-invariant (indeed, let $Q \in \mathrm{Syl}_q(M)$; then there exists $x \in Q \backslash I_G(\theta)$ such that $G = N\langle x \rangle$). If H is abelian, then $p = q = 2$, a contradiction. Otherwise, setting $N = S_1 \times S_2$ and denoting by π_i the projection of H onto S_i for $i = 1, 2$, we have that σ restricts irreducibly to H^{π_i}. Since the only proper subgroups of $\mathrm{PSL}(2, 9)$ having an irreducible representation of degree 5 are those isomorphic to $\mathbb{A}_5$, it follows that either $H^{\pi_i} = S_i$, or H^{π_i} is in K^S or L^S. Since H is Y-invariant, $(H^{\pi_2})^{\omega_2} = H^{\pi_1}$; in particular H^{π_2} is isomorphic to H^{π_1}. Moreover $H \cap S_i$ is normal in H^{π_i}; as H^{π_i} is simple, it follows that either $H \cap S_i = 1$ or $H \cap S_i = H^{\pi_i}$. Assume that both the projections are isomorphic to S. If $H \cap S_1 = S_1$, then $H \cap S_2 = S_2$. However, since $MN = G$, H cannot equal N. Thus $H \cap S_i = 1$ for $i = 1, 2$, which readily implies that H is a full diagonal subgroup. Let (s, s^α) be a typical element of H. Since H is Y-invariant, by Lemma 3.17 we get $(\alpha\omega_2)^2 = \Omega_2$, contradicting the fact that $\mathrm{Out}(S)$ is of exponent 2. Thus without loss of generality we may assume that $H^{\pi_2} = K$, whence $H^{\pi_1} = K^{\omega_2}$. If $H \cap S_2 = K$, then $H \cap S_1 = K^{\omega_2}$ and $H = K^{\omega_2} \times K$. However, the latter group is not Y-invariant. For, suppose the contrary. Then

$K = K^{\omega_2\omega_1} = K^{\Omega_2[\omega_2,\omega_1]}$. Since $\mathrm{Out}(S)$ is abelian, it follows that $K^{\Omega_2} \in K^S$, a contradiction, since K^S is not stabilized by ∂. Thus $H \cap S_2 = 1$ and H is diagonal with typical element (k, k^α), $k \in K^{\omega_2}$. This forces $\Omega_2 = (\alpha\omega_2)^2\xi$, where $\xi \in C_{\mathrm{Aut}(S)}(K^{\omega_2})$, and hence $\Omega_2 \equiv \xi \bmod \mathrm{Inn}(S)$. But this implies that K^∂ is conjugate to K in S, which is not the case. We conclude that G is minimally irreducible.

(iv) $S = \mathrm{PSL}(2, r^f)$, $p = \frac{1}{2}(r^f - 1)$, $T = N_S(U)$, where $R \in \mathrm{Syl}_r(U)$. Obviously S has a single conjugacy class of subgroups isomorphic to T and we are done.

(v) $S = \mathrm{PSp}(2^{t+1}, u)$, $u = r^{2^s}$, $p = \frac{1}{2}(u^{2^t} + 1)$, $T \cong \mathrm{PSp}(2^t, u^2).\mathbb{Z}_2$, a maximal subgroup belonging to the Aschbacher class $\mathscr{C}_3$. It is known (cf. [12, (4.3.10)]) that S has a single conjugacy class of subgroups isomorphic to T, and we are done.

(vi) $S = \mathrm{PSp}(2n, 3)$, $p = \frac{1}{2}(3^n - 1)$, n odd, $T \cong \mathrm{PSL}(2, 3^n)$. As in (v), T is a maximal subgroup belonging to the Aschbacher class $\mathscr{C}_3$ and we are done by [12, (4.3.10)].

(vii) $S = \mathrm{PSU}(3,3)$, $p = 7$, $T \cong \mathrm{PSL}(2,7)$. As already observed in the Remarks following Table 1, $\mathrm{PSU}(3,3)$ has only one conjugacy class of subgroups isomorphic to $\mathrm{PSL}(2,7)$.

(viii) $S = M_{12}$, $p = 11$, T a maximal subgroup isomorphic to $\mathrm{PSL}(2,11)$. It is well known that M_{12} has exactly one conjugacy class of such subgroups.

(ix) $S = M_{24}$ or $\mathrm{PSp}(6,2)$. In both cases $\mathrm{Out}(S)$ is trivial.

Next we deal with the case when S is σ-minimally irreducible. Suppose, under this assumption on S, that G is not χ-minimally irreducible and M is a maximal irreducible subgroup of G. As in Proposition 2.7, set $H = M \cap N$. As already indicated in the proof of Theorem 3.19, we may assume that Y is contained in M, and therefore H is Y-invariant. Moreover,

$$M = M \cap G = M \cap NY = (M \cap N)Y = HY.$$

We prove that in this situation H cannot be abelian.

Theorem 3.20. *Suppose that $G = NY$ and $N = S^{\times q}$ satisfy the assumptions of this subsection. Suppose furthermore that S is σ-minimally irreducible, but that G is not χ-minimally irreducible. If M is a maximal irreducible subgroup of G, then $H = M \cap N$ is non-abelian.*

Proof. By the above remarks, $M = HY$. Suppose that H is abelian. Then, by Proposition 2.7, $q = p$, any irreducible constituent of $\chi_{|H}$ has q^2 conjugates under Y, and $\theta_{i|H} = \sum_{j=1}^{q} \eta_{ij}$ for $i \in [q]$, where a double index is used to parametrize the linear constituents of $\chi_{|H}$. It is also convenient to label the characters η with the elements of $\mathbb{Z}_{q^2}$, defining $\eta_{(i-1)+(j-1)q} = \eta_{ij}$. Furthermore, up to reordering we may also assume that y acts as $\eta_a^y = \eta_{a+1}$, where $a \in \mathbb{Z}_{q^2}$. Since $\ker\theta_i = \ker\pi_i$, we may view η_{ij} as a character of $H/(H \cap \ker\pi_i) \cong H^{\pi_i}$. Setting $T = H^{\pi_1}$, with slight abuse of notation we may view $\tau_j = \eta_{1j}$ as an element of $\mathrm{Irr}(T)$. Then, by the above, $\tau_j^{y^q} = \tau_{j+1}$. Also, since σ is the unique non-trivial factor of θ, T is an abelian subgroup of S such that $\sigma_{|T} = \Sigma_1^q \tau_j$ is the sum of q distinct linear characters forming an orbit under $\langle y^q \rangle$.

Since y^q stabilizes H and each simple factor of N, and acts on S_1 as Ω_q, it follows that Ω_q normalizes T and all the characters τ are Ω_q-conjugate. By Lemma 2.6 we have $\gcd(|\mathrm{Out}(S)|, q) = 1$. Thus, since Ω_q is q-singular, it is inner. Let $z \in N_S(T)$ induce Ω_q. Set $L = T\langle z\rangle$ and $R = I_L(\tau)$. Thus $|L : R| = q$, since z permutes transitively the characters τ. By Lemma 2.2 we have $|S|_q = q$. Thus q does not divide $|T|$, $o(z) = q$ and $L = T : \langle z\rangle$, a split extension. It follows that $\lambda = \tau^L \in \mathrm{Irr}(L)$. As $\sigma_{|T} \geqslant \tau$ we have $\lambda^S = \tau^S \geqslant \sigma$ and $\sigma_{|L} \geqslant \lambda$, whence, by degree considerations, $\sigma_{|L} = \lambda$. Since S is σ-minimally irreducible, it follows that $L = S$. Since S is simple and non-abelian, we conclude that $T = 1$, a contradiction.

By Proposition 2.7, we are now left with the case when $\chi_{|H} = \sum_1^q \eta_i$, where $\eta_i = \theta_{i|H}$ and $\eta_1, \ldots, \eta_q$ are distinct irreducible characters of H. We will show that in this case H is a full diagonal subgroup of N. Our basic ingredients are an 'intermediate' definition and a lemma based on a nice result of L. L. Scott.

Definition 3.21. A subgroup H of $N = S^{\times q}$ is said to be *multidiagonal* if it is the direct product $\prod D_i$ of full diagonal subgroups of subproducts $\prod_{j \in J_i} S_j$, where the sets J_i form a partition of $J = \{1, \ldots, q\}$. If, in addition, $|J_i| > 1$ for each i, then H is said to be *thick*.

Lemma 3.22. *Let H be a subgroup of $N = S^{\times q}$. Then H is multidiagonal if and only if H projects surjectively to each simple factor of N, i.e. $H^{\pi_i} = S_i$ for each $i \in \{1, \ldots, q\}$. Moreover H is thick if and only if $H \cap S_i = 1$ for each i.*

Proof. It is clear that a thick multidiagonal subgroup H satisfies the conditions $H^{\pi_i} = S_i$ and $H \cap S_i = 1$ for each i. It was proven by Scott [21, p. 328] that $H^{\pi_i} = S_i$ for each i forces H to be multidiagonal. The additional condition that $H \cap S_i = 1$ for each i forces H to be thick.

Theorem 3.23. *Suppose that G, N and S satisfy the assumptions of Theorem* 3.20. *If M is a maximal irreducible subgroup of G, then $H = M \cap N$ is a full diagonal subgroup.*

Proof. By Theorem 3.20, H is non-abelian, and hence $\chi_{|H} = \sum_1^q \eta_i$, where $\eta_i = \theta_{i|H}$ and the characters η_i are distinct irreducible characters of H. Denoting by π_i the projection of N onto the ith factor S_i, for any $h \in H$ we have $\eta_i(h) = \theta_i(h) = \sigma_i(h^{\pi_i})$. Since S is minimally irreducible, it follows that $H^{\pi_i} = S_i \cong S$. As noted above, we may assume that $M = HY$, where H is Y-invariant and Y transitively permutes the groups S_i. Since $H^{\pi_i} = S_i$ is simple and $H \cap S_i \trianglelefteq H^{\pi_i}$, either $H \cap S_i = 1$ or $H \cap S_i = S_i$. In the latter case H would contain S_i and therefore M would contain N, forcing $M = G$, a contradiction. Thus, for any $i \in [q]$, $H \cap S_i = 1$. By Lemma 3.22, we conclude that H is a thick multidiagonal subgroup. Thus $H = \prod_1^r D_i$, where $r < q$ and each D_i is full diagonal. In particular, H is semisimple and $D_1, \ldots, D_r$ are its only minimal normal subgroups: thus y acts on these subgroups. However, since $r < q$ and y is q-singular, each D_i must be fixed by y. As y acts as a q-cycle on the simple factors of N, we conclude that $r = 1$, that is, H is diagonal.

By Theorem 3.23, if G is not χ-minimally irreducible, but S is σ-minimally irreducible, then there exists a full diagonal subgroup H of N isomorphic to S such that Y has an orbit of size q on $\mathrm{Irr}_p(H)$. In particular, q divides $|\mathrm{Out}(S)|$. A check of the minimally irreducible groups of prime degree (see Table 1) and of those listed in Theorem 3.9 shows that there are only two possibilities: either $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u+1)$ or $S = \mathrm{PSL}(n,u)$, $p = (u^n - 1)/(u-1)$. In both cases, the occurrence of non-minimally irreducible groups G arises from the nature of the action of $\overline{\Omega_q}$ or properties of any irreducible constituent η of $\chi_{|H}$, as shown by our final result:

Theorem 3.24. *Suppose that $G = NY$ and $N = S^{\times q}$ satisfy the assumptions of this subsection. Suppose furthermore that S is σ-minimally irreducible. Then G is χ-minimally irreducible, except in the following cases*:

(1) $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u+1)$, $q = 2$, $\overline{\Omega_2} \in \mho_1\langle\overline{\mathrm{Fr}}\rangle$;

(2) $S = \mathrm{PSL}(2,u)$, $p = u+1$, $\eta \neq \theta_{(u-1)/3}$, $q = 2$ *and* $\langle\overline{\Omega_2}\rangle = \overline{I_{\mathrm{Aut}(S)}(\eta)}$;

(3) $S = \mathrm{PSL}(n,u)$, $p = (u^n-1)/(u-1)$, $\eta \neq \theta_{(u-1)/2}$, $q = 2$ *and* $\overline{\Omega_2} \in \overline{I_{\mathrm{Aut}(S)}(\eta)}$;

(4) $S = \mathrm{PSL}(n,u)$, $p = (u^n-1)/(u-1)$, $\eta \neq \theta_{(u-1)/2}$, $q = n$ *and* $\langle\overline{\Omega_n}\rangle = \overline{I_{\mathrm{Aut}(S)}(\eta)}$.

Conversely, in each of these exceptional cases, there exist a full diagonal subgroup H and a cyclic q-group Y_1 such that HY_1 is a proper χ-irreducible subgroup of G.

Proof. As usual, for $x \in \mathrm{Aut}(S)$ we write $\bar{x}$ for the automorphism x taken modulo $\mathrm{Inn}(S)$. First observe that the conditions on $\overline{\Omega_q}$ in (1)–(4) do not depend on the choice of Y, by Lemma 3.13. Suppose that G is not χ-minimally irreducible and consider $M = HY$, where the diagonal subgroup $H = M \cap N$ is identified with S. Set $z = \overline{\Omega_q}$ and $\bar{I} = \overline{I_{\mathrm{Aut}(S)}(\eta)}$. Then, by Proposition 3.18, there exists $w \in \mathrm{Out}(S) \setminus \bar{I}$ such that $z = w^q \in \bar{I}$. We check whether these conditions hold in each case, using the notation and results of Theorem 3.9.

(i) If $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u+1)$, then $u = r^{2^s}$, $\mathrm{Out}(S) = \langle\bar{\partial}\rangle \times \langle\overline{\mathrm{Fr}}\rangle$ is an abelian 2-group of type $(2, 2^s)$ and $q = 2$. Since η_1 is fixed by Fr but not by ∂, z satisfies the required conditions if and only if $z \in \mho_1\langle\overline{\mathrm{Fr}}\rangle$.

(ii) If $S = \mathrm{PSL}(2,u)$, $p = u+1$, then $u = 2^{2^s}$, $\mathrm{Out}(S) = \langle\overline{\mathrm{Fr}}\rangle \cong \mathbb{Z}_{2^s}$ and $q = 2$. Since $\mathrm{Out}(S)$ is a cyclic 2-group, z satisfies the required conditions if and only if $\langle z\rangle = \bar{I}$ and $\eta_1 \neq \theta_{(u-1)/3}$ (see Theorem 3.9).

(iii) If $S = \mathrm{PSL}(n,u)$, $p = (u^n-1)/(u-1)$, $n \geqslant 3$, then $u = r^{n^s}$,

$$\mathrm{Out}(S) = \langle\overline{\mathrm{Fr}}\rangle \times \langle\bar{\gamma}\rangle,$$

where γ denotes the graph automorphism of S, and η is one of $u-2$ monomial characters θ_j parametrized by $\mathbb{Z}_{u-1}^*$ and described in Theorem 3.9. If u is odd and $\eta = \theta_{(u-1)/2}$, then η is $\mathrm{Aut}(S)$-invariant. Therefore we may assume that $j \neq \frac{1}{2}(u-1)$. It follows from Theorem 3.9 that $\eta = \theta_j$ is moved by γ, and hence $\bar{I} \leqslant \langle\overline{\mathrm{Fr}}\rangle$. If $q = 2$, since any element of $\langle\overline{\mathrm{Fr}}\rangle$ is a square, z satisfies the required conditions if and only if $z \in \bar{I}$. If $q = n$, then z satisfies the conditions if and only if $\langle z\rangle = \bar{I}$.

The converse part of the statement follows immediately from Proposition 3.18.

4 The homogeneous case

In this section, except in Theorem 4.3, we assume that G is irreducible with respect to a character χ of degree pq, and has a proper non-abelian minimal normal subgroup N such that $\chi_{|N}$ is homogeneous. Thus $\chi_{|N} = q\theta$, where $\theta \in \mathrm{Irr}(N)$ and $\theta(1) = p$. We first show that N must be simple.

Lemma 4.1. *Suppose that G is a finite group, χ is a faithful irreducible character of G of degree pq, and N is a minimal normal subgroup of G such that $\chi_{|N} = q\theta$, where $\theta \in \mathrm{Irr}(N)$. Then N is a non-abelian simple group.*

Proof. Let $N = S_1 \times \cdots \times S_m$, where each S_i is isomorphic to the non-abelian simple group S, and suppose that $m > 1$. Then θ is the product of m irreducible characters $\varphi_1, \ldots, \varphi_m \in \mathrm{Irr}(S)$. As $\theta(1) = p$, we may assume $\phi_1(1) = p$ and $\varphi_i(1) = 1$ for $i > 1$. Since S is simple, $\varphi_i = 1_S$ (the principal character of S) for all $i > 1$. Thus all factors S_i with $i > 1$ lie in $\ker \theta$. As θ is faithful, this is clearly impossible. We conclude that $m = 1$ and $N = S$.

Next we show that if G is χ-minimally irreducible then it has either one or two non-abelian minimal normal subgroups. In the latter case, the structure of G is easily obtained.

Proposition 4.2. *Suppose that G is χ-minimally irreducible and has two distinct non-abelian minimal normal subgroups N and M, and $\chi_{|N} = q\theta$, where $\theta \in \mathrm{Irr}(N)$.*

(i) *N, M are both simple, and $G = N \times M$.*

(ii) *N, M are minimally irreducible groups of degree p and q, respectively.*

Proof. Clearly $NM = N \times M \trianglelefteq G$. Thus $\chi_{|M}$ is also homogeneous. Otherwise, by Lemma 2.1, we would get $G = M\langle y\rangle$, with y a q-element, and therefore N would be abelian (indeed cyclic). In particular, by Lemma 4.1, both N, M are simple. We claim that $G = N \times M$. Suppose the contrary. Then, by the minimality assumption, $\chi_{|N\times M}$ splits into the sum of r irreducible characters of degree s, where $(r, s) = (q, p)$ or (p, q). Since $\chi_{|N} = (\chi_{|N\times M})_{|N}$, this forces $r = q$, $s = p$, and $\chi_{|N\times M} = \bar{\theta}_1 + \cdots + \bar{\theta}_q$, where $\bar{\theta}_i = \theta \cdot \psi_i$, $\psi_i \in \mathrm{Irr}(M)$. As $\bar{\theta}_i(1) = \theta(1) = p$, it follows that $\psi_i = 1_M$, which in turn implies that $\chi_{|M} = (\chi_{|N\times M})_{|M} = \chi(1)1_M$, a contradiction. Thus (i) is proven. Now we have the following situation: $G = N \times M$, where N and M are both simple; $\chi = \theta \cdot \psi$, where $\theta \in \mathrm{Irr}(N)$ has degree $p, \psi \in \mathrm{Irr}(M)$ has degree $q, \chi_{|N} = q\theta, \chi_{|M} = p\psi$. Suppose that H is a proper subgroup of N such that $\theta_{|H}$ is irreducible. Then $\theta_{|H} \cdot \psi$ is an irreducible character of $H \times M \lneq G$. But $\theta_{|H} \cdot \psi = \chi_{|H\times M}$, contradicting the minimality of G. By symmetry, the same argument works for M, and (ii) is proven.

By Proposition 4.2 (ii), if $G = N \times M$, where N, M are non-abelian simple groups, is minimally irreducible of degree pq, then N, M are minimally irreducible of degrees p, q, say. Conversely, given two minimally irreducible simple groups of prime degrees

p, q, their direct product is a minimally irreducible group of degree pq. More generally, we show that minimal irreducibility is preserved under direct products, with any number of factors and without requiring the factors to be simple, provided that the centre of the whole group is cyclic, the latter condition being equivalent to requiring that all factors of the same prime degree, except possibly one, have trivial centres.

Theorem 4.3. *Let $s \geqslant 2$, and $G = \prod_1^s N_i$. Then G is minimally irreducible of degree $\prod_{i=1}^s p_i$, p_i a prime, if and only if each N_i is minimally irreducible of degree p_i and* $\gcd(|Z(N_i)|, |Z(N_j)|) = 1$ *for $i \neq j$.*

Proof. We first observe that the coprimality condition on the centres implies that $Z(G)$ is cyclic, a necessary requirement for G to admit a faithful irreducible representation. Conversely, if each N_i is minimally irreducible of degree p_i, then $Z(G)$ cyclic implies the coprimality condition. Indeed, let us say that N_i is associated to the prime p if $p = p_i$. If N is a minimally irreducible group of degree p, then by Lemma 2.2 and Theorem 2.4 either N is simple, hence centreless, or its centre has p-power order. Thus $Z(G)$ cyclic implies that at most one factor among those associated to a given p has non-trivial centre. This is just the coprimality condition.

Assume first that $G = \prod_1^s N_i$ is minimally irreducible with respect to a character χ of degree $\prod_{i=1}^s p_i$. Then $\chi = \prod_{i=1}^s \theta_i$, where each θ_i is a faithful irreducible character of N_i. Moreover $\theta_i(1)$ cannot be composite for any i, since otherwise $\theta_j(1) = 1$ for some $j \neq i$, and therefore $\prod_{i \neq j} N_i$ would be a proper χ-irreducible subgroup of G. Thus, up to reordering, $\theta_i(1) = p_i$. Clearly N_i is θ_i-minimally irreducible for each i. For let T_i be a proper irreducible subgroup of N_i. Then $T_i \times \prod_{j \neq i} N_j$ would be a proper χ-irreducible subgroup of G. Finally, the coprimality condition follows from the remarks at the beginning of the proof.

We now turn to converse part of the statement. We want to prove that, if each N_i is minimally irreducible of prime degree p_i with respect to an irreducible character θ_i and $\gcd(|Z(N_i)|, |Z(N_j)|) = 1$ for $i \neq j$, then G is minimally irreducible of degree $\prod_{i=1}^s p_i$ with respect to the character $\chi = \prod_{i=1}^s \theta_i$. The assertion is trivially true if $s = 1$, and so we assume that $s > 1$ and argue by induction on s.

Suppose that G is not minimally irreducible and let M be a maximal irreducible subgroup of G. Let $[s] = I \dot{\cup} J$ $(I, J \neq \emptyset)$, $P = \prod_I N_i$, $Q = \prod_J N_j$, and write $G = P \times Q$. We claim that $M \not\geqslant P$. Otherwise, $M = P \times (M \cap Q)$ and $M \cap Q$ is a proper subgroup of Q. Now $\chi_{|M} = \chi_{|P} \cdot \chi_{|M \cap Q}$. However, by induction, Q is minimally irreducible with respect to $\chi_{|Q}$. It then follows that $\chi_{|M \cap Q}$ splits into irreducible constituents, and therefore also $\chi_{|M}$ is reducible, a contradiction. Thus the maximality of M implies that $G = MP$. Let π and ρ denote the projections of G onto P and Q, respectively. Then $P = G^\pi = (MQ)^\pi = M^\pi$; similarly, $Q = M^\rho$. Moreover $M \cap P$ is normalized by M and centralized by Q. Since by symmetry $G = MQ$, it follows that $M \cap P$ is a normal subgroup of G, hence of P, and similarly $M \cap Q$ is a normal subgroup of Q.

With the above information available, we first assume that none of the factors N_i are soluble.

Suppose that $s = 2$. Assume that $G = N_1 \times N_2$, where N_1 and N_2 are simple. Since N_1 is a minimal normal subgroup of G and $M \not\geqslant N_1$, it follows that $M \cap N_1 = 1$. (Similarly, $M \cap N_2 = 1$.) We readily see that $M \cong N_1 \cong N_2$. Indeed,

$$N_2 \cong G/N_1 = MN_1/N_1 \cong M/M \cap N_1 \cong M$$

(and similarly, interchanging N_1, N_2). Thus M is a simple group with minimally irreducible complex representations of prime degrees p_1, p_2. By Lemma 2.2, p_1, p_2 occur to the first power in $|M|$. On the other hand, as $\chi_{|M}$ is irreducible, $p_1 p_2 \mid |M|$, which implies that p_1, p_2 are distinct. Table 1 now shows that $M \cong \mathrm{PSL}(2, u)$ for some u. Moreover, if u is odd, then u is prime and $p_1 p_2 = u(u+1)/2$. Now M has an irreducible complex representation of degree $p_1 p_2$; however, the degree of an irreducible character of $\mathrm{PSL}(2, u)$ belongs to $\{1, u, \frac{1}{2}(u \pm 1), u \pm 1\}$ (cf. [6, Chapter 38]). If u is even, then $p_1 p_2 = u^2 - 1 > u + 1$, where the latter is the maximum character degree of $\mathrm{PSL}(2, u)$.

Next assume that $s > 2$ and let M be as above. Thus $M \not\geqslant N_i$ and $G = MN_i$ for $i = 1, \ldots, s$. Set $P_i = \prod_{j \neq i} N_j$. Then $M \cap P_i \trianglelefteq P_i$. But this implies that either $M \cap P_i = 1$, or $M \cap P_i$ is the product of some of the groups N_j with $j \neq i$. The latter is impossible, and so

$$N_i \cong G/P_i = MP_i/P_i \cong M/(M \cap P_i) \cong M$$

for all i (in particular, the groups N_i are isomorphic). Since $M \cap N_1 = 1$, we also have $G/P_1 \cong M \cong G/N_1$. It follows that $|P_1| = |N_1|^{s-1} = |N_1|$; this forces $s = 2$, a contradiction.

We now show that, if some subgroups N_i are simple then none of them can be soluble. Suppose the contrary, and write $G = P \times Q$, where P is the direct product of all the simple subgroups N_i, and Q is the product of the soluble ones, and both P, Q are non-trivial. Then (e.g., see [25, (II.4.19)])

$$P/(M \cap P) = M^{\pi}/(M \cap P) \cong M^{\rho}/(M \cap Q) = Q/(M \cap Q).$$

Since $M \not\geqslant P$, both the first and the last terms are non-trivial, but the first, being a quotient of a semisimple group, is semisimple, while the last one is soluble, a contradiction.

We are left now with the case when G is soluble. By Theorem 2.4 we have $G = P \times Q$, where P is the direct product of all the subgroups N_i which are non-nilpotent and Q is the direct product of those that are nilpotent. We prove that $P = 1$. Suppose the contrary and let A be a non-nilpotent direct factor of G. Set $G = A \times B$. By Theorem 2.4, $A = W : X$, where W is an elementary abelian r-group for some prime r and X acts irreducibly on W. Since $M \cap W$ is normalized by M and centralized by B, it is normal in $G = MB$ (and *a fortiori* in A). We claim that $M \geqslant W$. Suppose otherwise. Then $G = MW$ and $M \cap W$ is a normal subgroup of A properly contained in W. Thus, by irreducibility, $M \cap W = 1$. Since $G = MA$, it follows that

$$|W| = |MW : M| = |G : M| = |MA : M| = |A : (A \cap M)|.$$

But $A \cap M \trianglelefteq A$. Since it does not contain W, by the Remarks following Theorem 2.4 we conclude that

$$A \cap M \leqslant \mho_1(X) = Z(A).$$

Thus $A/(A \cap M)$ is not a primary group, and we get a contradiction. Next we claim that $Z = Z(A) \leqslant M$. Otherwise, $G = MZ$, whence

$$|G : M| = |MZ : M| = |MA : M| \quad \text{and} \quad |(A \cap M)Z| = \frac{|Z|\,|A \cap M|}{|M \cap Z|} = |A|.$$

Thus $A = (A \cap M)Z$. Since $A \cap M \lneq A$, this implies that $A \cap M$, and hence also A, is abelian, a contradiction. We have now obtained that $A \cap M = WZ(A)$ for any non-nilpotent direct factor $A = W : X$ of G. In particular,

$$|A/(A \cap M)| = |B/(B \cap M)| = p,$$

where p is the prime associated to A. Thus

$$T = (A \cap M) \times (B \cap M) < M < G,$$

$|G : T| = p^2$ and $|G : M| = p$. Now $\chi = \alpha\eta$, where $\alpha \in \mathrm{Irr}(A)$ and $\eta \in \mathrm{Irr}(B)$. Since both A and B are minimally irreducible (by induction), both α and β split over $A \cap M$ and $B \cap M$, respectively. In fact, by a well-known result on characters (see [10, Theorem 6.18]) they both split into p distinct irreducible constituents. On the other hand, since χ restricts irreducibly to M, and $|M : T| = p$, by the same result χ must split into p irreducible constituents, a contradiction. This proves that none of the subgroups N_i is non-nilpotent, and G itself is nilpotent. By the coprimality assumption on the centres, it now follows that the subgroups N_i have coprime orders. Write $G = P \times H$, where P is the Sylow p-subgroup of G and H its Hall p'-subgroup. Since each of P, H is obviously a product of some of the subgroups N_i, we have $G = MP = MH$, whence $|P/(P \cap M)| = |H/(H \cap M)|$. But the former is a non-trivial p-power while the latter is coprime to p. This is the final contradiction.

We now sum up the situation so far. If G has more than one non-abelian minimal normal subgroup, then $G = N \times M$, where N, M are non-abelian simple groups, minimally irreducible of degrees p, q, say and both $\chi_{|N}$ and $\chi_{|M}$ are homogeneous. Conversely, by Theorem 4.3 any such direct product is minimally irreducible of degree pq. So let us suppose that G has a unique non-abelian minimal normal subgroup N, such that $\chi_{|N}$ is homogeneous: say, $\chi_{|N} = q\theta$, $\theta(1) = p$. By Lemma 4.1, $N = S$ is simple. Set $C = C_G(S)$. If $G = SC = S \times C$, then, by Theorem 4.3, C is minimally irreducible of degree q. By Lemma 2.2 the subgroup C is soluble (and therefore is one of the groups listed in Theorem 2.4). Conversely, any such direct product is minimally irreducible of degree pq, again by Theorem 4.3. We are left to examine the case when SC is a *proper* normal subgroup of G. We show that SC behaves well: in particular it provides, also in the homogeneous case, a 'natural' normal subgroup of G with cyclic factor group.

Lemma 4.4. *Suppose that G satisfies the assumptions of Lemma* 4.1. *Set $N = S$ and $C = C_G(S)$. Then G/SC is isomorphic to a cyclic subgroup of $I_{\mathrm{Out}(S)}(\theta)$.*

Proof. Observe that, via obvious identifications, $\mathrm{Inn}(S) \leqslant G/C \leqslant \mathrm{Aut}(S)$, whence $G/SC \leqslant \mathrm{Out}(S)$. Thus the homogeneity assumption implies that $gSC \in I_{\mathrm{Out}(S)}(\theta)$ for each $g \in G$. All groups S listed in Table 1 have a cyclic outer automorphism group, except for the following cases: $S = \mathrm{PSL}(2,u)$, $p = \frac{1}{2}(u+1)$, $u > 11$; $S = \mathrm{PSL}(2,9)$, $p = 5$; $S = \mathrm{PSp}(2^{t+1},u)$, $p = \frac{1}{2}(u^{2^t}+1)$. In all cases, $\mathrm{Out}(S) \cong \langle \mathrm{Fr} \rangle \times \langle \partial \rangle$ and θ is a Weil representation of S (see the Remarks following Table 1). As shown in Theorem 3.9, θ is fixed by Fr and moved by ∂. Thus $I_{\mathrm{Out}(S)}(\theta)$ is a subgroup of the cyclic group $\langle \mathrm{Fr} \rangle$.

Lemma 4.5. *Let G be as in Lemma* 4.4. *Furthermore, suppose that G is χ-minimally irreducible and $SC \lneqq G$. Then $\chi_{|SC} = \theta \sum_1^q \gamma_i$, where $\gamma_1, \ldots, \gamma_q$ are distinct linear characters of C.* (*In particular, C is never trivial.*)

Proof. By Lemma 4.4, the subgroup SC has a cyclic supplement Y in G. Now $\chi_{|SC} = e \sum_1^t \varphi_i$, where $\varphi_i \in \mathrm{Irr}(SC)$. Since $SC = S \times C$, we have $\varphi_i = \theta_i \gamma_i$, where $\theta_i \in \mathrm{Irr}(S)$ and $\gamma_i \in \mathrm{Irr}(C)$. Restricting further to S, we deduce that $\theta_i = \theta$ for each i. It follows that $q = e \sum_1^t \gamma_i(1)$, so that either $e = q$ and $t = 1$, or $e = 1$ and $\sum_1^t \gamma_i(1) = q$. Suppose that the former case holds. Then $\chi_{|SC} = q\varphi$, where $\varphi \in \mathrm{Irr}_p(SC)$ is G-invariant. Since $G = SCY$, φ extends to a character $\psi \in \mathrm{Irr}_p(G)$. By Gallagher's Theorem, $\mathrm{Irr}(G|\varphi) = \{\psi\zeta \mid \zeta \in \mathrm{Irr}(G/SC)\}$. However, since χ lies in this set, we get $\chi(1) = q$, a contradiction. Thus the second option holds. Hence $\chi_{|C} = p \sum_1^t \gamma_i$. As the characters γ_i are G-conjugate and $q = \sum_1^t \gamma_i(1)$ is prime, it follows that either $t = 1$ and $\gamma_1(1) = q$, or $t = q$ and $\gamma_1, \ldots, \gamma_t$ are distinct linear characters of C. However, if γ_1 has degree q, then $\chi_{|SC} = \theta\gamma_1$ is irreducible, and the minimality assumption forces $G = SC$, a contradiction. The statement follows.

We can now describe the structure of the minimally irreducible groups G meeting the conclusions of Lemma 4.5. In particular, we show that S is a minimally irreducible group of degree p with non-trivial outer automorphism group, and point out some properties of CY.

Theorem 4.6. *Suppose that G satisfies the assumptions of Lemma* 4.1. *Set $N = S$ and $C = C_G(S)$. Then G is χ-minimally irreducible with $\chi_{|S} = q\theta$ and $\chi_{|SC} = \theta \sum_1^q \gamma_i$ if and only if the following assertions hold.*

(1) *G/SC is isomorphic to a non-identity cyclic q-subgroup $\langle \bar{\alpha} \rangle$ of $I_{\mathrm{Out}(S)}(\theta)$.*

(2) *S is θ-minimally irreducible, and the quadruple (S, p, q, α) is one of the following*:

(a) $(\mathrm{PSL}(2,r), r, 2, \partial)$, θ *the Steinberg character*;

(b) $(\mathrm{PSL}(2,u), u+1, 2, \sigma)$, σ *a suitable field automorphism*;

(c) $(\mathrm{PSL}(2,u), \frac{1}{2}(u+1), 2, \sigma)$, σ *any field automorphism*;

(d) $(\mathrm{PSL}(n,u), (u^n-1)/(u-1), n, \sigma)$, n *an odd prime*, σ *a suitable field automorphism*;

(e) $(\mathrm{PSL}(n,u), (u^n-1)/(u-1), 2, \gamma)$, n *an odd prime*, γ *the graph automorphism*, $\theta = \theta_{(u-1)/2}$;

(f) $(\mathrm{PSU}(n,u), (u^n+1)/(u+1), n, \sigma)$, n *an odd prime*, σ *any field automorphism of n-power order*;

(g) $(\mathrm{PSU}(n,u), (u^n+1)/(u+1), 2, \sigma)$, n *an odd prime*, σ *any field automorphism of 2-power order*.

(3) *There exists a cyclic q-subgroup Y of G such that $G = S : CY$.*

(4) $\{\gamma_i \mid 1 \leqslant i \leqslant q\}$ *is a Y-orbit, and* $\bigcap_i \ker \gamma_i = 1$. *In particular, $C\mho_1(Y)$ is abelian, whilst CY is not.*

(5) $C_C(Y)$ *is a q-group, and for any Y-invariant subgroup T of C, either $T \leqslant C_C(Y)$ or $TY = CY$.*

Proof. Suppose first that G is χ-minimally irreducible. By assumption, $SC \lneqq G$. Thus, by Lemma 2.1 applied to $N = SC$, there exists a cyclic q-supplement $Y_0 = \langle y_0 \rangle$ to SC in G. Furthermore, by Lemma 4.4, Y_0 maps to a non-trivial subgroup of $I_{\mathrm{Out}(S)}(\theta)$. So (1) holds. Now assume that S is not θ-minimally irreducible. Arguing as in Theorem 3.19, we may determine an irreducible subgroup T of S such that $T^{y_0} = T^s$ with $s \in S$. Let $Y_1 = \langle y_1 \rangle$, where y_1 is the q-part of $y_0 s^{-1}$. Then T is Y_1-invariant and $M = TCY_1$ is a subgroup of G. Since $\chi_{|TC} = \sum \theta_{|T}\gamma_i$, a sum of q distinct Y_1-conjugate irreducible characters, M is a proper χ-irreducible subgroup of G by Proposition 2.8, a contradiction. (Observe that the exceptional case arising in Theorem 3.19, namely $S = \mathrm{PSL}(2,9)$, $p = 5$ and $q = 2$, is excluded here, as both ∂ and $\partial \circ \mathrm{Fr}$ move θ.) As a result, Table 1 together with the Orbit Theorem 3.9 gives (2). We have already seen that $\mathrm{Inn}(S) \lesssim G/C \lesssim I_{\mathrm{Aut}(S)}(\theta)$. Next, we claim that $\mathrm{Inn}(S)$ is complemented in $I_{\mathrm{Aut}(S)}(\theta)$ whenever S runs over the minimally irreducible groups listed in Table 1 which have non-trivial outer automorphisms. Since our claim is obviously true if only field automorphisms are involved in $I_{\mathrm{Aut}(S)}(\theta)$, this leaves us to check the following two cases:

(i) $S = \mathrm{PSL}(2,p)$, $p = r$, θ the Steinberg character. Here $\mathrm{Out}(S)$ is generated by the diagonal automorphism ∂, which is induced by the matrix $D = ve_{12} + e_{21}$, where v is a primitive element of $\mathbb{F}_u$. Since $D^2 = v1_2$, we are done.

(ii) $S = \mathrm{PSL}(n,u)$, n an odd prime, $u = r^{n^s}$, $p = (u^n-1)/(u-1)$. Here

$$\mathrm{Out}(S) = \langle \mathrm{Fr} \rangle \times \langle \gamma \rangle.$$

Since γ is realized by the invert-transpose automorphism, we are done.

Since $\mathrm{Inn}(S)$ is complemented in $I_{\mathrm{Aut}(S)}(\theta)$, it also follows that $\mathrm{Inn}(S)$ is complemented in G/C. Now pick any cyclic q-subgroup Y of G such that CY/C is such a complement. (Clearly such a Y does exist, since any complement of $\mathrm{Inn}(S)$ in G/C is a cyclic q-group.) Then

$$C = SC \cap YC = (S \cap YC)C,$$

whence $YC \cap S \leqslant C \cap S = 1$. Thus Y fulfils (3). In order to prove (4), observe that, since $\theta\gamma_1, \dots, \theta\gamma_q$ are Y-conjugate, so are $\gamma_1, \dots, \gamma_q$. Moreover, since $\chi_{|C} = p\sum_{i=1}^{q} \gamma_i$ and χ is faithful, $\bigcap \ker \gamma_i = 1$. Finally, $[\mho_1(Y), C] = 1$. Indeed, set $Y = \langle y \rangle$ and $z = y^q$. Since z stabilizes each γ_i and $\sum_{i=1}^{q} \gamma_i$ is a faithful character of C, it follows that $[z, C] = 1$.

In order to prove (5), we first observe that, for a subgroup T of C, we have $T \not\leqslant C_C(Y)$ if and only if T is not central in G, that is, is not scalar. By Clifford theory, this is equivalent to requiring that $\chi_{|T} = p\sum_1^q \tau_i$, where the characters τ_i are distinct. Now assume that $T \leqslant C$ is a Y-invariant subgroup not contained in $C_C(Y)$. Set $M = STY$, $D = ST$, and $\chi_{|M} = \sum_1^r \varphi_i$, $\varphi_i \in \mathrm{Irr}(M)$. Note that M is not *a priori* normal in G, but it is Y-invariant. Since, by the above, $\chi_{|D} = \theta\sum_1^q \tau_i$, we have $\varphi_{1|D} \geqslant \theta\tau_1$. Since φ_1 and D are both Y-invariant and the characters τ_i are Y-conjugate, we also deduce that $\varphi_{1|D} \geqslant \theta\tau_i$ for every $i \in [q]$. Thus $\varphi_{1|D} \geqslant \chi_{|D}$. By comparison of degrees we conclude that $\chi_{|M} = \varphi_1$, whence $M = G$ and $TY = CY$. Therefore, for any Y-invariant subgroup T of C, either T is contained in $C_C(Y)$ (equivalently, T is scalar), or $TY = CY$.

We now show that $C_C(Y)$ is a q-group. Indeed, since C is abelian, $C = A \times B$, where $A \in \mathrm{Syl}_q(C)$ and $B \in \mathrm{Hall}_{q'}(C)$. If B is trivial, C and *a fortiori* $C_C(Y)$ are q-groups. So suppose that $B \neq 1$. Since AY is a q-group, $AY < CY$. This forces $A \leqslant C_C(Y)$ and $C_C(Y) = A \times C_B(Y)$. Now B decomposes as $C_B(Y) \times [B, Y]$ (see [25, Chapter 2, (5.17)]). Thus $C = A \times C_B(Y) \times [B, Y]$. We have $[B, Y] \neq 1$, since otherwise CY would be abelian. Since $[B, Y]$ is Y-invariant and does not centralize Y, we must have $[B, Y]Y = CY$, and hence $[B, Y](C \cap Y) = C$. It follows that $C_B(Y) = 1$, and we conclude that $C_C(Y) = A$.

Conversely, assume that properties (1)–(5) hold. By (4) and Proposition 2.9, there exists a faithful character $\chi \in \mathrm{Irr}_{pq}(G)$ such that $\chi_{|SC} = \theta\sum_1^q \gamma_i$. Let M be a maximal irreducible subgroup of G and set $H = M \cap SC$. By (1), q divides $|\mathrm{Out}(S)|$. Thus $p \neq q$ by Lemma 2.6. Using (3), we then conclude from Proposition 2.7 (with $N = SC$) that H is an irreducible group of degree p with respect to the character $\eta_i = (\theta\gamma_i)_{|H}$. Let π be the projection of H to S. Since $H \leqslant H^\pi \times C$, it follows that H^π is a θ-irreducible subgroup of S, whence, by (2), we have $H^\pi = S$. Now let ρ be the projection of H to C. The map sending $s \in S$ to the coset $c_h(H \cap C)$, where $sc_h \in H$, is an epimorphism from S to $H^\rho/(C \cap H)$ with kernel $S \cap H$. Thus

$$S/(S \cap H) \cong H^\rho/(C \cap H).$$

Since the latter group is abelian, we obtain that $S = S \cap H$ and $H = S \times (H \cap C)$. Arguing as in Theorem 3.19, we see that, without loss of generality, we may assume that M contains Y. Thus both H and $T = H \cap C$ are Y-invariant. Set $\tau_i = (\gamma_i)_{|T}$. Then $\eta_i = \theta\tau_i$ and, as $\eta_1, \dots, \eta_q$ are distinct, $[T, Y] \neq 1$. By (5) it follows that $TY = CY$, whence $M = G$. We conclude that G is χ-minimally irreducible.

Theorem 4.6 provides fairly detailed information on the structure of G. However, we can further refine the analysis and show that CY belongs to a quite restricted list of isomorphism types. We need the following lemma.

Lemma 4.7. *Let C be an abelian group acted upon non-trivially by a cyclic q-group Y. Assume that $[\mho_1(Y), C] = 1$ and that for each Y-invariant subgroup T of C, either T centralizes Y or $TY = CY$. Then there exists a Y-invariant subgroup W of C such that* (1) *$WY = CY$ and* (2) *the only Y-invariant subgroup of W not centralizing Y is W itself.*

Proof. Let $C = A \times B$, where A and B are the Sylow q-subgroup and the Hall q'-subgroup of G, respectively. Suppose first that $[A, Y] \neq 1$; then, by assumption, $CY = AY$ is a q-group, and hence $B = B \cap CY = 1$. Set $Q = CY$. Since C and $C_C(Y)$ are both normal in Q, there exists a normal subgroup L of Q such that $C_C(Y) \leqslant L \leqslant C$ and $|L : C_C(Y)| = q$. Thus $L = \langle a, C_C(Y)\rangle$ for some $a \in C \backslash C_C(Y)$. Set $Y = \langle y \rangle$. Since Y must act trivially on $L/C_C(Y)$, there exists $c \in C_C(Y)$ such that $a^y = ac$. It follows that $a^{y^i} = ac^i$ for any i. In particular, as $[\mho_1(Y), C] = 1$, we have $a = a^{y^q} = ac^q$, whence $c^q = 1$. Now set $W = \langle a, c \rangle$. Then $W \geqslant [W, Y] \neq 1$. Thus by assumption $WY = CY$, and (1) is proven. Let $T \not\leqslant C_W(Y)$ be a Y-invariant subgroup of W. Since $[WY, Y] = [W, Y] = \langle c \rangle$, it follows that $[T, Y] = \langle c \rangle$. But $W/[W, Y]$ is cyclic, and its (unique) maximal subgroup coincides with $C_W(Y)/[W, Y]$. It follows that $T = W$, and (2) is proven. Next, suppose that $[A, Y] = 1$, so that $[B, Y] \neq 1$. Since $B = C_B(Y) \times [B, Y]$, it follows that $[B, Y]$ does not centralize Y, and therefore $[B, Y]Y = CY$. But this obviously implies that $[B, Y] = B$, and hence $C_B(Y) = 1$. Set $W = B$. Then W satisfies both (1) and (2). (As for (2), observe that if X is a non-trivial Y-invariant subgroup of B then by assumption $XY = BY$, and hence $X = B$.)

Conditions (4), (5) in Theorem 4.6 lead us to classify the non-abelian groups which are the product of an abelian normal subgroup W and a cyclic q-subgroup Y, satisfying the following conditions: (1) $[\mho_1(Y), W] = 1$; (2) for each Y-invariant subgroup T of W, either T centralizes Y or $T = W$; (3) there exists $\gamma \in \mathrm{Irr}(W)$ such that the Y-orbit $\gamma^Y = \{\gamma = \gamma_1, \dots, \gamma_q\}$ has length q and $\bigcap_i \ker \gamma_i = 1$. Since these groups will turn out to be generalizations of the Suprunenko groups in Theorem 2.4, we shall call them *almost Suprunenko* groups. In particular, they turn out to be either q-primary or minimally irreducible of degree q. To indicate the relation with the prime q, it may sometimes be convenient to call them *almost q-Suprunenko* groups.

Theorem 4.8. *A group X is almost Suprunenko if and only if it is one of the following*:

(1) *a non-nilpotent Suprunenko group* (*of type* (iii) *in Theorem* 2.4);

(2) *a generalized modular q-group*

$$U_{l,m}(q) = \langle a, b \mid a^{q^l} = b^{q^m} = 1, a^{q^{l-1}} = [a, b] \rangle, \quad 1 < l \neq m;$$

(3) *a group of order* q^{l+2} *with presentation*

$$V_l(q) = \langle a, b \mid a^q = b^{q^l} = [a, b, b] = [a, b, a] = 1 \rangle, \quad l \geqslant 1;$$

(4) *the quaternion group* Q_8 *of order* 8.

Proof. We first show that all our candidates are indeed almost Suprunenko. It is useful to recall Brauer's Permutation Lemma ([10, (6.32)]) which relates the actions of Y on W and $\mathrm{Irr}(W)$. Namely, if m_i and n_i denote the number of Y-orbits of length q^i on W and $\mathrm{Irr}(W)$ respectively, then $n_0 = m_0$ and $\sum_{i \geqslant 0} n_i = \sum_{i \geqslant 0} m_i$ (that is, the two actions have the same number of orbits). The following simple observation will also be useful throughout. Let X be a group, and $x \in X$ be such that $[X, x]$ is contained in $Z(X)$. Then the map $g \mapsto [g, x]$ is a homomorphism from X to $Z(X)$.

Assume first that X is a non-nilpotent Suprunenko group. Then, by Theorem 2.4, $X = W : Y$, where W is an elementary abelian irreducible $Y/\mho_1(Y)$-module and Y is a cyclic q-group. Thus $[\mho_1(Y), W] = 1$. Moreover, as W is Y-irreducible, it has no proper Y-invariant subgroup and so condition (2) obviously holds. Finally, observe that, as $C_W(Y) = 1$, by Brauer's Lemma the only Y-invariant irreducible character of W is the trivial one. Since $[\mho_1(Y), W] = 1$, this implies that, if $1 \neq \gamma \in \mathrm{Irr}(W)$, then $|\gamma^Y| = q$. Set $\gamma^Y = \{\gamma = \gamma_1, \dots. \gamma_q\}$. Then $K = \ker \gamma$ is an hyperplane of W and the irreducibility of W under Y forces $\bigcap \ker \gamma_i = \mathrm{Core}_X(K) = 1$.

Next, suppose that $X = U_{l,m}(q)$ and set $W = \langle a \rangle$, $Y = \langle b \rangle$. It is easy to see that b^q centralizes a, that is, $[\mho_1(Y), W] = 1$. Also $C_W(Y) = \mho_1(W)$. Therefore any proper subgroup of W is centralized by Y, and (1), (2) are proved. Finally, let γ be a faithful irreducible character of W. Since $\gamma^y = \gamma$ if and only if $[y, w] = 1$ for every $w \in W$, that is, if and only if $X' = \Omega_1(W) \leqslant \ker \gamma$, it follows that γ^Y has length q.

Now suppose that $X = V_l(q)$. Since $[a, b]$ is central and a has order q, by the observation above $[a, b]$ also has order q. Thus, setting $W = \langle a \rangle \times \langle [a, b] \rangle$ and $Y = \langle b \rangle$, we see that W is elementary abelian of order q^2 and $C_W(Y) = \langle [a, b] \rangle = X'$. Observe that $b^q a = ab^q [b, a]^q = ab^q$; hence condition (1) is met. Now let T be a Y-invariant subgroup of W not contained in $\langle [a, b] \rangle$. Then $1 \neq [T, Y] \leqslant T$. It follows that $[T, Y] = C_W(Y)$ and therefore $T = W$. So condition (2) is met. Let $\gamma \in \mathrm{Irr}(W)$ with $\ker \gamma = \langle a \rangle$. Then γ is not Y-stable. Set $\gamma^Y = \{\gamma = \gamma_1, \dots \gamma_q\}$. Then $\bigcap \ker \gamma_i$, being a Y-invariant subgroup of $\langle a \rangle$, is necessarily trivial.

Finally, suppose that $X = Q_8 = \langle a, b \mid a^4 = 1,\ a^2 = b^2 = [a, b] \rangle$, and set $W = \langle a \rangle$, $Y = \langle b \rangle$. Since b^2 centralizes a, and the only proper subgroup of $\langle a \rangle$ coincides with $Z(Q_8)$, both (1) and (2) hold. Also, any faithful character of W satisfies (3).

We now turn to the more difficult converse implication. Assume that $X = WY$ is almost Suprunenko. Since obviously any Sylow subgroup of W is Y-invariant, and WY is not abelian, W must be a primary group. If W is a q'-group, then $W = C_W(Y) \times [W, Y]$. Since $1 \neq [W, Y]$ is a Y-stable subgroup of W, it follows that $W = [W, Y]$ and $C_W(Y) = 1$. In particular, since any characteristic subgroup of

W would obviously be Y-stable, W is characteristically simple. Hence W is an elementary abelian Y-irreducible module, and so WY is a non-nilpotent Suprunenko group, by Theorem 2.4. Now assume that W is a q-group, so that X itself is a q-group. (Recall that $W\mho_1(Y)$ is a normal abelian subgroup of $X = WY$, and therefore $\mathrm{cd}(X) = \{1, q\}$.) Observe that

$$Z(X) \geqslant C_W(Y)\mho_1(Y) \quad \text{and} \quad X' = [W, Y] \leqslant C_W(Y).$$

Also observe that W has a subgroup L normal in X such that $|W : L| = q$ (see [9, (III.7.2.d)]). By our assumptions on $X, L \leqslant C_W(Y)$, and hence $L = C_W(Y)$. Since X is not abelian, we readily deduce that $Z(X) = C_W(Y)\mho_1(Y)$ and $|X : Z(X)| = q^2$. Thus X is nilpotent of class 2 and $X = \langle a, b, Z(X)\rangle$ for suitable elements a, b. Clearly $X' = \langle [a, b]\rangle$. Moreover, since a^q and $[a, b]$ are both central, again by the observation above we have $1 = [a^q, b] = [a, b]^q$; hence $X' = \langle [a, b]\rangle$ has order q.

Write $K = \ker\gamma$, where $\gamma \in \mathrm{Irr}(W)$ satisfies condition (3), and as above set $L = C_W(Y)$. Obviously $L \cap K \leqslant \ker\gamma_i$ for each $i \in [q]$. Thus $\bigcap_i \ker\gamma_i = 1$ forces $L \cap K = 1$. Suppose first that $K \neq 1$. Since $|W : L| = q$ we have $K \times L = W$ and $|K| = q$. Now $K[K, Y]$ is a Y-invariant subgroup of W not contained in L, and hence $W = K \times [K, Y]$. It follows that $L = [K, Y]$ has order q, and therefore W is an elementary abelian group of order q^2. We observe that, since $X = WY$, any further relation among the generators of X is controlled by the intersection $W \cap Y$. If $W \cap Y \neq 1$, then $W \cap Y = [K, Y] = \Omega_1(Y)$, and we may assume that $[y, k] = y^{q^{l-1}}$, where $Y = \langle y\rangle$, $o(y) = q^l$ and $K = \langle k\rangle$ (see [9, (I.14.9)]). Therefore in this case $WY \cong U_{l,1}(q)$, the modular group of order q^{l+1}. On the other hand, if $W \cap Y = 1$, then $WY \cong V_l$, a group of order q^{l+2}, where l defined as before. So we are left with the case when $K = 1$. Then γ faithfully represents W and so W is cyclic, say generated by w. Since W is normal in X, y acts as an automorphism of order q on w, and we may assume that $[w, y] = w^{q^{l-1}}$, where $o(w) = q^l$. If $W \cap Y = 1$, then $X \cong U_{l,m}(q)$. Otherwise, $\Omega_1(Y) = \Omega_1(W) = \langle [w, y]\rangle$, and we are facing a symmetric situation. Assume that $q^l = |W| \geqslant |Y| = q^m$. Let $W \cap Y = \langle w^{q^s}\rangle = \langle y^{q^t}\rangle$. If $y^{q^t} = w^{xq^s}$, with $\gcd(x, q) = 1$, we may replace w by w^x and obtain $y^{q^t} = w^{q^s}$. Moreover, $m = l - s + t$ and $s \geqslant t$. We claim that $y_1 = w^{-q^{s-t}}y$ has order q^t, unless $s = t = 1$ and $q = 2$ (note that $s, t \geqslant 1$, as X is not cyclic). Indeed, $y_1^{q^t} = [w^{-1}, y]^{q^{s-t}\binom{q^t}{2}} \neq 1$ if and only if $q \nmid q^{s-t}\binom{q^t}{2}$, that is, if and only if $q = 2$ and $s = t = 1$. If the latter occurs, and furthermore $l \geqslant 3$, set $y_2 = w^{-2^{l-2}}y_1$. Then $y_2^2 = w^{-2^{l-1}}y_1^2 = 1$. Notice that we have deformed y by an element of W. Thus either $X = \langle w, y_1\rangle$ or $X = \langle w, y_2\rangle$, and in both cases $X \cong U_{l,t}(q)$. Finally, if $l = q = 2$ and $s = t = 1$, then $X \cong Q_8$.

Remarks. (1) By Theorem 4.8 (1), a non-nilpotent almost Suprunenko group is actually a Suprunenko group of type (iii). However if q is odd then $V_1(q)$ is the extraspecial group of order q^3 and exponent q, whilst $U_{2,1}(q)$ is the extraspecial group of order q^3 and exponent q^2. Furthermore, $V_1(2)$ is the dihedral group D_8 of order 8. Since $U_{l,1}$ is the modular q-group, we see that all nilpotent Suprunenko groups are also almost Suprunenko. Therefore the class of almost Suprunenko groups strictly contains the class of Suprunenko groups.

(2) Theorem 4.8 shows that if X is a nilpotent almost q-Suprunenko group, then $|X : Z(X)| = q^2$ and $|X'| = q$; in particular, X is of nilpotency class 2. It is not difficult to check that almost q-Suprunenko groups of different types are not isomorphic, apart from the exception of $U_{2,1}(2) \cong V_1(2) \cong D_8$.

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